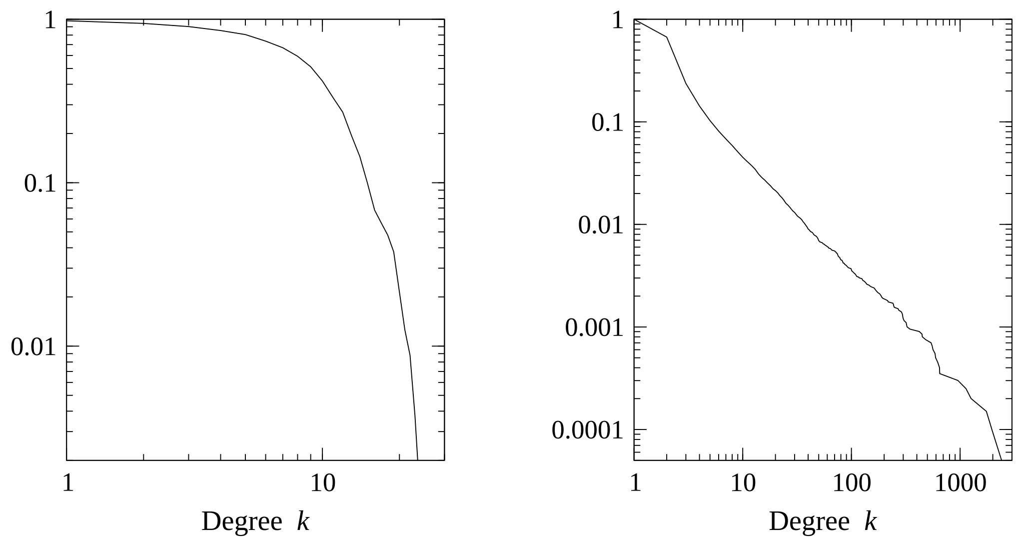
Complex Systems 535/Physics 508: Homework 5

1. Here are plots of the cumulative distribution function of degrees in two undirected networks:



   (a) One of these networks is approximately scale-free, the other is not. Which is which and how can you tell?

   (b) For the scale-free network give an estimate of the exponent α of the degree distribution.

   (c) If there are 10 000 nodes in the scale-free network, what approximately is the degree of the highest-degree node?

2. Consider the random graph $G(n, p)$ with n large.

   (a) If the network has a giant component that fills exactly half of the network, what is the average degree of a node?

   (b) For this same random graph what is the probability that a node has degree exactly 5?

   (c) What is the probability that a node belongs to the giant component if it has degree exactly 5?

   (d) Hence or otherwise, calculate the fraction of nodes in the giant component that have degree exactly 5.

3. Consider the random graph $G(n, p)$.

   (a) Show that in the limit of large n the expected number of triangles in the network is $\frac{1}{6}c^3$, where c is the mean degree. In other words, the number of triangles is constant, neither growing nor vanishing in the limit of large n.

   (b) Show that the expected number of connected triples in the network (as in Eq. (7.31)) is $\frac{1}{2}nc^2$.

   (c) Hence calculate the clustering coefficient C, as defined in Eq. (7.31), and confirm that it agrees for large n with the value given in Eq. (11.11).

4. Given that the clustering coefficient for $G(n, p)$ goes to zero for large n, can we make a different random graph model that has nonzero clustering coefficient? Indeed we can, as follows. Take n nodes and go through each distinct trio of three nodes, of which there are $\binom{n}{3}$, and with independent probability p connect the members of the trio together using three edges to form a triangle, where $p = c/\binom{n-1}{2}$ with c a constant. You can assume that n is very large.

   (a) Show that the mean degree of a node in this model network is $2c$.

   (b) Show that the degree distribution is $p_k = \mathrm{e}^{-c} c^{k/2}/(k/2)!$ if k is even and $p_k = 0$ if k is odd.

   (c) Show that the clustering coefficient, Eq. (7.31), is $C = 1/(2c + 1)$ (which does not go to zero as n becomes large).

   (d) Show that as a fraction of network size, the expected size S of the giant component, if there is one, satisfies $S = 1 - \mathrm{e}^{-cS(2-S)}$.

   (e) What is the value of the clustering coefficient when the giant component fills half of the network?

5. **Extra credit:** Write a computer program in the language of your choice that generates a random graph drawn from the model $G(n, m)$ for given values of n and the average degree $c = 2m/n$, then calculates the size of its largest component. Use your program to find the size of the largest component in a random graph with $n = 1\,000\,000$ and $c = 2 \ln 2 = 1.3863\ldots$ and compare your answer to the analytic prediction for the giant component of $G(n, p)$ with the same parameter values. You should find good agreement, even though the models are not identical. (As we discussed in class, the models become essentially the same for large values of m and n because the number of edges in $G(n, p)$ becomes tightly peaked around the expected value $\frac{1}{2}n(n-1)p$.)