Homework Set 10

Solutions are due Thursday, March 29.

Problem 1. Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on a variety X. Show that the following hold:

i) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are globally generated $\mathcal{O}_X$-modules, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is globally generated.
ii) If $\mathcal{L}$ and $\mathcal{M}$ are ample, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is ample.
iii) If m is a positive integer, then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is ample.
iv) If $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is ample.
v) If $\mathcal{L}$ is ample, then there is q_0 such that $\mathcal{L}^q \otimes_{\mathcal{O}_X} \mathcal{M}$ is ample for all $q \geq q_0$.

Problem 2. Let X be an algebraic variety. Show that the following hold:

i) If $\mathcal{M}$ is a globally generated $\mathcal{O}_X$-module, then for every morphism $f\colon Y \to X$, the $\mathcal{O}_Y$-module $f^*(\mathcal{F})$ is globally generated.
ii) If $\mathcal{L}$ is an ample line bundle on the algebraic variety X and Z is a locally closed subset of X, then $\mathcal{L}|_X$ is ample on Z.

Problem 3. If we denote by $\mathrm{Aut}(\mathbb{P}^n)$ the group of automorphisms of $\mathbb{P}^n$, we have seen that we have a group homomorphism

$$PGL_{n+1}(k) \to \mathrm{Aut}(\mathbb{P}^n).$$

Show that this is an isomorphism.

Problem 4. Let $f\colon X \to Y$ be a proper morphism of algebraic varieties.

i) Show that there is a morphism of Abelian groups $f_*\colon K_0(X) \to K_0(Y)$ such that $f_*([\mathcal{F}]) = \sum_{i \geq 0} (-1)^i [R^i f_*(\mathcal{F})]$.
ii) Show that we have the following version of the projection formula: if $\alpha \in K^0(Y)$ and $\beta \in K_0(X)$, then

$$f_*(f^*(\alpha) \cap \beta) = \alpha \cap f_*(\beta)$$

(where we denote by $- \cap -$ the action of K^0 on K_0.
iii) Show that if $g\colon Y \to Z$ is another proper morphism, then $g_* \circ f_* = (g \circ f)_*$.