Douglass Houghton Workshop, Section 2, Thu 03/12/26

Worksheet Never Gonna Give You Up

1. (Adapted from a Fall, 2003 Math 116 Exam)

   (a) Express the number

   $$.135135135\overline{135}$$

   as the sum of a geometric series.

   (b) Use the infinite geometric series formula to express that same number as a fraction in lowest terms.

2. We learned recently that

   $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

   as long as $|x| < 1$. You can think of both sides of the equation as *functions of* x, and so we have the suprising new idea that a common function we're familar with, $\frac{1}{1-x}$, can be expressed as an infinite series, for certain values of x.

   Maybe other functions we know and like can also be expressed as series? Suppose that:

   $$e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

   for some constants $a_0, a_1, a_2, \ldots$, for all x.

   (a) What must a_0 be? Hint: plug in 0 to both sides.

   (b) Take the derivative of both sides. Now deduce a_1.

   (c) Repeat to find $a_2, a_3, a_4, \ldots$.

   (d) Can it really be true?!? Try to test with the first 10 terms of the series and $x = 1$.

3. (This problem appeared on a Winter, 2003 Math 116 exam) Fred likes to juggle. So does Jason. The number of minutes Fred can juggle five balls without dropping one is a random variable, with probability density function $f(t) = 0.8e^{-0.8t}$. Similarly, the function $j(t) = 1.5e^{-1.5t}$ describes Jason's skill. Here t is time *in minutes*.

   (a) Find $\int_0^\infty f(t)\,dt$.

   (b) What percentage of Jason's juggling attempts are "embarrassing," meaning they last for 10 seconds or less?

   (c) How long can Fred juggle, on average?

   (d) Who is the better juggler? Give a good reason for your decision.

Recall that for a function of the form

$$f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots \\ + b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \cdots \tag{1}$$

integrating against a sine or cosine function makes almost all the terms 0, so for $n \geq 1$:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

Writing a function in the form of Equation (1) is called finding the **Fourier Series** of the function.

4. Let's compute the Fourier series for $f(x) = x^2$.
   (a) Compute a_0.
   (b) Fill in the table to the right.
   (c) Find the a_n and b_n for $f(x) = x^2$.

n	1	2	3	4	5	6
$\sin(n\pi)$						
$\sin(-n\pi)$						
$\cos(n\pi)$						
$\cos(-n\pi)$						

5. Between 1980 and 2009, there were 7015 Division I basketball games which were tied at the end of regulation play, resulting in at least one overtime period. Suppose p is the probability that an overtime period ends in a tie.
   (a) How many 2nd overtimes do you expect there were? How many 3rd overtimes? How many kth overtimes?
   (b) Write a sum (in terms of p) that's equal to the expected total number of overtimes. Then find a closed form for your sum.
   (c) There were in fact 8568 total overtime periods between 1980 and 2009. Use that to estimate p.
   (d) About how many games went 6 or more overtimes, do you guess?

6. (This problem appeared on a Winter, 2009 Math 116 exam) The quantity

$$\int_1^{\infty} \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$$

roughly models the resistance that football-shaped plankton encounter when falling through water. Note that $a = 1$, $b = 2$, and $c = 3$ are constants that describe the dimensions of the plankton. Find a value of M for which

$$\int_1^{M} \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$$

differs from the original model of resistance by at most 0.001. Hint: make use of the integral and the comparison test.