Physics 390: Homework 8

For full credit, show all your working.

1. **The ideal gas:** The ideal gas is defined as a set of noninteracting particles in a box, so that the total energy E_s of a state of the complete set is given by $E_s = E_1 + E_2 + E_3 + \ldots$, where E_1 is the energy of the first particle, and so on.

   (a) Suppose there are N particles in our gas. The partition function of the complete gas is $Z_N = \sum_s \mathrm{e}^{-E_s/kT}$. Show that
   $$Z_N = [Z_1]^N,$$
   where Z_1 is the partition function for a single particle.

   (b) The pressure of a gas in a box is given by
   $$p = kT\frac{\partial \ln Z_N}{\partial V},$$
   where V is the volume of the box. (The derivation of this equation is straightforward, but involves the concept of free energy, which we haven't covered—if you take Physics 406 you'll see where the equation comes from.) Using the expression for Z_1 that we found in class, calculate the pressure of the ideal gas of N particles and hence prove that for n moles of gas the pressure and volume are related by
   $$pV = nRT,$$
   where R is a constant.

   (c) From the results of your derivation, find a value for R and check that it agrees with the known value of the gas constant given in the back of the book.

2. **The nucleus:** The protons and neutrons in a nucleus are fermions with spin $\frac{1}{2}$, meaning that at most two of them can be in any energy state at the same time—one with spin up and one with spin down. Consider the nucleus of a ^{22}Ne atom, which has 10 protons and 12 neutrons and a radius of 3.1×10^{-15} m.

   (a) Approximating the nucleus as a cubic box of the appropriate size, estimate the total kinetic energy of the particles in the nucleus. Express your results in MeV.

   (b) How does your figure compare with the total relativistic energy $E = mc^2$ of ^{22}Ne?

3. **White dwarf stars:** Living stars such as our Sun hold their shape against gravity because of the ordinary (but very high) hydrodynamic pressure created by their heat. When stars die and stop shining, however, they become cool and collapse under their own weight to form white dwarf stars. White dwarfs are held up not by conventional pressure but by the degeneracy pressure of the Fermi gas formed by their electrons. In the calculations

below, assume that the electrons can be treated as a non-interacting Fermi gas of the kind discussed in class, filling the entire volume of the star.

Consider a spherical white dwarf star of mass M, radius R, and uniform density.

(a) Essentially all of the mass of the star is in the form of positively charged protons with mass m_p and charge $+e$. (The electrons are so much lighter that they make almost no contribution to the mass.) Given that the star is electrically neutral, give an expression for the number of electrons it contains. Hence write an expression for the number density ρ of electrons (the number of electrons per unit volume).

(b) The total kinetic energy of a non-interacting Fermi gas at $T = 0$ is given by

$$E_e = \int_0^{E_F} E\, g(E)\, \mathrm{d}E,$$

where E_F is the Fermi energy (also sometimes denoted μ). Using the formulas for the Fermi energy and the density of states $g(E)$ from the class (and remembering the factor of 2 because of the spin states), show that

$$E_e = \frac{3}{5} N E_F.$$

(c) Hence, assuming the star to be at temperature $T = 0$, show that the total kinetic energy of the electrons in the star is

$$E_e = \frac{3\hbar^2}{10 m_e R^2}\left(\frac{9\pi}{4}\right)^{2/3}\left(\frac{M}{m_p}\right)^{5/3},$$

where m_e is the mass of the electron.

(d) **Extra credit:** It can be shown by simple mechanics that the gravitational potential energy of the star is

$$E_g = -\frac{3GM^2}{5R},$$

where G is Newton's gravitational constant. By minimizing the total energy $E = E_g + E_e$ of the star, show that the radius of the star depends on its mass as

$$R = \frac{\hbar^2}{G m_e}\left(\frac{81\pi^2}{16 m_p^5}\right)^{1/3} M^{-1/3}.$$

When the Sun finally dies and becomes a white dwarf, what will its radius be? How does this compare to its current radius?