Douglass Houghton Workshop, Section 2, Thu 04/09/26

Worksheet Seek, and Ye Shall Find

1. Last time we found a remarkable power series:

$$\frac{x}{1-x-x^2} = x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+21x^8+34x^9+55x^{10}+\cdots = \sum_{n=0}^{\infty} F_n x^n$$

where F_n is the nth Fibonacci number, defined by $F_n = F_{n-1} + F_{n-2}$. That's already a pretty cool result. But if we play our cards right, we can parlay it into a non-recursive formula for the Fibonacci numbers.

(a) If a is a constant, what is the power series for $\frac{1}{1-ax}$ about $x = 0$?

(b) Verify that if $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ then $(1-\alpha x)(1-\beta x) = 1 - x - x^2$.

(c) Now suppose we could split the generating function above like this:

$$\frac{x}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

for some constants A and B. Find what A and B must be to make the equation above work for all values of x.

(d) Now find the series for $\frac{A}{1-\alpha x}$ and $\frac{B}{1-\beta x}$, in Σ form, and add them together to get a formula for the Fibonacci numbers.

2. Find the full Taylor series for $f(x) = \dfrac{1}{\sqrt{1-4x}}$ about $x = 0$. Also find the radius of convergence.

3. Consider an epidemic. On each day t, divide the population into three types:

$$\begin{aligned} S(t) &= \text{The fraction of people who are susceptible to infection,} \\ I(t) &= \text{The fraction of people currently infected, and} \\ R(t) &= \text{The fraction of people who have been removed from the pool.} \end{aligned}$$

(People are removed from the pool after they have the disease and can no longer become infected.) Suppose that the probability that the disease is transmitted during a single interaction between an infected and susceptible person is .01 (1%).

(a) Suppose Bob is a susceptible person, and on day t he interacts with n other people, chosen randomly. About how many *infected* people did he interact with?

(b) What's the probability Bob was NOT infected the first time he met an infected person AND not infected the second time he met an infected person? Assume the interactions are independent.

(c) What's the probability that Bob wasn't infected by any of his interactions?

(d) What's the last answer if the probability of transmission is p instead of .01?

4. So the probability that Bob becomes infected on day t is

$$1 - (\text{Your answer from the previous problem}).$$

   (a) Find the first two nonzero terms of the Taylor series about $x = 0$ for

   $$f(x) = (1 - x)^k$$

   where k is some constant.

   (b) Use that to approximate the probability that Bob becomes infected.

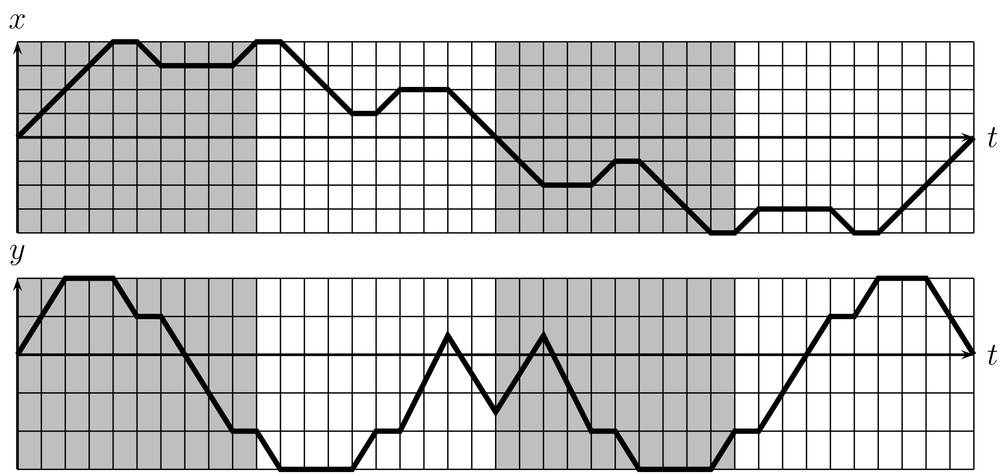
   (c) There are $S(t)$ susceptible people at time t, so take your answer from the last part and multiply it by $S(t)$ to get the expected number of infections that occur on day t.

5. So we can model the disease's spread by saying that the three types of people change accoring to these equations:

$$\begin{aligned} S(t+1) &= S(t) - (\text{your answer to the last problem}) \\ I(t+1) &= I(t) + (\text{your answer to the last problem}) - vI(t) \\ R(t+1) &= R(t) + vI(t) \end{aligned}$$

where v is the recovery rate. Write out those three equations—that's our model for the spread of the epidemic. We'll run it together with a spreadsheet.

6. Plot the positions of (x, y) given the graphs of $x(t)$ and $y(t)$ below:



7. (This problem appeared on a Fall, 2004 Math 116 exam.)

   (a) Find the second order Taylor polynomial of $f(x) = \sqrt{4 + x}$ for x near 0.

   (b) Find the Taylor series about $x = 0$ of $\sin(2x)$, either from scratch or by using a series you know already.

   (c) Using your answers to parts (a) and (b) and *without computing any derivatives*, find the second order Taylor polynomial that approximates $g(x) = \sqrt{4 + \sin(2x)}$ for x near 0.

   (d) The error in a Taylor polynomial approximation is mostly in the first term omitted, in this case the third order term. So compute that, and give an estimate of the maximum error in the approximation when $-0.1 \le x \le 0.1$.