Douglass Houghton Workshop, Section 2, Tue 04/07/26

Worksheet Rendezvous with Destiny

1. Previously we found formulas for converting from latitude (ϕ), longitude (θ), and distance-from-the-center-of-the-earth (ρ) to Cartesian coordinates:

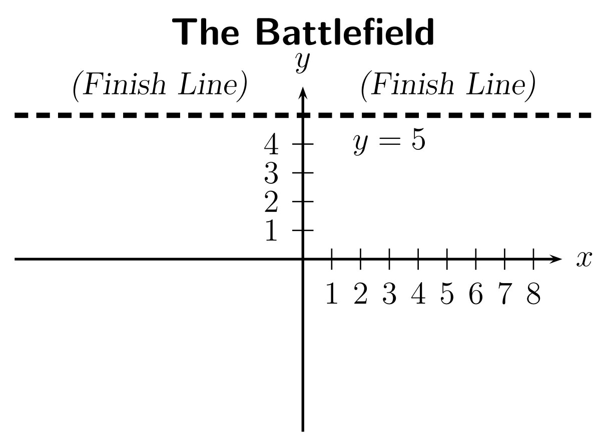
$$x = \rho \cos\phi \sin\theta, \qquad y = \rho\cos\phi\cos\theta, \qquad z = \rho\sin\phi$$

   (a) How can you find ρ if you know x, y, and z? (Hint: At first assume $z = 0$.)

   (b) Likewise find formulas for ϕ and θ in terms of x, y, and z.

2. There is *still* nothing special at latitude 14°38′53″ N, longitude 78°6′28″ W. It's just a point in the ocean. But, if you were to shoot a neutrino from the middle of the Diag (latitude 42°16′36″ N, longitude 83°44′15″ W) to that point, through the earth's crust, its deepest point would be directly under a very interesting place. Find that place, using one of the Google spreadsheets linked at the bottom of our homepage.

3. (This problem appeared on a Winter, 2003 Math 116 exam) The newest FOX reality show, "BattleBugs: Clash of the Beetles" begins (at $t = 0$) with eight assorted insects placed randomly on a large mat (the "battlefield", pictured here), on which is marked a "finish line". The producers hoped that the bugs would battle to be first to cross the finish line, but instead they wander around, each according to its nature. The motion of each bug is described by the equations below. Both x and y are measured in inches.



Hercules Beetle	Ladybug	Tiger Beetle	Longhorned Beetle
$x(t) = \cos(t/2)$ $y(t) = \sin(t/2)$	$x(t) = e^{-t}$ $y(t) = e^{-2t}$	$x(t) = 1 + t$ $y(t) = -1 + 8t$	$x(t) = 3 + t$ $y(t) = 4 - t$
Dung Beetle	**Scarab**	**June Beetle**	**African Ground Beetle**
$x(t) = t$ $y(t) = -2$	$x(t) = 2 - 7t$ $y(t) = -1 - 7t$	$x(t) = 0$ $y(t) = -1$	$x(t) = \sin(t)$ $y(t) = \cos(t)$

Which bug (or bugs)...

(a) move repetitively?

(b) begin closest to the finish line?

(c) move fastest?

(d) will move very slowly (or not at all), in the long run?

(e) will reach the finish line first?

(f) gets the dizziest?

4. Last time we found a remarkable power series:

$$\frac{x}{1-x-x^2} = x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+21x^8+34x^9+55x^{10}+\cdots = \sum_{n=0}^{\infty} F_n x^n$$

where F_n is the nth Fibonacci number, defined by $F_n = F_{n-1} + F_{n-2}$. That's already a pretty cool result. But if we play our cards right, we can parlay it into a non-recursive formula for the Fibonacci numbers.

(a) If a is a constant, what is the power series for $\frac{1}{1-ax}$ about $x = 0$?

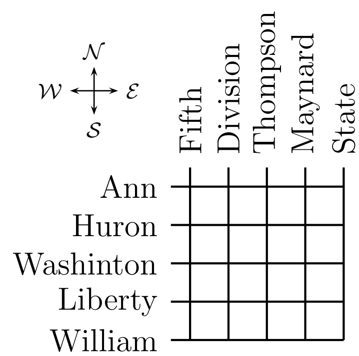
(b) Verify that if $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ then $(1-\alpha x)(1-\beta x) = 1 - x - x^2$.

(c) Now suppose we could split the generating function above like this:

$$\frac{x}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

for some constants A and B. Find what A and B must be to make the equation above work for all values of x.

(d) Now find the series for $\frac{A}{1-\alpha x}$ and $\frac{B}{1-\beta x}$, in Σ form, and add them together to get a formula for the Fibonacci numbers.



5. Last time we found that the number of ways to get from one point to another in the grid at right was an entry in Pascal's Triangle, as long as we only ever go east and south.

(a) Explain how the number of ways to go, say, 3 blocks east and 4 blocks south is related to counting strings of E's and S's.

(b) So how can we count the number of strings of a E's and b S's? You can use the notation $\binom{n}{k}$ for the kth entry in the nth row of Pascal's Triangle, where the top row is row 0 and the left column is column 0.

(c) Now suppose we play 5 games of Roulette, betting on red each time. We know the probability of winning each game is 9/19. What is the probability we win all 5? What's the probability we win 4 and lose 1, in any order? Write out the probabilities for all the possible outcomes, and check that your answers add up to 1. Hint: You can think of a series of wins and losses as a string of W's and L's.