Douglass Houghton Workshop, Section 2, Thu 04/02/26

Worksheet Quintessential

1. Last time we found that latitudes and longitudes can't be averaged to find a midpoint. This left us bitter and disillusioned, but undaunted. If we could just convert to (x, y, z) coordinates, then we'd be in business.

   Write ϕ for latitude and θ for longitude. Define the (x, y, z) coordinate system as:

   - The origin is at the center of the earth.
   - The radius of the earth has length 1.
   - The x-axis goes through the point $(\phi = 0, \theta = 90\text{W})$, near the Galápagos Islands.
   - The y axis goes through the point $(\phi = 0, \theta = 0)$, off the coast of Nigeria.
   - The z axis goes through the North Pole.

   (a) Find z in terms of ϕ and θ. (One of them doesn't matter.)

   (b) Now find x and y. Hint: the plane at latitude ϕ intersects the earth in a circle. Draw it on the board. What is its radius?

2. (Adapted from a Winter, 2010 exam problem)

   (a) Find the first four nonzero terms of the Taylor series for $\ln(1 + x)$ about $x = 0$.

   (b) Find the first three nonzero terms of the Taylor series for $g(x) = \ln\left(\frac{1+x}{1-x}\right)$ about $x = 0$. Hint: Rules of logarithms.

   (c) Find the exact value of the sum of the series $2\left(\frac{3}{4}\right) + \frac{2}{3}\left(\frac{3}{4}\right)^3 + \frac{2}{5}\left(\frac{3}{4}\right)^5 + \cdots$

3. It's an interesting idea to start with a sequence of numbers $a_0, a_1, a_2, \ldots$ and try to find a formula for the function with Taylor series $a_0 + a_1x + a_2x^2 + \cdots$. Consider the Fibonacci numbers:

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34

   where, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

   Suppose $f(x) = F_0 + F_1x + F_2x^2 + \cdots$. (It's called the *generating function* for the Fibonacci numbers.)

   (a) Write down the first 10 terms of the series for $f(x)$ and $xf(x)$.

   (b) What happens when you add those two together? Compare with $f(x)/x$.

   (c) Deduce a simple formula for $f(x)$.

4. Write down the Taylor series about $a = 0$ for the following functions, either from memory or by working them out.

(a) $e^x =$	(c) $\cos(x) =$	(e) $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) =$
(b) $e^{-x} =$	(d) $\sin(x) =$	(f) $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) =$

5. The symbol i is often used to represent $\sqrt{-1}$. *It is not a real number*, because of course any real number, when squared, is positive, but $i^2 = -1$. Just the same, it is often very useful (not just in math, but in physics and engineering) to form the set of **complex numbers**

$$\{x + iy : x \text{ and } y \text{ are real numbers}\}$$

and then try to do with complex numbers everything we're used to doing with real numbers. (Most things will work, some won't, and some will work better.)

   (a) We know that $i^2 = -1$, so $i^3 = i^2 \cdot i = (-1) \cdot i = -i$. Write down some more powers of i until you have a general formula for i^n.

   (b) Use the power series you found in the last problem above to find $\cosh(i\theta)$, where θ is a real number.

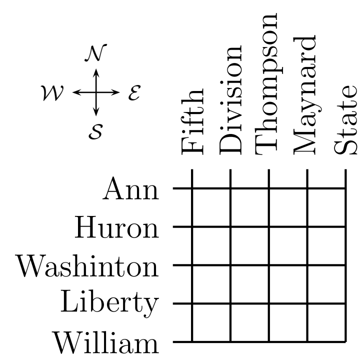
   (c) Find $\sinh(i\theta)$.

   (d) Add them together to get $e^{i\theta}$. Now you've defined what it means to take a number to an imaginary power!

   (e) Evaluate at $\theta = \pi$.

   (f) Admire your work, with wonder and amazement.

6. Last time we found that the number of ways to get from one point to another in the grid at right was an entry in Pascal's Triangle, as long as we only ever go east and south.



   (a) Explain how the number of ways to go, say, 3 blocks east and 4 blocks south is related to counting strings of E's and S's.

   (b) So how can we count the number of strings of a E's and b S's? You can use the notation $\binom{n}{k}$ for the kth entry in the nth row of Pascal's Triangle, where the top row is row 0 and the left column is column 0.

   (c) Now suppose we play 5 games of Roulette, betting on red each time. We know the probability of winning each game is 9/19. What is the probability we win all 5? What's the probability we win 4 and lose 1, in any order? Write out the probabilities for all the possible outcomes, and check that your answers add up to 1. Hint: You can think of a series of wins and losses as a string of W's and L's.

7. (From the Fall, 2013 Math 116 Final Exam) Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{2^n}{3n}(x-5)^n.$$

and rigorously justify your answer, explaining what convergence test(s) you use and how you used them.