Physics 406: Homework 9

1. **Internal energy in the grand ensemble:** We have seen that the grand potential Ω is given by

$$\Omega = -\tau \log \mathcal{Z} = U - \tau\sigma - \mu N.$$

   (a) Using the expressions for σ and N in terms of Ω show that the internal energy U is given by

$$\frac{U}{\tau} = \tau \frac{\partial \log \mathcal{Z}}{\partial \tau}\bigg|_{\mu} + \mu \frac{\partial \log \mathcal{Z}}{\partial \mu}\bigg|_{\tau}.$$

   (b) Prove that for any function $f(\tau,\lambda)$, where $\lambda = e^{\mu/\tau}$ is the activity of the system

$$\frac{\partial f}{\partial \tau}\bigg|_{\lambda} = \frac{\partial f}{\partial \tau}\bigg|_{\mu} + \frac{\partial f}{\partial \mu}\bigg|_{\tau}\frac{\partial \mu}{\partial \tau}\bigg|_{\lambda} = \frac{\partial f}{\partial \tau}\bigg|_{\mu} + \frac{\partial f}{\partial \mu}\bigg|_{\tau}\frac{\mu}{\tau}.$$

   (c) Hence show that U can also be written

$$U = \tau^2 \frac{\partial \log \mathcal{Z}}{\partial \tau}\bigg|_{\lambda}.$$

   This is the equivalent of the familiar expression $U = \tau^2(\partial \log Z/\partial\tau)$ for the canonical ensemble.

   (d) Use this result to show that the internal energy of the classical perfect gas is $U = \frac{3}{2}N\tau$. You will need the result for the value of N in a perfect gas that we derived in class.

2. **Black-body radiation again:** Photons are bosons, and so obey Bose-Einstein statistics.

   (a) Photons of frequency ω have energy $\hbar\omega$. Using the results we derived in class for the average number of bosons in a single-particle state, what is the average number $\langle s\rangle$ of photons in a mode of frequency ω in a cavity?

   (b) Compare this result with our previous derivation of the same quantity within the Boltzmann ensemble, and hence find a value for the chemical potential of a photon.

   One can repeat the whole derivation of the black-body spectrum in this way, regarding radiation as a gas of bosons.

3. **White dwarf stars:** Living stars such as our Sun hold their shape against gravity because of the ordinary (but very high) hydrodynamic pressure created by their heat. When stars die and stop shining however, they become cool and collapse under their own weight to form white dwarf stars. White dwarfs are held up not by conventional pressure but by the degeneracy pressure of the Fermi gas formed by their electrons.

   Consider a spherical white dwarf star of mass M, radius R, and uniform density.

   (a) Most of the mass of the star is in the form of protons with mass m_p. Given that the star is electrically neutral, how many electrons does it contain? Hence write an expression for the number density $\rho = N/V$ of electrons.

(b) Assuming the star to be at temperature $\tau = 0$, show that the total kinetic energy E_e of the degenerate Fermi gas of electrons in the star is

$$E_e = \frac{3\hbar^2}{10 m_e R^2} \left(\frac{9\pi}{4} \right)^{2/3} \left(\frac{M}{m_p} \right)^{5/3},$$

where m_e is the mass of the electron.

(c) It can be shown by simple mechanics that the gravitational potential energy of the star is

$$E_g = -\frac{3GM^2}{5R},$$

where G is Newton's gravitational constant. By minimizing the total energy $E = E_g + E_e$ of the star, show that the radius of the star depends on its mass as

$$R = \frac{\hbar^2}{G m_e} \left(\frac{81\pi^2}{16 m_p^5} \right)^{1/3} M^{-1/3}.$$

(d) When the Sun finally dies and becomes a white dwarf, what will its radius be? How does this compare to its current radius?