Douglass Houghton Workshop, Section 2, Tue 03/31/26

Worksheet Prank

1. On April first, Katie & Hao like to play practical jokes on people. Their rule, of course, is that every practical joke done on them demands an equal and opposite retaliation. From year to year they keep a mental "balance sheet" that records how much grief they "owe" or are "owed" by their rivals. Due to various interventions by authority, they have been very good the last few years, and they currently "owe" 100 practical jokes.

   They decide that every year, they will pay off 20% of their "debt", by playing pranks on their friends. At the same time, their friends play 5 pranks on them each year.

   (a) What will the "balance" be at the end of 4/1/2026?

   (b) Fill in the table with the balance B_n at the end of $4/1/(2025+n)$.

n	0	1 (2026)	2 (2027)	3	4	5
B_n	100					

   (c) Find a formula for B_n in terms of n.

   (d) What happens in the long run? (Does the sequence $B_0, B_1, B_2, \ldots$ converge?)

2. There is nothing special at latitude 14°38′53″ N, longitude 78°6′28″ W. It's just a point in the ocean. But, if you were to shoot a neutrino from the middle of the Diag (latitude 42°16′36″ N, longitude 83°44′15″ W) to that point, through the earth's crust, its deepest point would be directly under a very interesting place. Find that place.

3. It's an interesting idea to start with a sequence of numbers $a_0, a_1, a_2, \ldots$ and try to find a formula for the function with Taylor series $a_0 + a_1x + a_2x^2 + \cdots$. Consider the Fibonacci numbers:

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34

   where, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

   Suppose $f(x) = F_0 + F_1x + F_2x^2 + \cdots$. (It's called the *generating function* for the Fibonacci numbers.)

   (a) Write down the first 10 terms of the series for $f(x)$ and $xf(x)$.

   (b) What happens when you add those two together? Compare with $f(x)/x$.

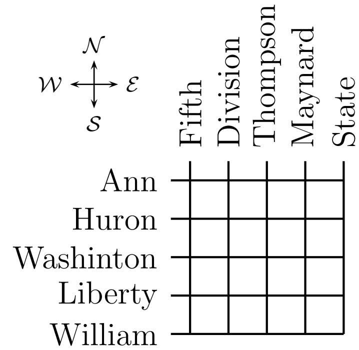
   (c) Deduce a simple formula for $f(x)$.

4. (From the Fall, 2013 Math 116 Final Exam) Find the interval of convergence of

   $$\sum_{n=1}^{\infty} \frac{2^n}{3n}(x-5)^n.$$

   and rigorously justify your answer, explaining what convergence test(s) you use and how you used them.

5. Last time we found that the number of ways to get from one point to another in the grid at right was an entry in Pascal's Triangle, as long as we only ever go east and south.



   (a) Explain how the number of ways to go, say, 3 blocks east and 4 blocks south is related to counting strings of E's and S's.

   (b) So how can we count the number of strings of a E's and b S's? You can use the notation $\binom{n}{k}$ for the kth entry in the nth row of Pascal's Triangle, where the top row is row 0 and the left column is column 0.

   (c) Now suppose we play 5 games of Roulette, betting on red each time. We know the probability of winning each game is 9/19. What is the probability we win all 5? What's the probability we win 4 and lose 1, in any order? Write out the probabilities for all the possible outcomes, and check that your answers add up to 1. Hint: You can think of a series of wins and losses as a string of W's and L's.

6. Write down the Taylor series about $a = 0$ for the following functions, either from memory or by working them out.

(a) $e^x =$	(c) $\cos(x) =$	(e) $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) =$
(b) $e^{-x} =$	(d) $\sin(x) =$	(f) $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) =$

7. The symbol i is often used to represent $\sqrt{-1}$. *It is not a real number*, because of course any real number, when squared, is positive, but $i^2 = -1$. Just the same, it is often very useful (not just in math, but in physics and engineering) to form the set of **complex numbers**

$$\{x + iy : x \text{ and } y \text{ are real numbers}\}$$

and then try to do with complex numbers everything we're used to doing with real numbers. (Most things will work, some won't, and some will work better.)

   (a) We know that $i^2 = -1$, so $i^3 = i^2 \cdot i = (-1) \cdot i = -i$. Write down some more powers of i until you have a general formula for i^n.

   (b) Use the power series you found in the last problem above to find $\cosh(i\theta)$, where θ is a real number.

   (c) Find $\sinh(i\theta)$.

   (d) Add them together to get $e^{i\theta}$. Now you've defined what it means to take a number to an imaginary power!

   (e) Evaluate at $\theta = \pi$.

   (f) Admire your work, with wonder and amazement.