Homework Set 2

Solutions are due Monday, October 26th.

Problem 1. Let X and Y be quasi-affine varieties. We have seen on the previous problem set that if $f\colon X \to Y$ is a morphism, and if $p \in X$ is a point such that $f(p) = q$, then f induces a local ring homomorphism $\phi\colon \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$.

i) Show that if $f'\colon X \to Y$ is another morphism with $f'(p) = q$, and induced homomorphism $\phi'\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, then $\phi = \phi'$ if and only if there is an open neighborhood U of p such that $f|_U = g|_U$.
ii) Show that given any local morphism of local k-algebras $\psi\colon \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$, there is an open neighborhood W of p, and a morphism $g\colon W \to Y$ with $g(p) = q$, and inducing ψ.
iii) Deduce that $\mathcal{O}_{X,p}$ and $\mathcal{O}_{Y,q}$ are isomorphic as k-algebras if and only if there are open neighborhoods W of p and V of q, and an isomorphism $h\colon W \to V$, with $h(p) = q$.

Problem 2. Show that for every quasi-affine varieties X and Y, we have $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Problem 3. Show that if X is an affine variety such that $\mathcal{O}(X)$ is a UFD, then every closed subset $Y \subset X$ of pure codimension one (that is, such that every irreducible component has codimension one) is a hypersurface (recall that by definition, this means that $I(Y)$ is a principal ideal).

Problem 4. Show that if X and Y are irreducible closed subsets of $\mathbf{A}^n$, then every irreducible component of $X \cap Y$ has dimension $\geq \dim(X) + \dim(Y) - n$ (Hint: describe $X \cap Y$ as the intersection of $X \times Y \subseteq \mathbf{A}^n \times \mathbf{A}^n$ with the diagonal $\Delta = \{(x,x) \mid x \in \mathbf{A}^n\}$).

Problem 5. Let X be a (quasi-affine) variety, and p a point on X. Show that $\dim_p(X) := \dim(\mathcal{O}_{X,p})$ is equal to the largest dimension of an irreducible component of X that contains p.

Problem 6. Let X, Y, Z, and W be affine varieties.

i) Show that if $f\colon X \to Y$ and $g\colon Y \to Z$ are finite morphisms, then the composition $g \circ f$ is a finite morphism.
ii) Show that if $f\colon X \to Y$ and $g\colon Z \to W$ are finite morphisms, then the map $X \times Z \to Y \times W$, given by $(x,z) \to (f(x), g(z))$ is a finite morphism.

Problem 7. Let R be a commutative ring. The set of prime ideals of R is called the *spectrum* of R, and it is denoted by $\mathrm{Spec}(R)$. If I is an ideal in R, then we put

$$V(I) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq I\}.$$

i) Show that if we define the sets $V(I)$ to be the closed sets, $\mathrm{Spec}(R)$ becomes a topological space.
ii) Show that $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$.
iii) Show that a set $V(I)$ is irreducible if and only if $\sqrt{I}$ is a prime ideal.
iv) Show that a point $\mathfrak{p} \in \mathrm{Spec}(X)$ is closed if and only if $\mathfrak{p}$ is a maximal ideal.
v) Show that if X is an affine variety, then we have a homeomorphism of X with the subspace $\mathrm{Specm}(\mathcal{O}(X))$ of $\mathrm{Spec}(\mathcal{O}(X))$ consisting of the closed points.