Homework Set 11

Solutions are due Thursday, April 5.

Problem 1. Let $(X,\mathcal{O}_X)$ be a ringed space, U an open subset of X, and $i\colon U\hookrightarrow X$ the inclusion map. For an $\mathcal{O}_U$-module $\mathcal{G}$ on U, we define the $\mathcal{O}_X$-module $i_!(\mathcal{G})$, the *extension by* 0 *of* $\mathcal{G}$, as follows. We put $\Gamma\big(V,i_!(\mathcal{G})\big)=\mathcal{G}(V)$ if $V\subseteq U$ and $\Gamma\big(V,i_!(\mathcal{G})\big)=0$, otherwise (with the non-zero restriction maps given by those for $\mathcal{G}$).

i) Show that for $x\in X$, we have $i_!(\mathcal{G})_x\simeq\mathcal{G}_x$ if $x\in U$ and $i_!(\mathcal{G})_x=0$, otherwise. In particular, the map taking $\mathcal{G}$ to $i_!(\mathcal{G})$ is an exact functor from the category of $\mathcal{O}_U$-modules to that of $\mathcal{O}_X$-modules.

ii) Show that for every $\mathcal{O}_X$-module $\mathcal{F}$ on X, we have a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G},\mathcal{F}|_U)\simeq\mathrm{Hom}_{\mathcal{O}_X}\big(i_!(\mathcal{G}),\mathcal{F}\big).$$

In other words, $(i_!,i^*)$ is an adjoint pair.

iii) In particular, for every $\mathcal{O}_X$-module $\mathcal{F}$, we have a canonical morphism $i_!(\mathcal{F}|_U)\to\mathcal{F}$. Show that this is injective.

Problem 2. Let $(X,\mathcal{O}_X)$ be a ringed space and U an open subset of X.

i) Show that if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module. Hint: you can use the functor in Problem 1.

ii) Deduce that if $\mathcal{F}$ and $\mathcal{G}$ are arbitrary $\mathcal{O}_X$-modules, then we have a functorial isomorphism

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})|_U\simeq\mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{F}|_U,\mathcal{G}|_U).$$

Problem 3. Let $(X,\mathcal{O}_X)$ be a ringed space.

i) Show that if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then for every $\mathcal{O}_X$-module $\mathcal{M}$, the $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{I})$ is flasque. Hint: you can use the functor in Problem 1.

ii) Deduce that for every $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, there is a spectral sequence

$$E_2^{p,q}=H^p\big(X,\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})\big)\Rightarrow_p\mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}).$$

Problem 4. Show that if $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on the algebraic variety X, then the $\mathcal{O}_X$-modules $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ are coherent. Moreover, if U is an affine open subset in X and $x\in X$ is an arbitrary point, then the following hold:

i) $\Gamma\big(U,\mathcal{E}xt^i(\mathcal{F},\mathcal{G})\big)\simeq\mathrm{Ext}^i_{\mathcal{O}_X(U)}\big(\mathcal{F}(U),\mathcal{G}(U)\big)$.

ii) $\mathrm{Ext}^i_{\mathcal{O}_U}(\mathcal{F}|_U,\mathcal{G}|_U)\simeq\mathrm{Ext}^i_{\mathcal{O}_X(U)}\big(\mathcal{F}(U),\mathcal{G}(U)\big)$.

iii) $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})_x\simeq\mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x)$.