Douglass Houghton Workshop, Section 1, Mon 12/09/24

Worksheet Sweet Sorrow

1. (This problem appeared on a Fall, 2008 Math 115 exam) Determine a and b for the function of the form $y = f(t) = at^2 + b/t$, with a local minimum at $(1, 12)$.

2. (From the Winter, 2007 Math 115 Final Exam) Suppose that f and g are continuous functions with
$$\int_0^2 f(x)\,dx = 5 \qquad \text{and} \qquad \int_0^2 g(x)\,dx = 13.$$
Compute the following. If the computation cannot be made because something is missing, explain clearly what is missing.
   (a) $\int_4^6 f(x-4)\,dx$
   (b) $\int_{-2}^0 2g(-t)\,dt$
   (c) $\int_2^0 (f(y)+2)\,dy$
   (d) $\int_2^2 g(x)\,dx$
   (e) Suppose that f is an even function. Find the average value of f from -2 to 2.

3. (Fall, 2014) Maggie and Makaela decide to make a business out of their love for all things Japanese. So they begin importing plush toys made in the images of the ubiquitous Japanese mascots (like Hello Kitty or Totoro). After careful study, Maggie and Makaela have determined that they can produce up to 160 toys of plush toys in a year. They can sell the first 100 toys to comic book stores and any remaining plush toys to wholesalers. The revenue in dollars from selling x toys of plush toys will be

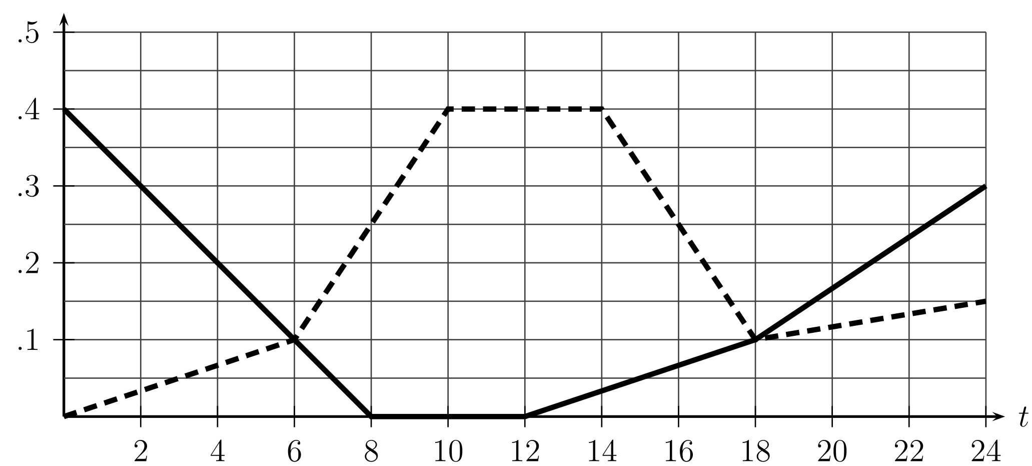
$$R(x) = \begin{cases} 6x & \text{if } 0 \le x \le 100 \\ 4x + 200 & \text{if } 100 < x \le 160. \end{cases}$$
   (a) What is the price comic book stores pay for each toy of plush toys?
   (b) What is the price wholesalers pay?
   (c) It costs $C(x) = 20 + 3x + 24\sqrt{x}$ to produce x toys of plush toys. (Use that formula for the rest of the problem.) What is the fixed cost of Maggie and Makaela's operation?
   (d) At what production levels does marginal revenue equal marginal cost?
   (e) How many toys of plush toys should Maggie and Makaela produce to maximize their profit, and what is the maximum possible profit?

4. (Fall, 2011) For positive A and B, the force between two atoms is a function of the distance, r, between them:
$$f(r) = -\frac{A}{r^2} + \frac{B}{r^3}.$$
   (a) Find the zeroes of f in terms of A and B.
   (b) Find all critical points and inflection points of f in terms of A and B.
   (c) If f has a local minimum at $(1, -2)$ find the values of A and B. Using your values for A and B, justify that $(1, -2)$ is a local minimum.

5. (Fall, 2009) After an unusual winter storm, the EPA is concerned about potential contamination of a river. A new researcher has been assigned the task of taking a sample to test the water quality. She tried to get as close to the river as possible in her car, but was forced to park a feet away. She also cannot get closer to the lab by car. She needs to walk to the river, retrieve a water sample, and then walk the sample to a lab located $4a$ feet down the river and $2a$ feet from the river bank. If the researcher wants to walk as short a distance as possible, what path should she take as she walks from her car to the river and then from the river to the lab?

Blake is planning a back-country skiing trip in the UP, and is therefore very concerned with the results of the story below.

6. (Fall, 2011) Suppose the graph below shows the rate of snow melt and snowfall on Mount Arvon, the highest peak in Michigan (at a towering 1970 ft), during a day (24 hour period) in April of last year. The function $m(t)$ (the solid curve) is the rate of snow melt, in inches per hour, t hours after the beginning of the day. The function $p(t)$ (the dashed curve) is the snowfall rate in inches per hour t hours after beginning of the day. There were 18 inches of snow on the ground at the beginning of the day.



(a) Over what period(s) was the snowfall rate greater than the snow melt rate?

(b) When was the amount of snow on Mount Arvon increasing the fastest? When was it decreasing the fastest?

(c) When was the amount of snow on Mount Arvon the greatest? Explain.

(d) How much snow was there on Mount Arvon at the end of the day (at $t = 24$)?

(e) Sketch a well-labeled graph of $P(t)$, an antiderivative of $p(t)$ satisfying $P(0) = 0$. Label and give the coordinates of the points on the graph of $P(t)$ at $t = 10$ and $t = 18$.

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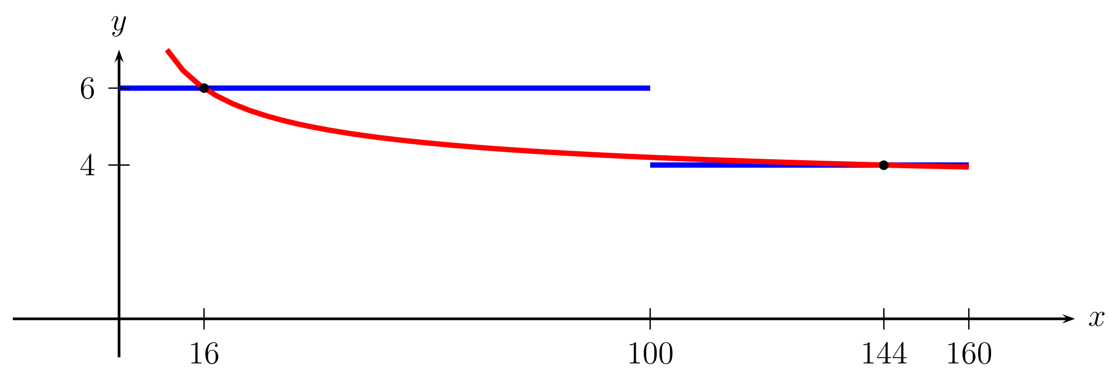
1. $\boxed{a = 4, b = 8}$. Check: $f(t) = 4t^2 + 8/t$, $f'(t) = 8t - 8/t^2$, $f''(t) = 8 + 16/t^3$. So $f(1) = 12$, $f'(1) = 0$, and $f''(1) = 24 > 0$.

2. (a) 5, by shifting
   (b) 26
   (c) -9
   (d) 0
   (e) 2.5

3. (a) The retailers pay \$6 per unit
   (b) The wholesalers pay \$4 per unit
   (c) Fixed cost $= C(0) = 20$.
   (d)



   (e) Check all the critical points and endpoints:

x	0	16	100	144	160
$R(x)$	0	96	600	776	840
$C(x)$	20	128	560	740	$500 + 96\sqrt{10} \approx 803.58$
$\pi(x)$	-20	-32	40	36	36.42

   So max profit comes when $x = 100$, and $\pi(100) = 40$.

4. (a) $f(r) = 0$ only when $r = B/A$.
   (b) Critical point at $x = \frac{3}{2}\frac{B}{A}$, Inflection point at $x = 2\frac{B}{A}$.
   (c) $f(1) = 2$ and $f'(1) = 0$, so $-A + B = -2$ and $2A - 3B = 0$. So $\boxed{A = 6,\ B = 4}$. Check: $f(r) = -6r^{-2}+4r^{-3}$, so $f'(r) = 12r^{-3}-12r^{-4}$ and $f''(r) = -36r^{-4}+48r^{-5}$. The $f(1) = -4$, $f'(1) = 0$, and $f''(1) = 12 > 0$. So $(1, -2)$ is indeed a local minimum.

5. First, the hard way. Let x be the distance along the river from the point opposite the car to the collection site. Then the total distance walked is

$$L = \sqrt{a^2 + x^2} + \sqrt{(4a - x)^2 + (2a)^2} = \boxed{\sqrt{x^2 + a^2} + \sqrt{x^2 - 8ax + 20a^2}}.$$

So

$$\begin{aligned}\frac{dL}{dx} &= \frac{1}{2}\left(x^2 + a^2\right)^{-1/2}(2x) + \frac{1}{2}\left(x^2 - 8ax + 20a^2\right)^{-1/2}(2x - 8a) \\ &= \boxed{\frac{x}{\sqrt{x^2 + a^2}} + \frac{x - 4a}{\sqrt{x^2 - 8ax + 20a^2}}}.\end{aligned}$$

Critical points are where $dL/dx = 0$, so

$$\begin{aligned}\frac{x}{\sqrt{x^2 + a^2}} &= \frac{4a - x}{\sqrt{x^2 - 8ax + 20a^2}} \\ x\sqrt{x^2 - 8ax + 20a^2} &= (4a - x)\sqrt{x^2 + a^2} \\ x^2(x^2 - 8ax + 20a^2) &= (16a^2 - 8ax + x^2)(x^2 + a^2) \\ x^4 - 8ax^3 + 20a^2x^2 &= x^4 - 8ax^3 + 17a^2x^2 - 8a^3x + 16a^4 \\ 3a^2x^2 + 8a^3x - 16a^4 &= 0 \\ a^2(3x - 4a)(x + 4a) &= 0.\end{aligned}$$

So the only positive critical point is at $x = 4a/3$. (Since we squared both sides and possibly intrduced an extraneous solution, we should check to make sure $L'(4a/3)$ really is 0. It is.) It's a min since

$$\begin{aligned}L'(0) &= \frac{-4a}{\sqrt{20a^2}} = -\frac{2}{\sqrt{5}} < 0 \\ L'(4a) &= \frac{4a}{\sqrt{17a^2}} = \frac{4}{\sqrt{17}} > 0.\end{aligned}$$

Since it's the only critical point and it's a local min, it's a global min.
Can make this a lot less arduous by realizing that all the lengths are in terms of a.

That was done to keep you from using your calculator, but if you let $y = x/a$, i.e. make y be x in units of a, then all the a's go away and everything looks much better. Also, there's a shortcut to this problem. Consider a point that's the mirror image of

the lab across the river edge. The distance from the collection spot to the mirror point is the same as the distance from the collection point to the lab. Therefore we might as well pick the spot on the shore that minimizes the total distance from the car to the shore to the mirror point. But that should clearly be a straight line. The collection point will be a third of the way $(a/(a + 2a))$ along the shore, or $4a/3$.

6. (a) $6 < t < 18$

   (b) Increasing fastest when $10 \le t \le 12$, decreasing fastest at $t = 0$

(c) Max snow at $t = 18$.

(d) $18\,\text{in} + [-12\,\text{boxes} + 32\,\text{boxes} - 4.5\,\text{boxes}] \cdot \frac{(.05\,\text{in/hr})(2\,\text{hr})}{1\,\text{box}} = 19.5\,\text{in}$

(e) (0,0.0)MIN inc,cu (6,0.3) inc,cu (10,1.3) linear (14,2.9) inc,cd (18,3.9) inc,cu (24,4.65). Should be smooth.