Douglass Houghton Workshop, Section 2, Tue 02/24/26

Worksheet Karma

With a lot of hard work, we filled the table to the right with the values of $\int_{-\pi}^{\pi} f(x)g(x)\,dx$, where f is the row and g is the column, and m and n are positive integers.

	1	$\sin(nx)$	$\cos(nx)$
1	2π	0	0
$\sin(mx)$	0	$\begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$	0
$\cos(mx)$	0	0	$\begin{cases} \pi & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$

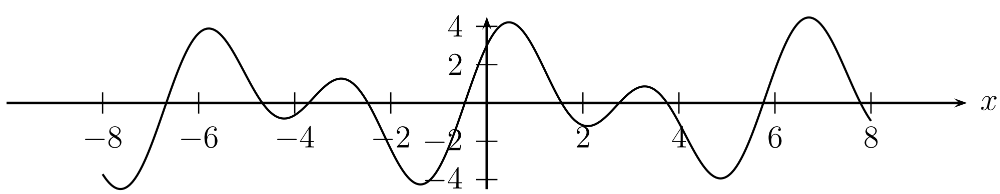
The implication was that for a function of the form

$$\begin{aligned} f(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) \; + \; \cdots \\ + \; b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) \; + \; \cdots \end{aligned} \tag{1}$$

integrating against a sine or cosine function makes almost all the terms 0, so for $n \geq 1$:

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0, \quad \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \pi a_n, \quad \text{and} \quad \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \pi b_n.$$

1. Generalize: Suppose that you have a function with the form of Equation (1), but you don't know the coefficients. You can, however, find integrals like the ones above.
   - How can you find a_1, the coefficient of $\cos(x)$?
   - How can you find a_n and b_n?

2. 

Suppose $h(x)$ is some function you measure in nature, and its graph looks like the one above. You do some numerical integration and discover that

$$\int_{-\pi}^{\pi} h(x)\,dx = 0 \qquad \int_{-\pi}^{\pi} h(x)\cos(2x)\,dx = 6.28$$

$$\int_{-\pi}^{\pi} h(x)\cos(x)\,dx = 3.14 \qquad \int_{-\pi}^{\pi} h(x)\sin(2x)\,dx = 4.71$$

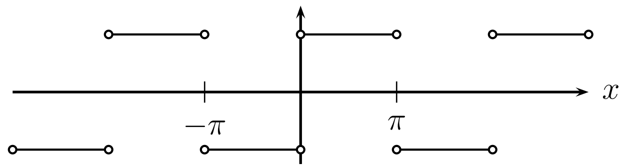
$$\int_{-\pi}^{\pi} h(x)\sin(x)\,dx = 6.28 \qquad \int_{-\pi}^{\pi} h(x)\cos(nx)\,dx = \int_{-\pi}^{\pi} h(x)\sin(nx)\,dx = 0 \text{ for } n \geq 3.$$

Can you guess a formula for $h(x)$? Use what you know, and check by graphing your formula on a calculator.

3. Consider the square wave:

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } \quad 0 < x < \pi \end{cases}$$

and that pattern is repeated every 2π.



Suppose the square wave can be written in terms of sines and cosines, as in Equation (1) above. Find the a_n and the b_n.

4. Let's find $\int \sec(\theta)\, d\theta$. Here's an idea:

$$\int \sec\theta\, d\theta = \int \frac{1}{\cos\theta}\, d\theta = \int \frac{\cos\theta}{\cos^2\theta}\, d\theta = \int \frac{\cos\theta}{1-\sin^2\theta}\, d\theta.$$

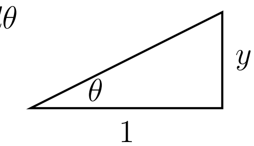
(a) Check and see if you agree with all the steps in the line above. Then do a substitution on the last integral to make it better.

(b) Now solve.

5. We are very close to finding the shape of a hanging chain. We let y be the slope of the chain, and by separating the variables of our differential equation and doing a trig substitution we found that

$$ax = \int \sec\theta\, d\theta$$

where a is some constant and $\theta = \tan^{-1} y$.



(a) Do the previous problem, then plug the result into the equation above to get an equation relating θ and x.

(b) Reverse the substitution, so you have an equation relating y and x. The triangle above might be useful.

(c) What is y when $x = 0$? Use that to resolve your constant of integration.

(d) Solve for y in terms of x! Simplify as much as possible!

(e) Remember that y is the *slope* of the chain, so do one more integral to get the shape of the chain.

6. Determine whether the following improper integrals converge or diverge.

(a) $\displaystyle\int_1^\infty \frac{1}{x+e^x}\, dx$ (b) $\displaystyle\int_1^e \frac{1}{x(\ln(x))^2}\, dx$