Homework Set 1

Solutions are due Monday, September 25.

Problem 1. Let X be an affine algebraic variety, and let $\mathcal{O}(X)$ be the ring of regular functions on X. For every subset I of $\mathcal{O}(X)$, let

$$V(I) := \{p \in X \mid f(p) = 0 \text{ for all } f \in I\}.$$

For $S \subseteq X$, consider the following subset of $\mathcal{O}(X)$

$$I_X(S) := \{f \in \mathcal{O}(X) \mid f(p) = 0 \text{ for all } p \in S\}.$$

Show that the maps $V(-)$ and $I_X(-)$ define order-reversing inverse bijections between the closed subsets of X and the radical ideals in $\mathcal{O}(X)$. This generalizes the case $X = \mathbf{A}^n$ that we discussed in class.

Problem 2. Let $Y \subseteq \mathbf{A}^2$ be the *cuspidal curve* defined by the equation $x^2 - y^3 = 0$. Construct a bijective morphism $f\colon \mathbf{A}^1 \to Y$. Is it an isomorphism ?

Problem 3. Show that if X and Y are topological spaces, with X irreducible, and $f\colon X \to Y$ is a continuous map, then $\overline{f(X)}$ is irreducible. Use this to show that the closed subset

$$M^r_{m,n}(k) = \{A \in M_{m,n}(k) \mid \operatorname{rank}(A) \leq r\}$$

of $\mathbf{A}^{mn}$ is irreducible.

Problem 4. Let X be a topological space, and consider a finite open cover

$$X = U_1 \cup \ldots \cup U_n,$$

where each U_i is nonempty. Show that X is irreducible if and only if the following hold:

i) Each U_i is irreducible.
ii) For every i and j, we have $U_i \cap U_j \neq \emptyset$.

Problem 5. Let $n \geq 2$ be an integer.

i) Show that the set

$$B_n = \left\{(a_0, a_1, \ldots, a_n) \in \mathbf{A}^{n+1} \mid \operatorname{rank}\begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_1 & a_2 & \ldots & a_n \end{pmatrix} \leq 1\right\}$$

is a closed subset of $\mathbf{A}^{n+1}$.

ii) Show that

$$B_n = \{(s^n, s^{n-1}t, \ldots, t^n) \mid s, t \in k\}.$$

Deduce that B_n is irreducible.

I don't expect you to submit a solution for the next problem. I include it here for those interested in the correct generalization of what we did in class to the case when the ground field is not algebraically closed.

Problem 6 Recall first the construction of the *maximal spectrum* of an arbitrary commutative ring. Given a ring R, let $\mathrm{Specm}(R)$ be the set of all maximal ideals in R. For every ideal I in R, we define

$$V(I) := \{\mathfrak{m} \in \mathrm{Specm}(R) \mid I \subseteq \mathfrak{m}\}.$$

1) Show that $\mathrm{Specm}(R)$ has a structure of topological space in which the closed subsets are the subsets of the form $V(I)$, for an ideal I in R (if instead of *maximal ideals*, we consider *prime ideals*, we obtain the *spectrum* $\mathrm{Spec}(R)$ of R).

For every subset $S \subseteq \mathrm{Specm}(R)$, we define

$$I(S) := \bigcap_{\mathfrak{m} \in S} \mathfrak{m}.$$

2) Show that for every subset S of $\mathrm{Specm}(R)$, we have $V(I(S)) = \overline{S}$.

3) Show that if R is a k-algebra of finite type over a field, then for every ideal J in R, we have $I(V(J)) = \sqrt{J}$. (Hint: show first that it is enough to check this when $R = k[x_1, \ldots, x_n]$. In this case, consider the *integral* ring extension

$$k[x_1, \ldots, x_n] \hookrightarrow \overline{k}[x_1, \ldots, x_n],$$

where $\overline{k}$ is the algebraic closure of k, in order to reduce the assertion to the case of an algebraically closed field. A useful fact is that if $R \hookrightarrow R'$ is an injective, integral ring homomorphism, then for every maximal ideal $\mathfrak{m}$ in R, there is a maximal ideal $\mathfrak{m}'$ such that $\mathfrak{m} = \mathfrak{m}' \cap R$.)

4) We now keep the assumption that R is a k-algebra of finite type. Show that we have an inclusion $k^n \hookrightarrow \mathrm{Specm}(R)$ whose image consists of all maximal ideals $\mathfrak{m}$ such that the canonical morphism $k \to R/\mathfrak{m}$ is an isomorphism.

5) Show that if $X = V(I) \subseteq \mathrm{Specm}(R)$ and if $X(\overline{k})$ is the closed subset of $\mathbf{A}^m_{\overline{k}}$ defined by $I \cdot \overline{k}[x_1, \ldots, x_n]$, then there is a surjective map $X(\overline{k}) \to X$.