

Worksheet Magnificent

1. Consider the “Hard Eight” bet in craps. The bet wins on double fours ($\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$) and loses on “soft eight” ($\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$ or $\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$) and on 7. If something other than a 7 or 8 is rolled, the bet stays through the next roll.

- (a) Draw the addition table below on the board and fill it in.

+	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

- (b) Calculate these probabilities:

- W = the probability of winning on the first roll.
- L = the probability of losing on the first roll.
- C = the probability that the game continues to a second roll.

- (c) Calculate the probability of winning on the *second* roll.

- (d) Calculate the probability of winning on the k th roll.

- (e) Calculate the probability of winning on *one of* the first n rolls.

- (f) Calculate the probability of winning the hard-eight bet.

2. (From the Winter, 2011 Math 116 final exam) For $n \geq 1$, identify the properties of the four sequences below.

n th term in the sequence	Bounded?	Increasing?	Converges?
$a_n = (-1)^n + \frac{1}{n}$			
$b_n = 1 + \frac{(-1)^n}{n}$			
$c_n = \left(\frac{6}{5}\right)^n$			
$s_n = \sum_{k=1}^n \frac{1}{k^2}$			

3. Write out the terms of each series. 4. Evaluate the following:

(a) $\sum_{i=0}^6 i$

(a) $\sum_{i=1}^{100} 2$

(b) $\sum_{k=3}^{10} (-1)^k k^2$

(b) $\sum_{i=1}^{100} 1/(i+3) - 1/(i+4)$

(c) $\sum_{\ell=1}^5 x^{2\ell-1} \ell!$

(c) $\sum_{i=1}^{\infty} 1/(i(i+1))$
(Hint: $(1/i) - (1/(i+1)) = ?$)

(d) $\sum_{m=0}^{\infty} \frac{(2m)!}{m!m!} x^m$

(d) $1/6 + 1/12 + 1/20 + 1/30 + 1/42 + \dots$

5. (Fall, 2015) Determine whether the following improper integrals converge or diverge.

(a) $\int_1^{\infty} \frac{\ln x}{x^{3/2}} dx$

(b) $\int_0^1 \frac{x}{x^2 + x^{3/2}} dx$

6. (This problem appeared on a Winter, 2003 Math 116 exam) Fred likes to juggle. So does Jason. The number of minutes Fred can juggle five balls without dropping one is a random variable, with probability density function $f(t) = 0.8e^{-0.8t}$. Similarly, the function $j(t) = 1.5e^{-1.5t}$ describes Jason's skill. Here t is time *in minutes*.

- Find $\int_0^\infty f(t) dt$.
- What percentage of Jason's juggling attempts are "embarrassing," meaning they last for 10 seconds or less?
- How long can Fred juggle, on average?
- Who is the better juggler? Give a good reason for your decision.

7. (This problem appeared on a Winter, 2009 Math 116 exam) The quantity

$$\int_1^\infty \frac{dx}{\sqrt{(a^2 + x)(b^2 + x)(c^2 + x)}}$$

roughly models the resistance that football-shaped plankton encounter when falling through water. Note that $a = 1$, $b = 2$, and $c = 3$ are constants that describe the dimensions of the plankton. Find a value of M for which

$$\int_1^M \frac{dx}{\sqrt{(a^2 + x)(b^2 + x)(c^2 + x)}}$$

differs from the original model of resistance by at most 0.001. Hint: make use of the integral and the comparison test.

Recall that for a function of the form

$$(1) \quad f(x) = a_0 + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) + \cdots \\ + b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + \cdots$$

integrating against a sine or cosine function makes almost all the terms 0, so for $n \geq 1$:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx.$$

Writing a function in the form of Equation (??) is called finding the **Fourier Series** of the function.

8. Let's compute the Fourier series for $f(x) = x^2$.

- Compute a_0 .
- Fill in the table to the right.
- Find the a_n and b_n for $f(x) = x^2$.

n	1	2	3	4	5	6
$\sin(n\pi)$						
$\sin(-n\pi)$						
$\cos(n\pi)$						
$\cos(-n\pi)$						