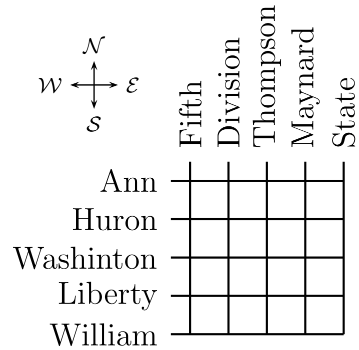
Douglass Houghton Workshop, Section 2, Thu 03/26/26

Worksheet One Ring to Rule Them All

1. A small section of downtown Ann Arbor is shown to the right. Copy the map onto the board.



   (a) Suppose Sally lives at the corner of Washington and Thompson, and she needs to get to class at Mason Hall, which is at State and William. She doesn't want to walk out of her way, so she will only go east and south. Still, she has some choices. How many ways are there to get to class?

   (b) Interesting, I wonder what that number means? Write your answer to part (a) at the corner of Washington and Thompson. Now pick a different starting corner, and figure out how many ways there are to get to class from there. Repeat, writing your answers on the board at the relevant corner.

   (c) What's the pattern?

   (d) Explain why the pattern must continue to hold, no matter how big the city is.

2. (Fall, 2007) Find the interval of convergence of $\displaystyle\sum_{n=3}^{\infty} \frac{(3-x)^{3n}}{8^n(n-2)}$.

3. There is nothing special at latitude 14°38′53″ N, longitude 78°6′28″ W. It's just a point in the ocean. But, if you were to shoot a neutrino from the middle of the Diag (latitude 42°16′36″ N, longitude 83°44′15″ W) to that point, through the earth's crust, its deepest point would be directly under a very interesting place. Find that place.

4. Last time it appeared we showed that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$. Crazy!

   (a) We got that series by considering derivatives and plugging in $x = 0$. See if you can deduce a series for $\cos(x)$ the same way, by starting with

$$\cos(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

   (b) Test it at $x = \pi$, by adding up all the terms through x^{10}. Is it close to what you expect it to be?

   (c) Do the same for $\sin(x)$.

   (d) Now systemetize the result: if we have a function $f(x)$ which has derivatives, how do we find its series? Find formulas for a_0, a_1, etc. in terms of f.

5. It's an interesting idea to start with a sequence of numbers $a_0, a_1, a_2, \ldots$ and try to find a formula for the function with Taylor series $a_0 + a_1 x + a_2 x^2 + \cdots$. Consider the Fibonacci numbers:

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34

where, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

Suppose $f(x) = F_0 + F_1 x + F_2 x^2 + \cdots$. (It's called the *generating function* for the Fibonacci numbers.)

(a) Write down the first 10 terms of the series for $f(x)$ and $xf(x)$.

(b) What happens when you add those two together? Compare with $f(x)/x$.

(c) Deduce a simple formula for $f(x)$.

6. For reasons unexplained, let's approximate the function $f(x) = x^2$ by its Fourier series.

(a) The b_n are all 0. Why?

(b) Find a_0.

(c) Find a_n for $n \geq 1$. Hint: Ladder Method.

7. We know by the integral test that $\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$ converges. But what does it converge to?

(a) Use your calculator to find the first dozen or so partial sums. Can you guess what the limit is? If you like, type in the calculator program on the right and let it run, to see how the partial sums change.

(b) Plug in $x = \pi$ to the result of the last problem and see if you can find $\zeta(2)$.

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0 → S
1 → N
Lbl 10
S+1/N² → S
N+1 → N
Disp S
Goto 10
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