Homework Set 1

Solutions are due Monday, October 5th.

Problem 1. Let X be a topological space, and consider a finite open cover

$$X = U_1 \cup \ldots \cup U_n,$$

where each U_i is nonempty. Show that X is irreducible if and only if the following hold:

i) Each U_i is irreducible.
ii) For every i and j, we have $U_i \cap U_j \neq \emptyset$.

Problem 2. Let Y be the closed algebraic subset of $\mathbf{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find the corresponding prime ideals.

Problem 3. Show that if f is a nonconstant polynomial in $k[x_1, \ldots, x_n]$, then the corresponding hypersurface $V(f)$ is irreducible if and only if f has no distinct prime factors.

Problem 4. Let X be an affine algebraic variety, and let $\mathcal{O}(X)$ be the ring of regular functions on X. For every subset I of $\mathcal{O}(X)$, let

$$V(I) := \{p \in X \mid f(p) = 0 \text{ for all } f \in I\}.$$

For $S \subseteq X$, consider the following subset of $\mathcal{O}(X)$

$$I_X(S) := \{f \in \mathcal{O}(X) \mid f(p) = 0 \text{ for all } p \in S\}.$$

Show that the maps $V(-)$ and $I_X(-)$ define order-reversing inverse bijections between the closed subsets of X and the radical ideals in $\mathcal{O}(X)$. This generalizes the case $X = \mathbf{A}^n$ that we discussed in class.

Problem 5. Suppose that $\mathrm{char}(k) = p > 0$, and consider the map $f\colon \mathbf{A}^n \to \mathbf{A}^n$ given by $f(x_1, \ldots, x_n) = (x_1^p, \ldots, x_n^p)$. Show that f is a morphism of affine algebraic varieties, and that it is a homeomorphism, but it is not an isomorphism. This morphism, called the (k-linear) *Frobenius morphism* plays an important role in algebraic geometry in positive characteristic.

Problem 6. Let $Y \subseteq \mathbf{A}^2$ be the cuspidal curve defined by the equation $x^2 - y^3 = 0$. Construct a bijective morphism $f\colon \mathbf{A}^1 \to Y$. Is it an isomorphism ?

Problem 7. Show that the image of a morphism of affine algebraic varieties $f\colon X \to Y$ might not be locally closed in Y (you can use, for example, the morphism $f\colon \mathbf{A}^2 \to \mathbf{A}^2$ given by $f(x, y) = (x, xy)$).

Problem 8. If X is an affine algebraic variety, and if $u \in \mathcal{O}(X)$, then we denote by $D_X(u)$ the open subset of X

$$D_X(u) = \{x \in X \mid u(x) \neq 0\}$$

(we have seen in class that this is again an affine variety). Suppose that $f\colon X \to Y$ is a morphism of affine algebraic varieties, and denote by $f^\sharp\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ the induced ring homomorphism, that takes $\phi \in \mathcal{O}(Y)$ to $\phi \circ f$. Show that if $u \in \mathcal{O}(Y)$, then

i) We have $f^{-1}(D_Y(u)) = D_X(w)$, where $w = f^\sharp(u)$.
ii) The induced ring homomorphism

$$\mathcal{O}(D_Y(u)) \to \mathcal{O}(D_X(w))$$

can be identified with the homomorphism

$$\mathcal{O}(Y)_u \to \mathcal{O}(X)_w$$

induced by $f^\sharp$ by localization.

Problem 9. Let X be a quasiaffine variety, and p a point on X. Show that the local ring $\mathcal{O}_{X,p}$ of X at p is a domain if and only if p lies on a unique irreducible component of X.

Problem 10. Let $f\colon X \to Y$ be a morphism of quasiaffine varieties, and let $Z \subseteq X$ be a closed irreducible subset. Note that by the first problem in Problem Session 2, we know that $W := \overline{f(Z)}$ is irreducible. Show that we have an induced morphism of k-algebras

$$g\colon \mathcal{O}_{Y,W} \to \mathcal{O}_{X,Z}.$$

Show that g is a local homomorphism of local rings (that is, it maps the maximal ideal of $\mathcal{O}_{Y,W}$ inside the maximal ideal of $\mathcal{O}_{X,Z}$).