Math 632. Homework Set 1

Solutions are due Tuesday, February 2nd.

Recall that all our schemes are of finite type over an algebraically closed field k (for the purpose of this problem set, what matters is that they are Noetherian).

Problem 1. Let $f\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ be a morphism of ringed spaces. Show that if $\mathcal{F}$ is an $\mathcal{O}_X$-module, and $\mathcal{G}$ is an $\mathcal{O}_Y$-module, then there is a canonical isomorphism (of abelian groups)

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*(\mathcal{G}),\mathcal{F}) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*(\mathcal{F})).$$

Problem 2. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules on a scheme X.

i) Show that if $\mathcal{F}$ is coherent, and $\mathcal{G}$ is quasicoherent (coherent), then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is quasicoherent (coherent). Show also that if X is affine, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is the sheaf associated to the $\mathcal{O}(X)$-module $\mathrm{Hom}_{\mathcal{O}(X)}(\mathcal{F}(X),\mathcal{G}(X))$.

ii) Show that if $\mathcal{F}$ is quasicoherent, but not coherent, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ might not be quasicoherent, even if $\mathcal{G}$ is coherent.

Problem 3. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module on a scheme X. The *support* of $\mathcal{F}$ is the set

$$\mathrm{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

i) Describe this set when X is affine.

ii) Deduce that $\mathrm{Supp}(\mathcal{F})$ is closed in X.

iii) Show that this might fail if $\mathcal{F}$ is only quasicoherent, but not coherent.

Problem 4. Let $\mathcal{F}$ be a quasicoherent sheaf on a scheme X, and $\mathcal{I}$ a quasicoherent sheaf of ideals. Let $\mathcal{I}\cdot\mathcal{F}$ be the sheaf associated to the presheaf that takes an open subset U of X to $\Gamma(U,\mathcal{I})\cdot\Gamma(U,\mathcal{F})$.

i) Show that $\mathcal{I}\cdot\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-submodule of $\mathcal{F}$.

ii) Show that if X is affine, then

$$\Gamma(X,\mathcal{I}\cdot\mathcal{F}) = \Gamma(X,\mathcal{I})\cdot\Gamma(X,\mathcal{F}).$$

Problem 5. Show that if $f\colon X \to Y$ is an affine morphism of schemes, then the functor $f_*\colon \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$ between the two categories of quasicoherent sheaves, is exact.

Problem 6. Let $i\colon Z \hookrightarrow X$ be a closed immersion, and consider the functor $i_*\colon \mathrm{Qcoh}(Z) \to \mathrm{Qcoh}(X)$.

i) Show that i_* is fully faithful (hence it gives an equivalence of $\mathrm{Qcoh}(Z)$ with a full subcategory of $\mathrm{Qcoh}(X)$ (recall that we have seen in class that if $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to i_*(\mathcal{O}_Z))$, then a sheaf $\mathcal{F} \in \mathrm{Qcoh}(X)$ is isomorphic to some $i_*(\mathcal{G})$ if and only if $\mathcal{I} \cdot \mathcal{F} = 0$).

ii) Show that if $\mathcal{F}$ is a coherent sheaf on X such that $\mathrm{Supp}(\mathcal{F}) \subseteq i(Z)$, then there are coherent subsheaves

$$\mathcal{F}_0 = (0) \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_r = \mathcal{F}$$

such that $\mathcal{F}_j/\mathcal{F}_{j-1}$ is isomorphic to some $i_*(\mathcal{G}_j)$ for every $j \geq 1$.