Math 632. Homework Set 2

Solutions are due Thursday, February 18th.

As usual, all schemes are assumed of finite type over k.

Problem 1. Let X be a scheme, and $\mathcal{F}$ a quasicoherent sheaf on X.

i) Show that if $(\mathcal{F}_i)_i$ is a family of $\mathcal{O}_X$-submodules of $\mathcal{F}$, then we have an $\mathcal{O}_X$-submodule $\bigcap_i \mathcal{F}_i$ of $\mathcal{F}$ given by
$$\Gamma(U, \cap_i \mathcal{F}_i) = \cap_i \Gamma(U, \mathcal{F}_i).$$
ii) Show that if all $\mathcal{F}_i$ are quasicoherent, and the family is finite, then $\bigcap_i \mathcal{F}_i$ is quasicoherent. Give an example to show that this is not necessarily true if the family is infinite.
iii) Use this to give a new proof of the fact that if $(Y_i)_i$ is a finite family of closed subschemes of X, then there is a unique smallest closed subscheme $\bigcup_i Y_i$ of X with the property that each Y_i is a closed subscheme of $\bigcup_i Y_i$.

Problem 2. Let X be a scheme, and $\mathcal{F}$ a quasicoherent sheaf on X.

i) Suppose that $(\mathcal{F}_i)_i$ is a family of quasicoherent $\mathcal{O}_X$-submodules of $\mathcal{F}$. Show that if $\sum_i \mathcal{F}_i$ is the sheaf associated to the presheaf $\mathcal{P}$ given by $\Gamma(U, \mathcal{P}) = \sum_i \Gamma(U, \mathcal{F}_i)$, then $\sum_i \mathcal{F}_i$ is a quasicoherent $\mathcal{O}_X$-submodule of $\mathcal{F}$. Furthermore, show that if $U \subseteq X$ is an affine open subset, then $\Gamma\left(U, \sum_i \mathcal{F}_i\right) = \sum_i \Gamma(U, \mathcal{F}_i)$.
ii) Use this to give a new proof of the fact that if $(Y_i)_i$ is a family of closed subschemes of X, then there is a unique largest closed subscheme $\bigcap_i Y_i$ of X with the property that it is a closed subscheme of each of the Y_i.

Problem 3. Let X be a scheme. Show that if
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
is an exact sequence of $\mathcal{O}_X$-modules on X and if $\mathcal{F}'$ and $\mathcal{F}''$ are (quasi)coherent, then so is $\mathcal{F}$.

Problem 4. Prove that if $f \colon X \to Y$ is a morphism of schemes, and if $\mathcal{F}$ is a quasicoherent sheaf on X, and $\mathcal{E}$ is a locally free sheaf on Y, then there is a canonical isomorphism
$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{E})) \simeq f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E}.$$
This is the *projection formula*.

Problem 5. Let X be a topological space. Show that if
$$0 \to \mathcal{F}' \xrightarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{F}'' \to 0$$

is an exact sequence of sheaves of abelian groups on X such that $\mathcal{F}'$ is flasque, then the following sequence is exact

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0.$$

(Hint: given $s'' \in \Gamma(X, \mathcal{F}'')$, consider the set

$$\{(U, s) \mid U \,\text{open in}\, X, s \in \mathcal{F}(U), v(s) = s''|_U\}$$

ordered by $(U, s) \leq (V, t)$ if $U \subseteq V$ and $t|_U = s$. Show that this set has a maximal element and that if $\mathcal{F}'$ is flasque and (V, t) is maximal, then $V = X$).

Problem 6. Let $(X, \mathcal{O}_X)$ be a ringed space.

(1) If U is an open subset of X and $i \colon U \to X$ the inclusion, show that there is a sub-$\mathcal{O}_X$-module $i_!(\mathcal{O}_U)$ of $\mathcal{O}_X$ such that $i_!(\mathcal{O}_U)(V) = \mathcal{O}_X(V)$ if $V \subseteq U$, and $i_!(\mathcal{O}_U)(V) = 0$, otherwise.
(2) Show that if $x \in U$, then we have a canonical isomorphism $\mathcal{O}_{U,x} \simeq i_!(\mathcal{O}_U)_x$ and if $x \notin U$, then $i_!(\mathcal{O}_U)_x = 0$. The sheaf $i_!(\mathcal{O}_U)$ is called the *extension by zero* of $\mathcal{O}_U$.
(3) Show that for every $\mathcal{O}_X$-module $\mathcal{F}$, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(i_!\mathcal{O}_U, \mathcal{F}) \simeq \mathcal{F}(U).$$

(4) Deduce that if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then it is flasque.
(5) Give an example of a scheme X and an open subset U such that $i_!\mathcal{O}_U$ is not quasicoherent.