CHAPTER 4. PROJECTIVE VARIETIES

In this chapter we introduce a very important class of algebraic varieties, the *projective varieties*.

1. The Zariski topology on the projective space

In this section we discuss the Zariski topology on the projective space, by building an analogue of the correspondence between closed subsets in affine space and radical ideals in the polynomial ring. As usual, we work over a fixed algebraically closed field k.

Definition 1.1. For a non-negative integer n, the projective space $\mathbf{P}^n = \mathbf{P}^n_k$ is the set of all 1-dimensional linear subspaces in k^{n+1}.

For now, this is just a set. We proceed to endow it with a topology and in the next section we will put on it a structure of algebraic variety. Note that a 1-dimensional linear subspace in k^{n+1} is described by a point $(a_0, \ldots, a_n) \in \mathbf{A}^{n+1} \smallsetminus \{0\}$, with two points $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_n)$ giving the same subspace if and only if there is $\lambda \in k^*$ such that $\lambda a_i = b_i$ for all i. In this way, we identify $\mathbf{P}^n$ with the quotient of the set $\mathbf{A}^{n+1} \smallsetminus \{0\}$ by the action of k^* given by

$$\lambda \cdot (a_0, \ldots, a_n) = (\lambda a_0, \ldots, \lambda a_n).$$

Let $\pi\colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n$ be the quotient map. We denote the image in $\mathbf{P}^n$ of a point $(a_0, \ldots, a_n) \in \mathbf{A}^{n+1} \smallsetminus \{0\}$ by $[a_0, \ldots, a_n]$.

Let $S = k[x_0, \ldots, x_n]$. The relevant structure on S, for the study of $\mathbf{P}^n$, is that of a graded k-algebra. Recall that a graded (commutative) ring R is a commutative ring that has a decomposition as an Abelian group

$$R = \bigoplus_{m \in \mathbf{Z}} R_m$$

such that $R_i \cdot R_j \subseteq R_{i+j}$ for all i and j. We say that R is $\mathbf{N}$-graded if $R_m = 0$ for $m < 0$.

Note that the definition implies that if R is a graded ring, then R_0 is a subring of R and each R_m is an R_0-module, making R an R_0-algebra. We say that R is a graded A-algebra, for a commutative ring A, if R is a graded ring such that R_0 is an A-algebra (in which case R becomes an A-algebra, too). If R and S are graded rings, a *graded homomorphism* $\varphi\colon R \to S$ is a ring homomorphism such that $\varphi(R_m) \subseteq S_m$ for all $m \in \mathbf{Z}$.

The polynomial ring S is an $\mathbf{N}$-graded k-algebra, with S_m being the set of homogeneous polynomials of degree m. In general, if R is a graded ring, a nonzero element of R_m is *homogeneous of degree m*. By convention, 0 is homogeneous of degree m for every

m. Given an arbitrary element $f \in R$, if we write

$$f = \sum_i f_i, \quad \text{with} \quad f_i \in R_i,$$

then the f_i are the *homogeneous componenets* of f.

Remark 1.2. Note that the action of k^* on $\mathbf{A}^{n+1} \smallsetminus \{0\}$ is an algebraic action: in fact, it is induced by the algebraic action of k^* on $\mathbf{A}^{n+1}$ corresponding to the homomorphism

$$S \to k[t, t^{-1}] \otimes_k S, \quad f \to f(tx_1, \ldots, tx_n).$$

Exercise 1.3. For an ideal I in a graded ring R, the following are equivalent:

i) The ideal I can be generated by homogeneous elements of R.

ii) For every $f \in I$, all homogeneous components of f lie in I.

iii) The decomposition of R induces a decomposition $I = \bigoplus_{m \in \mathbf{Z}} (I \cap R_m)$.

An ideal that satisfies the equivalent conditions in the above exercise is a *homogeneous* (or *graded*) ideal. Note that if I is a homogeneous ideal in a graded ring R, then the quotient ring R/I becomes a graded ring in a natural way:

$$R/I = \bigoplus_{m \in \mathbf{Z}} R_m/(I \cap R_m).$$

We now return to the study of $\mathbf{P}^n$. The starting observation is that while it does not make sense to evaluate a polynomial in S at a point $p \in \mathbf{P}^n$, it makes sense to say that a *homogeneous* polynomial vanishes at p: indeed, if f is homogeneous of degree d and $\lambda \in k^*$, then

$$f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d \cdot f(a_0, \ldots, a_n),$$

hence $f(\lambda a_0, \ldots, \lambda a_n) = 0$ if and only if $f(a_0, \ldots, a_n) = 0$. More generally, given any $f \in S$, we say that f vanishes at p if every homogeneous component of f vanishes at p.

Given any homogeneous ideal I of S, we define the *zero-locus* $V(I)$ of I to be the subset of $\mathbf{P}^n$ consisting of all points $p \in \mathbf{P}^n$ such that every polynomial f in I vanishes at p. Like the corresponding notion in the affine space, this notion satisfies the following basic properties. The proof is straightforward, hence we leave it as an exercise.

Proposition 1.4. *The following hold:*

1) $V(S) = \emptyset$.
2) $V(0) = \mathbf{P}^n$.
3) *If I and J are ideals in S with $I \subseteq J$, then $V(J) \subseteq V(I)$.*
4) *If $(I_\alpha)_\alpha$ is a family of ideals in S, we have*

$$\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right).$$

5) *If I and J are ideals in S, then*

$$V(I) \cup V(J) = V(I \cap J) = V(I \cdot J).$$

It follows from the proposition that we can put a topology on $\mathbf{P}^n$ (the *Zariski topology*) in which the closed subsets of $\mathbf{P}^n$ are the subsets of the form $V(I)$, where I is a homogeneous ideal in S.

Remark 1.5. A closed subset $Y \subseteq \mathbf{A}^{n+1}$ is invariant by the k^*-action (that is, $\lambda \cdot Y = Y$ for every $\lambda \in k^*$) if and only if the ideal $I_{\mathbf{A}^n}(Y) \subseteq S$ is homogeneous (cf. Lemma 7.22 in Chapter 1). Indeed, if f is homogeneous, then for every $\lambda \in k^*$ and every $u \in \mathbf{A}^{n+1}$, we have $f(\lambda u) = 0$ if and only if $f(u) = 0$. We thus see that if I is a homogeneous ideal, then its zero-locus in $\mathbf{A}^{n+1}$ is k^*-invariant. In particular, if $I_{\mathbf{A}^n}(Y)$ is homogeneous, then Y is k^*-invariant. Conversely, if Y is k^*-invariant and $f \in I_{\mathbf{A}^n}(Y)$, let us write $f = \sum_i f_i$, with $f_i \in S_i$. For every $u \in Y$ and every $\lambda \in k^*$, we have $\lambda u \in Y$, hence

$$0 = f(\lambda u) = \sum_{i \geq 0} \lambda^i \cdot f_i(u).$$

It is easy to see that since this property holds for infinitely many λ, we have $f_i(u) = 0$ for all i, hence $I_{\mathbf{A}^n}(Y)$ is homogeneous.

Remark 1.6. The topology on $\mathbf{P}^n$ is the quotient topology with respect to the k^*-action on $\mathbf{A}^{n+1} \smallsetminus \{0\}$. In other words, if $\pi\colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n$ is the quotient map, then a subset Z of $\mathbf{P}^n$ is closed if and only if its inverse image $\pi^{-1}(Z)$ is closed. For this, we may assume that Z is nonempty. If $\pi^{-1}(Z)$ is closed, then it is clear that $\pi^{-1}(Z) \cup \{0\}$ is closed, hence by the previous remark, there is a homogeneous ideal $I \subseteq S$ such that $\pi^{-1}(Z) \cup \{0\}$ is the zero-locus of I. In this case, it is clear that Z is the zero-locus of I in $\mathbf{P}^n$. The converse is clear.

We now construct a map in the opposite direction. Given any subset $S \subseteq \mathbf{P}^n$, let $I(S)$ be the set of polynomials in S that vanish at all points in S. Note that $I(S)$ is a homogeneous radical ideal of S (the fact that it is homogeneous follows from the fact that if $f \in I(S)$, then all homogeneous components of f lie in $I(S)$). This definition satisfies the following properties, that are straightforward to check.

Proposition 1.7. *The following hold:*

1) $I(\emptyset) = S$.
2) *If* $(W_\alpha)_\alpha$ *is a family of subsets of* $\mathbf{A}^n$, *then* $I\left(\bigcup_\alpha W_\alpha\right) = \bigcap_\alpha I(W_\alpha)$.
3) *If* $W_1 \subseteq W_2$, *then* $I(W_2) \subseteq I(W_1)$.

We now turn to the compositions of the two maps. The first property is tautological.

Proposition 1.8. *For every subset* S *of* $\mathbf{P}^n$, *we have* $V\big(I(S)\big) = \overline{S}$.

Proof. The proof follows verbatim the proof in the case of affine space (see Proposition 1.8 in Chapter 1). □

The more interesting statement concerns the other composition. This is the content of the next proposition, a graded version of the Nullstellensatz.

Proposition 1.9. *If* $J \subseteq S$ *is a radical ideal different from* $S_+ = (x_0, \ldots, x_n)$, *then* $I\big(V(J)\big) = J$.

Note that $V(S_+) = \emptyset$, hence $I\big(V(S_+)\big) = S$. The ideal S_+, which behaves differently in this correspondence, is the *irrelevant ideal*.

Proof of Proposition 1.9. The inclusion "$\supseteq$" is trivial, hence we only need to prove the reverse inclusion. It is enough to show that every homogeneous polynomial $f \in I\big(V(J)\big)$ lies in J.

We make use of the map $\pi\colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n$. Let Z be the closed subset of $\mathbf{A}^{n+1}$ defined by J, so that $Z \smallsetminus \{0\} = \pi^{-1}\big(V(J)\big)$. Our assumption on f says that f vanishes on $Z \smallsetminus \{0\}$. If $\deg(f) > 0$, then $f(0) = 0$, and we conclude by Hilbert's Nullstellensatz that $f \in J$. On the other hand, if $\deg(f) = 0$ and $f \neq 0$, then it follows that $V(J) = \emptyset$. This implies that $Z \subseteq \{0\}$ and another application of Hilbert's Nullstellensatz gives $S_+ \subseteq J$. Since $J \neq S_+$ by assumption, we have $J = S$, in which case $f \in J$. ☐

Corollary 1.10. *The two maps $V(-)$ and $I(-)$ give inclusion reversing inverse bijections between the set of homogeneous radical ideals in S different from S_+ and the closed subsets of $\mathbf{P}^n$.*

Proof. Note that for every closed subset Z of $\mathbf{P}^n$, we have $I(Z) \neq S_+$. Indeed, if $I(Z) = S_+$, then it follows from Proposition 1.8 that

$$Z = V\big(I(Z)\big) = V(S_+) = \emptyset.$$

However, in this case $I(Z) = I(\emptyset) = S$. The assertion in the corollary follows directly from Propositions 1.8 and 1.9. ☐

Exercise 1.11. Show that if I is a homogeneous ideal in a graded ring S, then the following hold:

i) The ideal I is radical if and only if for every *homogeneous* element $f \in S$, with $f^m \in I$ for some $m \geq 1$, we have $f \in I$.
ii) The radical $\mathrm{rad}(I)$ of I is a homogeneous ideal.

Exercise 1.12. Show that if I is a homogeneous ideal in a graded ring S, then I is a prime ideal if and only if for every homogeneous elements $f, g \in S$ with $fg \in I$, we have $f \in I$ or $g \in I$. Deduce that a closed subset Z of $\mathbf{P}^n$ is irreducible if and only if $I(Z)$ is a prime ideal. In particular, $\mathbf{P}^n$ is irreducible.

Definition 1.13. If X is a closed subset of $\mathbf{P}^n$ and I_X is the corresponding homogeneous radical ideal, then $S_X := S/I_X$ is the *homogeneous coordinate ring* of X. Note that this is an $\mathbf{N}$-graded k-algebra. In particular, S is the *homogeneous coordinate ring* of $\mathbf{P}^n$.

Suppose that X is a closed subset of $\mathbf{P}^n$, with homogeneous coordinate ring S_X. For every homogeneous $g \in S_X$ of positive degree, we consider the following open subset of X:

$$D_X^+(g) = X \smallsetminus V(\widetilde{g}),$$

where $\widetilde{g} \in S$ is any homogeneous polynomial which maps to $g \in S_X$. Note that if h is another homogeneous polynomial of positive degree, we have

$$D_X^+(gh) = D_X^+(g) \cap D_X^+(h).$$

Remark 1.14. Every open subset of X is of the form $X \smallsetminus V(J)$, where J is a homogeneous ideal in S. By choosing a system of homogeneous generators for J, we see that this is the union of finitely many open subsets of the form $D_X^+(g)$. Therefore the open subsets $D_X^+(g)$ give a basis of open subsets for the topology of X.

Definition 1.15. For every closed subset X of $\mathbf{P}^n$, we define the *affine cone* $C(X)$ over X to be the union in $\mathbf{A}^{n+1}$ of the corresponding lines in X. Note that if X is nonempty, then

$$C(X) = \pi^{-1}(Z) \cup \{0\}.$$

If $X = V(I)$ is nonempty, for a homogeneous ideal $I \subseteq S$, it is clear that $C(X)$ is the zero-locus of I in $\mathbf{A}^{n+1}$. Therefore $C(X)$ is a closed subset of $\mathbf{A}^n$ for every X. Moreover, we see that $\mathcal{O}\big(C(X)\big) = S_X$.

Exercise 1.16. Show that if G is an irreducible linear algebraic group acting on a variety X, then every irreducible component of X is invariant under the G-action.

Remark 1.17. Let X be a closed subset of $\mathbf{P}^n$, with corresponding homogeneous radical ideal $I_X \subseteq S$, and let $C(X)$ be the affine cone over X. Since $C(X)$ is k^*-invariant, it follows from the previous exercise that the irreducible components of $C(X)$ are k^*-invariant, as well. By Remark 1.5, this means that the minimal prime ideals containing I_X are homogeneous. They correspond to the irreducible components $X_1, \ldots, X_r$ of X, so that the irreducible components of $C(X)$ are $C(X_1), \ldots, C(X_r)$.

2. Regular functions on quasi-projective varieties

Our goal in this section is to define a structure sheaf on $\mathbf{P}^n$. The main observation is that if F and G are homogeneous polynomials of the same degree, then we may define a function $\frac{F}{G}$ on the open subset $\mathbf{P}^n \smallsetminus V(G)$ by

$$[a_0, \ldots, a_n] \to \frac{F(a_0, \ldots, a_n)}{G(a_0, \ldots, a_n)}.$$

Indeed, if $\deg(F) = d = \deg(G)$, then

$$\frac{F(\lambda a_0, \ldots, \lambda a_n)}{G(\lambda a_0, \ldots, \lambda a_n)} = \frac{\lambda^d \cdot F(a_0, \ldots, a_n)}{\lambda^d \cdot G(a_0, \ldots, a_n)} = \frac{F(a_0, \ldots, a_n)}{G(a_0, \ldots, a_n)}.$$

Let W be a locally closed subset in $\mathbf{P}^n$. A *regular function* on W is a function $f \colon W \to k$ such that for every $p \in W$, there is an open neighborhood $U \subseteq W$ of p and homogeneous polynomials of the same degree F and G such that $G(q) \neq 0$ for every $q \in U$ and

$$f(q) = \frac{F(q)}{G(q)} \quad \text{for all} \quad q \in U.$$

The set of regular functions on W is denoted by $\mathcal{O}(W)$. Note that $\mathcal{O}(W)$ is a k-algebra with respect to the usual operations on functions. For example, if $f_1(q) = \frac{F_1(q)}{G_1(q)}$ for $q \in U_1$ and

$f_2(q) = \frac{F_2(q)}{G_2(q)}$ for $q \in U_2$, where U_1 and U_2 are open neighborhoods of p, then $F_1G_2 + F_2G_1$ and G_1G_2 are homogeneous polynomials of the same degree and

$$f_1(q) + f_2(q) = \frac{(F_1G_2 + F_2G_1)(q)}{(G_1G_2)(q)} \quad \text{for} \quad q \in U_1 \cap U_2.$$

Moreover, it is clear that if V is an open subset of W, the restriction to V of a regular function on W is a regular function of V. We thus obtain in this way a subpresheaf $\mathcal{O}_W$ of k-algebras of $\mathcal{F}un_W$. In fact, this is a sheaf, as follows immediately from the fact that regular functions are defined in terms of a local property.

Remark 2.1. Note that if W is a locally closed subset of $\mathbf{P}^n$, then the sheaf $\mathcal{O}_W$ we defined is the one induced from $\mathcal{O}_{\mathbf{P}^n}$ as in Chapter 3.3.

Our first goal is to show that all spaces defined in this way are algebraic varieties. Let U_i be the open subset defined by $x_i \neq 0$. Note that we have

$$\mathbf{P}^n = \bigcup_{i=0}^{n} U_i.$$

The key fact is the following assertion:

Proposition 2.2. *For every i, with $0 \leq i \leq n$, the map*

$$\psi_i \colon \mathbf{A}^n \to U_i, \quad \psi(v_1, \ldots, v_n) = [v_1, \ldots, v_i, 1, v_{i+1}, \ldots, v_n]$$

is an isomorphism in $\mathcal{T}op_k$.

Proof. It is clear that ψ_i is a bijection, with inverse

$$\varphi_i \colon U_i \to \mathbf{A}^n, \quad [u_0, \ldots, u_n] \to (u_0/u_i, \ldots, u_{i-1}/u_i, u_{i+1}/u_i, \ldots, u_n/u_i).$$

In order to simplify the notation, we give the argument for $i = 0$, the other cases being analogous. Consider first a principal affine open subset of $\mathbf{A}^n$, of the form $D(f)$, for some $f \in k[x_1, \ldots, x_n]$. Note that if $\deg(f) = d$, then we can write

$$f(x_1/x_0, \ldots, x_n/x_0) = \frac{g(x_0, \ldots, x_n)}{x_0^d}$$

for a homogeneous polynomial $g \in S$ of degree d. It is then clear that $\varphi_0^{-1}\big(D(f)\big) = D^+_{\mathbf{P}^n}(x_0 g)$, hence this is open in U_0. Since the principal affine open subsets in $\mathbf{A}^n$ give a basis for the topology of $\mathbf{A}^n$, we see that φ_0 is continuous.

Consider now an open subset of U_0 of the form $D^+_{\mathbf{P}^n}(h)$, for some homogeneous $h \in S$, of positive degree. If we put $h_0 = h(1, x_1, \ldots, x_n)$, we see that $\varphi_0\big(D^+_{\mathbf{P}^n}(h)\big) = D(h_0)$ is open in $\mathbf{A}^n$. Since the open subsets of the form $D^+_{\mathbf{P}^n}(h)$ give a basis for the topology of $\mathbf{P}^n$, we conclude that φ_0 is a homeomorphism.

We now need to show that if U is open in $\mathbf{A}^n$ and $\alpha \colon U \to k$, then $\alpha \in \mathcal{O}_{\mathbf{A}^n}(U)$ if and only if $\alpha \circ \varphi_0 \in \mathcal{O}_{\mathbf{P}^n}\big(\varphi_0^{-1}(U)\big)$. If $\alpha \in \mathcal{O}_{\mathbf{A}^n}(U)$, then for every point $p \in U$, we have an open neighborhood $U_p \subseteq U$ of p and $f_1, f_2 \in k[x_1, \ldots, x_n]$ such that

$$f_2(u) \neq 0 \quad \text{and} \quad \alpha(u) = \frac{f_1(u)}{f_2(u)} \quad \text{for all} \quad u \in U_p.$$

As above, we can write

$$f_1(x_1/x_0,\ldots,x_n/x_0)=\frac{g_1(x_0,\ldots,x_n)}{x_0^d}\quad\text{and}\quad f_2(x_1/x_0,\ldots,x_n/x_0)=\frac{g_2(x_0,\ldots,x_n)}{x_0^d}$$

for some homogeneous polynomials $g_1, g_2 \in S$ of the same degree, in which case we see that

$$g_2(v)\neq 0\quad\text{and}\quad \alpha\big(\varphi_0(v)\big)=\frac{g_1(v)}{g_2(v)}\quad\text{for all}\quad v\in\varphi_0^{-1}(U_p).$$

Since this holds for every $p \in U$, we see that $\alpha \circ \varphi_0$ is a regular function on $\varphi_0^{-1}(U)$.

Conversely, suppose that $\alpha \circ \varphi_0$ is a regular function on $\varphi_0^{-1}(U)$. This means that for every $q \in \varphi_0^{-1}(U)$, there is an open neighborhood $V_q \subseteq \varphi_0^{-1}(U)$ of q and homogeneous polynomials $h_1, h_2 \in S$ of the same degree such that

$$h_2(v)\neq 0\quad\text{and}\quad \alpha\big(\varphi_0(v)\big)=\frac{h_1(v)}{h_2(v)}\quad\text{for all}\quad v\in V_q.$$

In this case, we have

$$h_2(1,u_1,\ldots,u_n)\neq 0\quad\text{and}\quad \alpha(u_1,\ldots,u_n)=\frac{h_1(1,u_1,\ldots,u_n)}{h_2(1,u_1,\ldots,u_n)}$$

for all $u = (u_1, \ldots, u_n) \in \varphi_0(V_q)$. Since this holds for every $q \in \varphi_0^{-1}(U)$, it follows that α is a regular function on U. This completes the proof of the fact that φ_0 is an isomorphism. □

Corollary 2.3. *For every locally closed subset W of $\mathbf{P}^n$, the space $(W, \mathcal{O}_W)$ is an algebraic variety.*

Proof. It is enough to show the assertion for $W = \mathbf{P}^n$: the general case is then a consequence of Propositions 3.5 and 5.4 in Chapter 3. We have already seen that $\mathbf{P}^n$ is a prevariety. In order to show that it is separated, using Proposition 5.6 in Chapter 3, it is enough to show that each $U_i \cap U_j$ is affine and that the canonical morphism

$$\tau_{i,j}\colon \mathcal{O}(U_i)\otimes_k \mathcal{O}(U_j)\to\mathcal{O}(U_i\cap U_j) \tag{1}$$

is surjective. Suppose that $i < j$ and let us denote by $x_1, \ldots, x_n$ the coordinates on the image of φ_i and by $y_1, \ldots, y_n$ the coordinates on the image of φ_j. Note that via the isomorphism φ_i, the open subvariety $U_i \cap U_j$ is mapped to the open subset

$$\{(u_1,\ldots,u_n)\in\mathbf{A}^n\mid u_j\neq 0\},$$

which is affine, being a principal affine open subset of $\mathbf{A}^n$. Similarly, φ_j maps $U_i \cap U_j$ to the open subset

$$\{(u_1,\ldots,u_n)\in\mathbf{A}^n\mid u_{i+1}\neq 0\}.$$

Furthermore, since we have

$$\varphi_j\circ\varphi_i^{-1}(u_1,\ldots,u_n)=\left(\frac{u_1}{u_j},\ldots,\frac{u_i}{u_j},\frac{1}{u_j},\frac{u_{i+1}}{u_j},\ldots,\frac{u_{j-1}}{u_j},\frac{u_{j+1}}{u_j},\ldots,\frac{u_n}{u_j}\right)$$

for all $(u_1, \ldots, u_n) \in \varphi_i(U_i \cap U_j)$, we see that the morphism

$$\tau_{i,j}\colon k[x_1,\ldots,x_n]\otimes_k k[y_1,\ldots,y_n]\to k[x_1,\ldots,x_n,x_j^{-1}]$$

satisfies $\tau(x_\ell) = x_\ell$ for all ℓ and $\tau(y_{i+1}) = x_j^{-1}$. Therefore $\tau_{i,j}$ is surjective for all i and j, proving that $\mathbf{P}^n$ is separated. □

Example 2.4. The map

$$\pi\colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n, \quad \pi(x_0, \ldots, x_n) = [x_0, \ldots, x_n]$$

is a morphism. Indeed, with the notation in the proof of Proposition 2.2, it is enough to show that for every i, the induced map $\pi^{-1}(U_i) \to U_i$ is a morphism. However, via the isomorphism $U_i \simeq \mathbf{A}^n$, this map becomes

$$\mathbf{A}^{n+1} \smallsetminus V(x_i) \to \mathbf{A}^n, \quad (x_0, \ldots, x_n) \to (x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i),$$

which is clearly a morphism.

Definition 2.5. A *projective variety* is an algebraic variety that is isomorphic to a closed subvariety of some $\mathbf{P}^n$. A *quasi-projective* variety is an algebraic variety that is isomorphic to a locally closed subvariety ofsome $\mathbf{P}^n$.

Remark 2.6. It follows from definition that if X is a projective variety and Y is a closed subvariety of X, then Y is a projective variety as well. Similarly, if X is a quasi-projective variety and Z is a locally closed subvariety of X, then Z is a quasi-projective variety.

Remark 2.7. Every quasi-affine variety is quasi-projective: this follows from the fact that $\mathbf{A}^n$ is isomorphic to an open subvariety of $\mathbf{P}^n$.

Remark 2.8. Note that unlike the coordinate ring of an affine variety, the homogeneous coordinate ring of a projective variety $X \subseteq \mathbf{P}^n$ is not an intrinsic invariant: it depends on the embedding in the projective space.

We next show that the distinguished open subsets $D_X^+(g)$ are all affine varieties[1].

Proposition 2.9. *For every closed subvariety X of $\mathbf{P}^n$ and every homogeneous element $g \in S_X$ of positive degree, the variety $D_X^+(g)$ is affine.*

Proof. Since X is a closed subvariety of $\mathbf{P}^n$ and $D_X^+(g) = D_{\mathbf{P}^n}^+(\widetilde{g}) \cap X$, where $\widetilde{g} \in S$ is any lift of g, it is enough to prove the assertion when $X = \mathbf{P}^n$. Let $U = D_{\mathbf{P}^n}^+(g)$ and put $d = \deg(g)$.

Consider the regular functions $f_0, \ldots, f_n$ on U given by $f_i(u_0, \ldots, u_n) = \frac{u_i^d}{g(u)}$. Note that they generate the unit ideal in $\Gamma(U, \mathcal{O}_{\mathbf{P}^n})$. Indeed, since $g \in S_+ = \mathrm{rad}(x_0^d, \ldots, x_n^d)$, it follows that there is m such that $g^m \in (x_0^d, \ldots, x_n^d)$. If we write $g^m = \sum_{i=1}^n h_i x_i^d$ and if we consider the regular functions $\alpha_i\colon U \to k$ given by

$$\alpha_i(u_1, \ldots, u_n) = \frac{h_i(u)}{g(u)^{m-1}},$$

then $\sum_{i=0}^n f_i \cdot \alpha_i = 1$, hence $f_0, \ldots, f_n$ generate the unit ideal in $\Gamma(U, \mathcal{O}_{\mathbf{P}^n})$. By Proposition 3.16 in Chapter 3, we see that it is enough to show that each subset $U \cap U_i$ is affine, where U_i is the open subset of $\mathbf{P}^n$ defined by $x_i \neq 0$. However, by the isomorphism

[1]For another proof of this proposition, making use of the Veronese embedding, see Exercise 2.23 below.

$U_i \simeq \mathbf{A}^n$ given in Proposition 2.2, the open subset $U \cap U_i$ becomes isomorphic to the subset

$$\{u = (u_1, \ldots, u_n) \in \mathbf{A}^n \mid g(u_1, \ldots, u_i, 1, u_{i+1}, \ldots, u_n) \neq 0\},$$

which is affine by Proposition 4.18 in Chapter 1. This completes the proof. □

Since the open subsets $D_X^+(g)$ are affine, they are determined by their rings of regular functions. Our next goal is to describe these rings.

We begin with some general considerations regarding localization in graded rings. If S is a graded ring and $T \subseteq S$ is a multiplicative system consisting of homogeneous elements of S, then the ring of fractions $T^{-1}S$ has an induced grading, in which

$$(T^{-1}S)_m = \left\{ \frac{f}{t} \mid t \in T, f \in S_{\deg(t)+m} \right\}.$$

Note that even if S is $\mathbf{N}$-graded, $T^{-1}S$ is not, in general, $\mathbf{N}$-graded. We will use two special cases. If $g \in S$ is a homogeneous element, then S_g is a graded ring, and we denote by $S_{(g)}$ its degree 0 part. Similarly, if $\mathfrak{p}$ is a homogeneous prime ideal in S and if we take T to be the set of homogeneous elements in $S \smallsetminus \mathfrak{p}$, then $T^{-1}S$ is a graded ring and we denote its degree 0 part by $S_{(\mathfrak{p})}$. Therefore $S_{(g)}$ is the subring of S_g consisting of fractions $\frac{h}{g^m}$, where h is a homogeneous element of S, of degree $m \cdot \deg(g)$. Similarly, $S_{(\mathfrak{p})}$ is the subring of $S_{(\mathfrak{p})}$ consisting of all fractions of the form $\frac{f}{h}$, where $f, g \in S$ are homogeneous, of the same degree, with $g \not\in \mathfrak{p}$. Note that $S_{(\mathfrak{p})}$ is a local ring, with maximal ideal

$$\{f/h \in S_{(\mathfrak{p})} \mid f \in \mathfrak{p}\}.$$

Let X be a closed subset of $\mathbf{P}^n$, with corresponding radical ideal I_X and homogeneous coordinate ring S_X. Note that if $h \in S_X$ is homogeneous, of positive degree, we have a morphism of k-algebras

$$\Phi \colon (S_X)_{(h)} \to \mathcal{O}\big(D_X^+(h)\big),$$

such that $\Phi(f/h^m)$ is given by the function $p \to \frac{\widetilde{f}(p)}{\widetilde{h}^m(p)}$, where $\widetilde{f}, \widetilde{h} \in S$ are elements mapping to $f, h \in S_X$, respectively (it is clear that $\Phi(f/h^m)$ is independent of the choice of $\widetilde{f}$ and $\widetilde{h}$).

Proposition 2.10. *For every X and h as above, the morphism Φ is an isomorphism.*

Proof. We will prove a more general version in Proposition 3.17 below. □

We end this section with the description of the dimension of a closed subset of $\mathbf{P}^n$ in terms of the homogeneous coordinate ring.

Proposition 2.11. *If $X \subseteq \mathbf{P}^n$ is a nonempty closed subset, with homogeneous coordinate ring S_X, then* $\dim(X) = \dim(S_X) - 1$.

Proof. Note that the morphism $\pi \colon \mathbf{A}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n$ induces a surjective morphism $f \colon C(X) \smallsetminus \{0\} \to X$ whose fibers are 1-dimensional (in fact, they are all isomorphic to

$\mathbf{A}^1 \smallsetminus \{0\}$). It follows from Corollary 4.3 in Chapter 2 that

$$\dim\big(C(X)\big) = 1 + \dim(X).$$

Since S_X is the coordinate ring of the affine variety $C(X)$, we obtain the assertion in the proposition. □

Corollary 2.12. *If X and Y are nonempty closed subsets of $\mathbf{P}^n$, with $\dim(X)+\dim(Y) \geq n$, then $X \cap Y$ is nonempty and every irreducible component of $X \cap Y$ has dimension $\geq \dim(X) + \dim(Y) - n$.*

Proof. Note that $\big(C(X) \cap C(Y)\big) \smallsetminus \{0\} = C(X \cap Y) \smallsetminus \{0\}$. It is clear $C(X) \cap C(Y)$ is nonempty, since it contains 0. In this case, it follows from Exercise 3.21 in Chapter 2 that every irreducible component of $C(X) \cap C(Y)$ has dimension

$$\geq \dim\big(C(X)\big) + \dim\big(C(Y)\big) - (n+1) = \dim(X) + \dim(Y) - n + 1.$$

This implies that $C(X) \cap C(Y)$ is not contained in $\{0\}$, hence $X \cap Y$ is non-empty. Moreover, the irreducible components of $C(X) \cap C(Y)$ are of the form $C(Z)$, where Z is an irreducible component of $X \cap Y$, hence

$$\dim(Z) = \dim\big(C(Z)\big) - 1 \geq \dim(X) + \dim(Y) - n.$$

□

Exercise 2.13. A *hypersurface* in $\mathbf{P}^n$ is a closed subset defined by

$$\{[x_0, \ldots, x_n] \in \mathbf{P}^n \mid F(x_0, \ldots, x_n) = 0\},$$

for some homogeneous polynomial F, of positive degree. Given a closed subset $X \subseteq \mathbf{P}^n$, show that the following are equivalent:

i) X is a hypersurface.
ii) The ideal $I(X)$ is a principal ideal.
iii) All irreducible component of X have codimension 1 in $\mathbf{P}^n$.

Note that if X is any irreducible variety and U is a nonempty open subset of X, then the map taking $Z \subseteq U$ to $\overline{Z}$ and the map taking $W \subseteq X$ to $W \cap U$ give inverse bijections (preserving the irreducible decompositions) between the nonempty closed subsets of U and the nonempty closed subsets of X that have no irreducible component contained in the $X \smallsetminus U$. This applies, in particular, to the open immersion

$$\mathbf{A}^n \hookrightarrow \mathbf{P}^n, \quad (x_1, \ldots, x_n) \to [1, x_1, \ldots, x_n].$$

The next exercise describes this correspondence at the level of ideals.

Exercise 2.14. Let $S = k[x_0, \ldots, x_n]$ and $R = k[x_1, \ldots, x_n]$. For an ideal J in R, we put

$$J^{\rm hom} := \left(f^{\rm hom} \mid 0 \neq f \in J\right),$$

where $f^{\rm hom} = x_0^{\deg(f)} \cdot f(x_1/x_0, \ldots, x_n/x_0) \in S$. On the other hand, if $\mathfrak{a}$ is a homogeneous ideal in S, then we put $\overline{\mathfrak{a}} := \{h(1, x_1, \ldots, x_n) \mid h \in \mathfrak{a}\} \subseteq R$.

An ideal $\mathfrak{a}$ in S is called x_0-saturated if $(\mathfrak{a} \colon x_0) = \mathfrak{a}$ (recall that $(\mathfrak{a} \colon x_0) := \{u \in S \mid x_0 u \in \mathfrak{a}\}$).

i) Show that the above maps give inverse bijections between the ideals in R and the x_0-saturated homogeneous ideals in S.
ii) Show that we get induced bijections between the radical ideals in R and the homogeneous x_0-saturated radical ideals in S. Moreover, a homogeneous radical ideal $\mathfrak{a}$ is x_0-saturated if and only if either no irreducible component of $V(\mathfrak{a})$ is contained in the hyperplane $(x_0 = 0)$, or if $\mathfrak{a} = S$.
iii) The above correspondence induces a bijection between the prime ideals in R and the prime ideals in S that do not contain x_0.
iv) Consider the open immersion

$$\mathbf{A}^n \hookrightarrow \mathbf{P}^n, \ \ (u_1, \ldots, u_n) \to (1\colon u_1\colon \ \ldots : u_n),$$

which allows us to identify $\mathbf{A}^n$ with the complement of the hyperplane $(x_0 = 0)$ in $\mathbf{P}^n$. Show that for every ideal J in R we have $\overline{V_{\mathbf{A}^n}(J)} = V_{\mathbf{P}^n}(J^{\mathrm{hom}})$.
v) Show that for every homogeneous ideal $\mathfrak{a}$ in S, we have $V_{\mathbf{P}^n}(\mathfrak{a}) \cap \mathbf{A}^n = V_{\mathbf{A}^n}(\overline{\mathfrak{a}})$.

Exercise 2.15. Recall that $GL_{n+1}(k)$ denotes the set of invertible $(n+1) \times (n+1)$ matrices with entries in k. Let $PGL_{n+1}(k)$ denote the quotient $GL_{n+1}(k)/k^*$, where k^* acts on $GL_{n+1}(k)$ by

$$\lambda \cdot (a_{i,j})_{i,j} = (\lambda a_{i,j})_{i,j}.$$

i) Show that $PGL_{n+1}(k)$ has a natural structure of linear algebraic group, and that it is irreducible.
ii) Prove that $PGL_{n+1}(k)$ acts algebraically on $\mathbf{P}^n$.

Definition 2.16. Two subsets of $\mathbf{P}^n$ are *projectively equivalent* if they differ by an automorphism in $PGL_{n+1}(k)$ (we will see later that these are, indeed, all automorphisms of $\mathbf{P}^n$).

Definition 2.17. A *linear subspace* of $\mathbf{P}^n$ is a closed subvariety of $\mathbf{P}^n$ defined by an ideal generated by homogeneous polynomials of degree one. A *hyperplane* is a linear subspace of codimension one.

Exercise 2.18. Consider the projective space $\mathbf{P}^n$.

i) Show that a closed subset Y of $\mathbf{P}^n$ is a linear subspace if and only if the affine cone $C(Y) \subseteq \mathbf{A}^{n+1}$ is a linear subspace.
ii) Show that if L is a linear subspace in $\mathbf{P}^n$ of dimension r, then there is an isomorphism $L \simeq \mathbf{P}^r$.
iii) Show that the hyperplanes in $\mathbf{P}^n$ are in bijection with the points of "another" projective space $\mathbf{P}^n$, usually denoted by $(\mathbf{P}^n)^*$. We denote the point of $(\mathbf{P}^n)^*$ corresponding to the hyperplane H by $[H]$.
iv) Show that the subset

$$\big\{\big(p, [H]\big) \in \mathbf{P}^n \times (\mathbf{P}^n)^* \mid p \in H\big\}$$

is closed in $\mathbf{P}^n \times (\mathbf{P}^n)^*$.
v) Show that given two sets of points in $\mathbf{P}^n$

$$\Gamma = \{P_0, \ldots, P_{n+1}\} \text{ and } \Gamma' = \{Q_0, \ldots, Q_{n+1}\},$$

such that no $(n+1)$ points in the same set lie in a hyperplane, there is a unique $A \in PGL_{n+1}(k)$ such that $A \cdot P_i = Q_i$ for every i.

Exercise 2.19. Let $X \subseteq \mathbf{P}^n$ be an irreducible closed subset of codimension r. Show that if $H \subseteq \mathbf{P}^n$ is a hypersurface such that X is not contained in H, then every irreducible component of $X \cap H$ has codimension $r+1$ in $\mathbf{P}^n$.

Exercise 2.20. Let $X \subseteq \mathbf{P}^n$ be a closed subset of dimension r. Show that there is a linear space $L \subseteq \mathbf{P}^n$ of dimension $(n-r-1)$ such that $L \cap X = \emptyset$.

Exercise 2.21. (The Segre embedding). Consider two projective spaces $\mathbf{P}^m$ and $\mathbf{P}^n$. Let $N = (m+1)(n+1)-1$, and let us denote the coordinates on $\mathbf{A}^{N+1}$ by $z_{i,j}$, with $0 \leq i \leq m$ and $0 \leq j \leq n$.

1) Show that the map $\mathbf{A}^{m+1} \times \mathbf{A}^{n+1} \to \mathbf{A}^{N+1}$ given by
$$((x_i)_i, (y_j)_j) \to (x_i y_j)_{i,j}$$
induces a morphism
$$\varphi_{m,n} \colon \mathbf{P}^m \times \mathbf{P}^n \to \mathbf{P}^N.$$
2) Consider the ring homomorphism
$$f_{m,n} \colon k[z_{i,j} \mid 0 \leq i \leq m, 0 \leq j \leq n] \to k[x_1, \ldots, x_m, y_1, \ldots, y_n], \quad f_{m,n}(z_{i,j}) = x_i y_j.$$
Show that $\ker(f_{m,n})$ is a homogeneous prime ideal that defines in $\mathbf{P}^N$ the image of $\varphi_{m,n}$ (in particular, this image is closed).
3) Show that $\varphi_{m,n}$ is a closed immersion.
4) Deduce that if X and Y are (quasi)projective varieties, then $X \times Y$ is a (quasi)projective variety.

Exercise 2.22. (The Veronese embedding). Let n and d be positive integers, and let $M_0, \ldots, M_N$ be all monomials in $k[x_0, \ldots, x_n]$ of degree d (hence $N = \binom{n+d}{d} - 1$).

1) Show that there is a morphism $\nu_{n,d} \colon \mathbf{P}^n \to \mathbf{P}^N$ that takes the point $[a_0, \ldots, a_n]$ to the point $\big[M_0(a), \ldots, M_N(a)\big]$.
2) Consider the ring homomorphism $f_d \colon k[z_0, \ldots, z_N] \to k[x_0, \ldots, x_n]$ defined by $f_d(z_i) = M_i$. Show that $\ker(f_d)$ is a homogeneous prime ideal that defines in $\mathbf{P}^N$ the image of $\nu_{n,d}$ (in particular, this image is closed).
3) Show that $\nu_{n,d}$ is a closed immersion.
4) Show that if Z is a hypersurface of degree d in $\mathbf{P}^n$ (this means that $I(Z) = (F)$, where F is a homogeneous polyomial of degree d), then there is a hyperplane H in $\mathbf{P}^N$ such that for every projective variety $X \subseteq \mathbf{P}^n$, the morphism $\nu_{n,d}$ induces an isomorphism between $X \cap Z$ and $\nu_{n,d}(X) \cap H$. This shows that the Veronese embedding allows to reduce the intersection with a hypersurface to the intersection with a hyperplane.
5) The *rational normal curve* in $\mathbf{P}^n$ is the image of the Veronese embedding $\nu_{1,d} \colon \mathbf{P}^1 \to \mathbf{P}^d$, mapping $[a,b]$ to $[a^d, a^{d-1}b, \ldots, b^d]$. Show that the rational normal curve is the zero-locus of the 2×2-minors of the matrix
$$\begin{pmatrix} z_0 & z_1 & \ldots & z_{d-1} \\ z_1 & z_2 & \ldots & z_d \end{pmatrix}.$$

Exercise 2.23. Use the Veronese embedding to deduce the assertion in Proposition 2.9 from the case when h is a linear form (which follows from Proposition 2.2).

Exercise 2.24. A *plane Cremona transformation* is a birational map of $\mathbf{P}^2$ into itself. Consider the following example of *quadratic* Cremona transformation: $\varphi\colon \mathbf{P}^2 \to \mathbf{P}^2$, given by $\varphi(x\colon y\colon z) = (yz\colon xz\colon xy)$, when no two of x, y, or z are zero.

1) Show that φ is birational, and its own inverse.
2) Find open subsets $U, V \subset \mathbf{P}^2$ such that φ induces an isomorphism $U \simeq V$.
3) Describe the open sets on which φ and φ^{-1} are defined.

3. A generalization: the MaxProj construction

We now describe a generalization of the constructions in the previous two sections. A key idea introduced by Grothendieck in algebraic geometry is that it is often better to study morphisms $f\colon X \to Y$, instead of varieties X (the case of a variety being recovered as the special case when Y is a point). More precisely, instead of studying varieties with a certain property, one should extend this property to morphisms and study it in this context. We begin with one piece of terminology.

Definition 3.1. Given a variety Y, a *variety over* Y is a morphism $f\colon X \to Y$, where X is another variety. A morphism between varieties $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ is a morphism of varieties $g\colon X_1 \to X_2$ such that $f_2 \circ g = f_1$. It is clear that we can compose morphisms of varieties over Y and we get, in this way, a category.

Following the above philosophy, we introduce in this section the Proj construction, that allows us to study projective varieties over Y, when Y is affine (as we will see, these are simply closed subvarieties of a product $Y \times \mathbf{P}^n$). We will return later to the case when Y is an arbitrary variety, after discussing quasi-coherent sheaves.

The setting is the following: we fix an $\mathbf{N}$-graded, reduced, finitely generated k-algebra $S = \bigoplus_{m \in \mathbf{N}} S_m$. This implies that S_0 is a finitely generated k-algebra and it is also easy to see that each S_m is a finitely generated S_0-module. We put $S_+ = \bigoplus_{m>0} S_m$.

Exercise 3.2. Given homogeneous elements $t_0, \ldots, t_n \in S_+$, show that they generate S as an S_0-algebra if and only if they generate S_+ as an ideal.

For the sake of simplicity, we always assume that S is generated as an S_0-algebra by S_1. This condition is equivalent with the fact that S is isomorphic, as a graded ring, to the quotient of $S_0[x_0, \ldots, x_n]$ by a homogeneous ideal, where the grading on this polynomial ring is given by the total degree of the monomials. Note that by the above exercise, our assumption implies that S_1 generates S_+ as an ideal.

Consider the affine varieties $W = \mathrm{MaxSpec}(S)$ and $W_0 = \mathrm{MaxSpec}(S_0)$ (see Exercise 2.18 in Chapter 3 for the notation). The inclusion $S_0 \hookrightarrow S$ corresponds to a morphism $f\colon W \to W_0$. The grading on S translates into an algebraic action of the torus k^* on W, as follows. We have a morphism

$$\alpha\colon k^* \times W \to W$$

corresponding to the k-algebra homomorphism $S \to k[t, t^{-1}] \otimes_k S$ mapping $\sum_i f_i$ to $\sum_i t^i f_i$, where $f_i \in S_i$ for all i. One can check directly that this gives an action of k^* on W, but we prefer to argue as follows: let us choose a surjective graded homomorphism of S_0-algebras $\varphi\colon S_0[x_0, \ldots, x_n] \to S$, corresponding to a closed immersion $j\colon W \hookrightarrow W_0 \times \mathbf{A}^{n+1}$ such that if $p\colon W_0 \times \mathbf{A}^{n+1} \to W_0$ is the first projection, we have $p \circ j = f$. As before, we have a morphism

$$\beta\colon k^* \times W_0 \times \mathbf{A}^{n+1} \to W_0 \times \mathbf{A}^{n+1}.$$

Since φ is a graded homomorphism, we see that the two morphisms are compatible via j, in the sense that

$$j\big(\alpha(\lambda, w)\big) = \beta\big(\lambda, j(w)\big) \quad \text{for all} \quad \lambda \in k^*, w \in W.$$

It is straightforward to check that

$$\beta(\lambda, w_0, x_0, \ldots, x_n) = (w_0, \lambda x_0, \ldots, \lambda x_n) \quad \text{for all} \quad \lambda \in k^*, w_0 \in W_0, (x_0, \ldots, x_n) \in \mathbf{A}^{n+1}.$$

Therefore β gives an algebraic action of k^* on $W_0 \times \mathbf{A}^{n+1}$, and thus α gives an algebraic action of k^* on W. We will keep using this embedding for describing the action of k^* on W. To simplify the notation, we will write $\lambda \cdot w$ for $\alpha(\lambda, w)$.

Lemma 3.3. *Given the above action of k^* on W, the following hold:*

i) *An orbit consists either of one point or it is 1-dimensional.*
ii) *A point is fixed by the k^*-action if and only if it lies in $V(S_+)$.*
iii) *If O is a 1-dimensional orbit, then O is a closed subset of $W \smallsetminus V(S_+)$, $\overline{O} \simeq \mathbf{A}^1$, and $\overline{O} \cap V(S_+)$ consists of one point.*

Proof. By embedding W in $W_0 \times \mathbf{A}^{n+1}$ as above, we reduce the assertions in the lemma to the case when $W = W_0 \times \mathbf{A}^{n+1}$, in which case they are all clear. Note that via this embedding, we have $V(S_+) = W_0 \times \{0\}$. □

Remark 3.4. By arguing as in Remark 1.5, we see that a closed subset $Z \subseteq W$ is invariant by the k^*-action (that is, $\lambda \cdot Z = Z$ for every $\lambda \in k^*$) if and only if the corresponding ideal $I_W(Z)$ is homogeneous.

Definition 3.5. Given S as above, we define $\mathrm{MaxProj}(S)$ to be the set of one-dimensional orbit closures for the action of k^* on W. Since every such orbit is clearly irreducible, being the image of a morphism $k^* \to W$, it follows from Lemma 3.3 and Remark 3.4 that these orbit closures are in bijection with the homogeneous prime ideals $\mathfrak{q} \subseteq S$ such that $S_+ \not\subseteq \mathfrak{q}$ and $\dim(S/\mathfrak{q}) = 1$.

We put a topology on $X = \mathrm{MaxProj}(S)$ by declaring that a subset is closed if it consists of all 1-dimensional orbit closures contained in some torus-invariant closed subset of W. Equivalently, the closed subsets are those of the form

$$V(I) = \{\mathfrak{q} \in \mathrm{MaxProj}(S) \mid I \subseteq \mathfrak{q}\},$$

for some homogeneous ideal $I \subseteq S$. The assertions in the next lemma, which are straightforward to prove, imply that this gives indeed a topology on $\mathrm{MaxProj}(S)$.

Lemma 3.6. *With the above notation, the following hold:*

i) *We have* $V(0) = \mathrm{MaxProj}(S)$ *and* $V(S) = \emptyset$.

ii) *For every two homogeneous ideals* I *and* J *in* S, *we have*

$$V(I) \cup V(J) = V(I \cap J) = V(I \cdot J).$$

iii) *For every family* $(I_\alpha)_\alpha$ *of homogeneous ideals in* S, *we have*

$$\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right).$$

Since every homogeneous ideal is generated by finitely many homogeneous elements, we see that every open set can be written as a finite union of sets of the form

$$D_X^+(f) = \{\mathfrak{q} \in \mathrm{MaxProj}(S) \mid f \not\in \mathfrak{q}\},$$

where $f \in S$ is a homogeneous element. In fact, we may take f of positive degree, since if $t_0, \ldots, t_n \in S_1$ generate S_+, we have

$$D_X^+(f) = \bigcup_{i=0}^n D_X^+(t_i f).$$

As a special case of this equality for $f = 1$, we have

$$\mathrm{MaxProj}(S) = D_X^+(t_0) \cup \ldots \cup D_X^+(t_n).$$

Remark 3.7. It is clear that if I is a homogeneous ideal in S, then $V(I) = V\big(\mathrm{rad}(I)\big)$. Moreover, if

$$I' = \{f \in S \mid f \cdot S_+ \subseteq \mathrm{rad}(I)\},$$

then $V(I) = V(I')$.

For future reference, we give the following variant of graded Nullstellensatz.

Proposition 3.8. *Let* S *be a graded ring as in the proposition. If* I *is a homogeneous, radical ideal in* S, *and* $f \in S$ *is homogeneous, such that* $f \in \mathfrak{q}$ *for all* $\mathfrak{q} \in \mathrm{MaxProj}(S)$ *with* $\mathfrak{q} \supseteq I$, *then* $f \cdot S_+ \subseteq I$. *If* $\deg(f) > 0$, *then* $f \in I$.

Proof. We first prove the last assertion, assuming $\deg(f) > 0$. After writing S as a quotient of a polynomial ring over S_0, we see that we may assume that $S = A[x_0, \ldots, x_n]$, with the standard grading. Recall that we take $W_0 = \mathrm{MaxSpec}(S_0)$ and $W = \mathrm{MaxSpec}(S) = W_0 \times \mathbf{A}^{n+1}$. Let $Y \subseteq W$ be the closed subset defined by I. Note that W is k^*-invariant. Our assumption says that f vanishes on $\{w_0\} \times L$, whenever L is a line in $\mathbf{A}^{n+1}$ with $\{w_0\} \times \mathbf{A}^{n+1} \subseteq Y$. On the other hand, since $\deg(f) > 0$, we see that f automatically vanishes along $W_0 \times \{0\}$, hence f vanishes along Y (we use the fact that Y is a union of k^*-orbits). We thus conclude that $f \in I$. The first assertion in the proposition now follows by applying what we know to each product fg, with $g \in S_1$. $\square$

Given an ideal $\mathfrak{q} \in \mathrm{MaxProj}(S)$, let T denote the set of homogeneous elements in $S \smallsetminus \mathfrak{q}$. Recall that the ring of fractions $T^{-1}S$ carries a natural grading, whose degree 0 part is denoted by $S_{(\mathfrak{q})}$. This is a local ring, with maximal ideal $\mathfrak{m}_\mathfrak{q} := \mathfrak{q} \cdot T^{-1}S \cap S_{(\mathfrak{q})}$. Similarly, given a homogeneous element $f \in S$, the localization S_f carries a natural grading, whose degree 0 part is denoted $S_{(f)}$.

Lemma 3.9. *For every $t \in S_1$, the following hold:*

i) *We have an isomorphism of graded rings $S_t \simeq S_{(t)}[x, x^{-1}]$.*
ii) *Every homogeneous ideal in S_t is of the form $\bigoplus_{m\in\mathbf{Z}}(I \cap S_{(t)})t^m$.*
iii) *We have a homeomorphism between $D^+(t)$ and $\mathrm{MaxSpec}(S_{(t)})$.*
iv) *For every $\mathfrak{q} \in \mathrm{MaxProj}(S)$, the residue field of $S_{(\mathfrak{q})}$ is equal to k.*

Proof. Since the element $\frac{t}{1} \in S_t$ has degree 1 and is invertible, it follows easily that the homomorphism of graded $S_{(t)}$-algebras

$$S_{(t)}[x, x^{-1}] \to S_t$$

that maps x to $\frac{t}{1}$ is an isomorphism. This gives i) and the assertion in ii) is straightforward to check.

It is clear that localization induces a bijection between the homogeneous prime ideals in S that do not contain t and the homogeneous prime ideals in S_t. Moreover, it follows from ii) that every such prime ideal in S_t is of the form $\bigoplus_{m\in\mathbf{Z}} \mathfrak{p}t^m$, for a unique prime ideal $\mathfrak{p}$ in $S_{(t)}$. If $\mathfrak{q} \subseteq S$ corresponds to $\mathfrak{p} \subseteq S_{(t)}$, then

$$(S/\mathfrak{q})_t \simeq (S_{(t)}/\mathfrak{p})[x, x^{-1}], \tag{2}$$

hence

$$\dim(S/\mathfrak{q}) = \dim\big((S/\mathfrak{q})_t\big) = \dim(S_{(t)}/\mathfrak{p}) + 1.$$

Therefore $\mathfrak{q}$ lies in $\mathrm{MaxProj}(S)$ if and only if $\mathfrak{p}$ is a maximal ideal in $S_{(t)}$. This gives the bijection between $D^+(t)$ and $\mathrm{MaxSpec}(S_{(t)})$ and it is straightforward to check, using the definitions of the two topologies, that this is a homeomorphism.

Finally, given any $\mathfrak{q} \in \mathrm{MaxProj}(S)$, we can find $t \in S_1$ such that $\mathfrak{q} \in D^+(t)$. If $\mathfrak{p}$ is the corresponding ideal in $S_{(t)}$, then the isomorphism (2) implies that the residue field of $S_{(\mathfrak{q})}$ is isomorphic as a k-algebra to the residue field of $(S_{(t)})_{\mathfrak{p}}$, hence it is equal to k. □

We now define a sheaf of functions on $X = \mathrm{MaxProj}(S)$, with values in k, as follows. For every open subset U in X, let $\mathcal{O}_X(U)$ be the set of functions $\varphi\colon U \to k$ with the following property: for every $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x and homogeneous elements $f, g \in S$ of the same degree such that for every $\mathfrak{q} \in U_x$, we have $g \not\in \mathfrak{q}$ and $\varphi(\mathfrak{q})$ is equal to the image of $\frac{f}{g}$ in the residue field of $S_{(\mathfrak{q})}$, which is equal to k by Lemma 3.9. It is straightforward to check that $\mathcal{O}_X(U)$ is a k-subalgebra of $\mathcal{F}un_X(U)$ and that, with respect to restriction of functions, $\mathcal{O}_X$ is a sheaf. This is the *sheaf of regular functions* on X. From now on, we denote by $\mathrm{MaxProj}(S)$ the object $(X, \mathcal{O}_X)$ in $\mathcal{T}op_k$.

Remark 3.10. It is clear from the definition that we have a morphism in $\mathcal{T}op_k$

$$\mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0)$$

that maps $\mathfrak{q}$ to $\mathfrak{q} \cap S_0$.

Proposition 3.11. *If we have a surjective, graded homomorphism $\varphi\colon S \to T$, then we have a commutative diagram*

$$\begin{array}{ccc} \operatorname{MaxProj}(T) & \xrightarrow{j} & \operatorname{MaxProj}(S) \\ \downarrow & & \downarrow f \\ \operatorname{MaxSpec}(T_0) & \xrightarrow{i} & \operatorname{MaxProj}(S_0), \end{array}$$

in which i is a closed immersion and j given an isomorphism onto $V(I)$ (with the induced sheaf from the ambient space)[2]*, where $I = \ker(\varphi)$.*

Proof. Note first that since φ is surjective, the induced homomorphism $S_0 \to T_0$ is surjective as well, hence the induced morphism $i\colon \operatorname{MaxSpec}(T_0) \to \operatorname{MaxSpec}(S_0)$ is a closed immersion. Since φ is graded and surjective, we have $T_+ = \varphi(S_+)$ and $S_+ = \varphi^{-1}(T_+)$, hence $S_+ \subseteq \varphi^{-1}(\mathfrak{p})$ if and only if $T_+ \subseteq \mathfrak{p}$. We can thus define $j\colon \operatorname{MaxProj}(T) \to \operatorname{MaxProj}(S)$ by $j(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. It is straightforward to see that the diagram in the proposition is commutative and that j gives a homeomorphism of $\operatorname{MaxProj}(T)$ onto the closed subset $V(I)$ of $\operatorname{MaxProj}(S)$. Furthermore, it is easy to see, using the definition, that if U is an open subset of $V(I)$, then a function $\varphi\colon U \to k$ has the property that $\varphi \circ j$ is regular on $j^{-1}(U)$ if and only if it can be locally extended to a regular function on open subsets in $\operatorname{MaxProj}(S)$. This gives the assertion in the proposition. □

We now consider in detail the case when $S = A[x_0, \ldots, x_n]$, with the standard grading. As before, let $W_0 = \operatorname{MaxSpec}(A)$. We have seen that a point $\mathfrak{p}$ in $X = \operatorname{MaxProj}(S)$ corresponds to a subset in $W_0 \times \mathbf{A}^{n+1}$, of the form $\{w_0\} \times L$, where L is a 1-dimensional linear subspace in k^{n+1}, corresponding to a point in $\mathbf{P}^n$. We thus have a bijection between $\operatorname{MaxProj}(S)$ and $W_0 \times \mathbf{P}^n$. Moreover, since $x_0, \ldots, x_n$ span S_1, we see that

$$X = \bigcup_{i=0}^{n} D_X^+(x_i).$$

The above bijection induces for every i a bijection between $D_X^+(x_i)$ and $W_0 \times D_{\mathbf{P}^n}^+(x_i)$. In fact, this is the same as the homeomorphism between $D_X^+(x_i)$ and

$$\operatorname{MaxSpec}\big(A[x_0, \ldots, x_n]_{(x_i)}\big) = \operatorname{MaxSpec}\big(A[x_0/x_i, \ldots, x_n/x_i]\big)$$

given by assertion iii) in Lemma 3.9. Furthermore, arguing as in the proof of Proposition 2.2, we see that each of these homeomorphisms gives an isomorphism of objects in $\mathcal{T}op_k$. We thus obtain the following

Proposition 3.12. *If $S = A[x_0, \ldots, x_n]$, with the standard grading, and $W_0 = \operatorname{MaxSpec}(A)$, then we have an isomorphism*

$$\operatorname{MaxProj}(S) \simeq W_0 \times \mathbf{P}^n$$

of varieties over W_0.

[2] Once we will show that $\operatorname{MaxProj}(S)$ and $\operatorname{MaxProj}(T)$ are algebraic varieties, this simply says that j is a closed immersion.

Corollary 3.13. *If S is a reduced, $\mathbf{N}$-graded, finitely generated k-algebra, generated as an S_0-algebra by S_1, then* $\mathrm{MaxProj}(S)$ *is a quasi-projective variety.*

Proof. By the assumption on S, we have a graded, surjective morphism of S_0-algebras

$$S_0[x_0, \ldots, x_n] \to S.$$

If $W_0 = \mathrm{MaxSpec}(S_0)$, then it follows from Propositions 3.11 and 3.12 that we have a closed immersion

$$\mathrm{MaxProj}(S) \hookrightarrow \mathrm{MaxProj}\big(S_0[x_0, \ldots, x_n]\big) \simeq W_0 \times \mathbf{P}^n,$$

which gives the assertion in the corollary, since a product of quasi-projective varieties is quasi-projective by Exercise 2.21. □

Remark 3.14. If X is a closed subset of $\mathbf{P}^n$, with homogeneous coordinate ring S_X, then $\mathrm{MaxProj}(S_X) \simeq X$. More generally, suppose that A is a reduced, finitely generated k-algebra, $W_0 = \mathrm{MaxSpec}(A)$, and X is a closed subvariety of $W_0 \times \mathbf{P}^n$. If I is a radical, homogeneous ideal in $A[x_0, \ldots, x_n]$ such that $X = V(I)$, then

$$X \simeq \mathrm{MaxProj}\big(A[x_0, \ldots, x_n]/I\big).$$

Indeed, the surjection

$$A[x_0, \ldots, x_n] \to A[x_0, \ldots, x_n]/I$$

induces by Proposition 3.11 a closed immersion

$$\iota\colon \mathrm{MaxProj}\big(A[x_0, \ldots, x_n]/I\big) \hookrightarrow \mathrm{MaxProj}\big(A[x_0, \ldots, x_n]\big).$$

It is then clear that, via the isomorphism $\mathrm{MaxProj}\big(A[x_0, \ldots, x_n]\big) \simeq W_0 \times \mathbf{P}^n$ provided by Proposition 3.12, the image of ι is equal to X.

Definition 3.15. Given an affine variety Y, a variety $f\colon X \to Y$ over Y is *projective* if there is a reduced, $\mathbf{N}$-graded, finitely generated k-algebra S, generated as an S_0-algebra by S_1, such that $Y \simeq \mathrm{MaxSpec}(S_0)$, and X is isomorphic (over Y) to $\mathrm{MaxProj}(S)$. It follows from the above remark, together with Propositions 3.11 and 3.12, that X is projective over Y if and only if it admits a closed immersion (over Y) in $Y \times \mathbf{P}^n$.

Proposition 3.16. *If S is a reduced, $\mathbf{N}$-graded, finitely generated k-algebra, generated as an S_0-algebra by S_1, then for every homogeneous $f \in S$, of positive degree, the open subset $D_X^+(f) \subseteq X =$* $\mathrm{MaxProj}(S)$ *is affine.*

Proof. By Proposition 3.11, it is enough to prove this when $S = S_0[x_0, \ldots, x_n]$. The argument in this case follows the one in the proof of Proposition 2.9. □

We now give a generalization of Proposition 2.10 describing the regular functions on the affine open subsets $D_X^+(f)$ in $\mathrm{MaxProj}(S)$.

Proposition 3.17. *Let S be a reduced, $\mathbf{N}$-graded, finitely generated k-algebra, generated as an S_0-algebra by S_1, and let $X =$* $\mathrm{MaxProj}(S)$*. For every homogeneous $f \in S$, of positive degree, consider the homomorphism*

$$\Phi\colon S_{(f)} \to \mathcal{O}\big(D_X^+(f)\big)$$

that maps $\frac{g}{f^m}$ *to the function taking* $\mathfrak{q} \in D_X^+(f)$ *to the image of* $\frac{g}{f^m}$ *in the residue field of* $S_{(\mathfrak{q})}$*, which is isomorphic to* k*. Then* Φ *is an isomorphism.*

Proof. The proof is similar to that of Proposition 4.6 in Chapter 1. We first show that Φ is injective. Suppose that $\frac{g}{f^m}$ lies in the kernel of Φ. In this case, for every $\mathfrak{q} \in X \smallsetminus V(f)$, we have $g \in \mathfrak{q}$. This implies that $fg \in \mathfrak{q}$ for every $\mathfrak{q} \in X$, hence $fg = 0$ by Proposition 3.8, hence $\frac{g}{f^m} = 0$ in $(S_X)_{(f)}$. This proves the injectivity of Φ.

In order to prove the surjectivity of Φ, consider $\varphi \in \mathcal{O}\big(D_X^+(f)\big)$. By hypothesis, and using the quasi-compactness of $D_X^+(f)$, we may write

$$D_X^+(f) = V_1 \cup \ldots \cup V_r,$$

for some open subsets V_i such that for every i, there are $g_i, h_i \in S$ homogeneous of the same degree such that for every $\mathfrak{q} \in V_i$, we have $h_i \not\in \mathfrak{q}$ and $\varphi(\mathfrak{q})$ is the image of $\frac{g_i}{h_i}$ in the residue field of $S_{(\mathfrak{q})}$. We may assume that $V_i = X \smallsetminus V(f_i)$ for $1 \leq i \leq r$, for some homogeneous $f_i \in S$, of positive degree. Since $h_i \not\in \mathfrak{q}$ for every $\mathfrak{q} \in X \smallsetminus V(f_i)$, it follows from Proposition 3.8 that $f_i \in \mathrm{rad}(h_i)$. After possibly replacing f_i by a suitable power, we may assume that $f_i \in (h_i)$ for all i. Finally, after multiplying both g_i and h_i by the same homogeneous element, we may assume that $f_i = h_i$ for all i.

We know that for $u \in X \smallsetminus V(g_i g_j)$ the two fractions $\frac{g_i(u)}{h_i(u)}$ and $\frac{g_j(u)}{h_j(u)}$ have the same image in the residue field of every $S_{(\mathfrak{q})}$. By the injectivity statement we have already proved, this implies that

$$\frac{g_i}{h_i} = \frac{g_j}{h_j} \quad \text{in} \quad S_{h_i h_j}.$$

Therefore there is a positive integer N such that

$$(h_i h_j)^N (g_i h_j - g_j h_i) = 0 \quad \text{for all} \quad i, j.$$

After replacing each g_i and h_i by $g_i h_i^N$ and h_i^{N+1}, respectively, we see that we may assume that

$$g_i h_j - g_j h_i = 0 \quad \text{for all} \quad i, j.$$

On the other hand, since

$$D_X^+(f) = \bigcup_{i=1}^r D_X^+(h_i),$$

we have

$$V(f) = V(h_1, \ldots, h_r),$$

and therefore Proposition 3.8 implies that $f \in \mathrm{rad}(h_1, \ldots, h_r)$. We can thus write

$$f^m = \sum_{i=1}^r a_i h_i \quad \text{for some} \quad m \geq 1 \quad \text{and} \quad a_1, \ldots, a_r \in S.$$

Moreover, by only considering the terms in $S_{m \cdot \deg(f)}$, we see that we may assume that each a_i is homogeneous, with $\deg(a_i) + \deg(h_i) = m \cdot \deg(f)$.

In order to complete the proof, it is enough to show that

$$\varphi = \Phi\left(\frac{a_1g_1 + \ldots + a_rg_r}{f^m}\right).$$

Note that for $\mathfrak{q} \in D_X^+(h_j)$, we have

$$\frac{g_j}{h_j} = \frac{a_1g_1 + \ldots + a_rg_r}{f^m} \quad \text{in} \quad S_{(\mathfrak{q})}$$

since

$$h_j \cdot \sum_{i=1}^r a_ig_i = \sum_{i=1}^r a_ih_ig_j = f^mg_j.$$

This completes the proof. □

Remark 3.18. Suppose that S is an $\mathbf{N}$-graded k-algebra as above and

$$f\colon X = \mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0) = Y$$

is the corresponding morphism. If $a \in S_0$ and we consider the $\mathbf{N}$-graded k-algebra S_a, then we have a map

$$j\colon \mathrm{MaxProj}(S_a) \to \mathrm{MaxProj}(S)$$

that maps $\mathfrak{q}$ to its inverse image in S. This gives an open immersion, whose image is $f^{-1}\big(D_Y(a)\big)$: this follows by choosing generators $t_1, \ldots, t_r \in S_1$ of S as an S_0-algebra, and by showing that for every i, the induced map

$$\mathrm{MaxSpec}\big((S_a)_{(t_i)}\big) \to \mathrm{MaxSpec}\big(S_{(t_i)}\big)$$

is an open immersion, with image equal to the principal affine open subset corresponding to $\frac{a}{1} \in S_{(t_i)}$.

Remark 3.19. Suppose again that S is an $\mathbf{N}$-graded k-algebra as above and $f\colon X = \mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0) = Y$ is the corresponding morphism. If J is an ideal in S_0, then the inverse image $f^{-1}\big(V(J)\big)$ is the closed subset $V(J \cdot S)$. This is the image of the closed immersion

$$\mathrm{MaxProj}\big(S/\mathrm{rad}(J \cdot S)\big) \hookrightarrow \mathrm{MaxProj}(S)$$

(see Proposition 3.11).

Remark 3.20. For every S as above, we have a surjective morphism

$$\pi\colon \mathrm{MaxSpec}(S) \smallsetminus V(S_+) \to \mathrm{MaxProj}(S).$$

Since all fibers are of dimension 1 (in fact, they are all isomorphic to $\mathbf{A}^1 \smallsetminus \{0\}$), we conclude that

$$\dim\big(\mathrm{MaxProj}(S)\big) = \dim\big(\mathrm{MaxSpec}(S) \smallsetminus V(S_+)\big) - 1 \leq \dim(S) - 1.$$

Moreover, this is an equality, unless every irreducible component of maximal dimension of $\mathrm{MaxSpec}(S)$ is contained in $V(S_+)$, in which case we have $\dim(S) = \dim(S_0)$.

Exercise 3.21. Show that if S is an $\mathbf{N}$-graded k-algebra as above and $X = \mathrm{MaxProj}(S)$, then for every $\mathfrak{q} \in X$, there is a canonical isomorphism

$$\mathcal{O}_{X,\mathfrak{q}} \simeq S_{(\mathfrak{q})}.$$