Douglass Houghton Workshop, Section 2, Tue 01/27/26

Worksheet Et tu, Brute?

1. Suppose you want to calculate

$$I = \int \frac{2x+6}{(x-3)(x-7)}\,dx.$$

   (a) Hmm, that looks hard. But what if you had something like this:

$$\int \left(\frac{A}{x-3} + \frac{B}{x-7}\right) dx$$

   where A and B are constants. You could find that, right? Do it, and get an answer in terms of A and B.

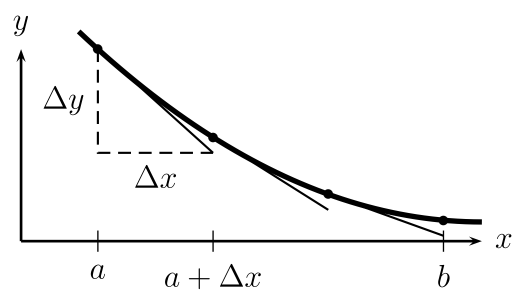
   (b) That wasn't so bad. And it's related to the initial problem. What do you get if you add the two fractions in part (a), by finding a common denominator?

   (c) Now, what do A and B have to be to make the answer to part (b) equal the integrand of I?

   (d) Now find I.

   (e) When does this trick work?

2. How can we compute the length of a curve $y = f(x)$? Consider cutting it up into small pieces, and approximating each piece with a line segment, as in the picture below.



   (a) How long is the first piece? It is tangent to the curve at a.

   (b) How long is the ith piece?

   (c) Write the left-hand Riemann sum for the length of the curve from a to b.

   (d) Now make it into an integral, which will be our formula for arc length.

3. Consider the **gamma function**: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt$, for $x > 0$.

   (a) Use integration by parts to prove that $\Gamma(x+1) = x\Gamma(x)$.

   (b) Show that $\Gamma(1) = 1$. Then fill in this chart, using part (a):

x	1	2	3	4	5	6
$\Gamma(x)$						

   (c) So if x is a positive integer, what is $\Gamma(x)$?

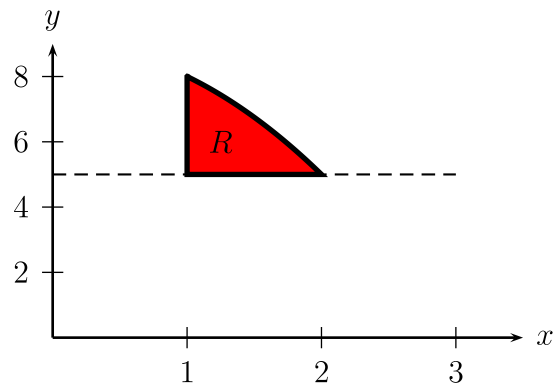
4. Evaluate $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx$ where m and n are positive integers. (You might want to graph a few examples.)

5. Find the area of the finite region that is bounded by the y-axis, the line $y = 1$, and the graph of $y = x^{1/4}$ in two ways:

   (a) By integrating with respect to x and

   (b) By writing x as a function of y and integrating with respect to y.

6. (Adapted from a Fall, 2011 Math 116 Exam) Consider the region R in the xy-plane bounded by the curves $y = 9 - x^2$, $x = 1$, and $y = 5$. This region is pictured below.

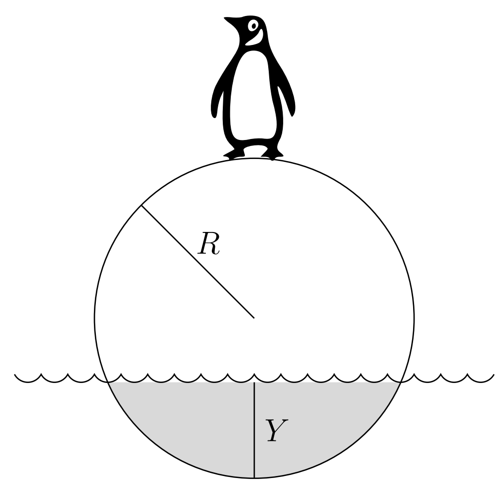


   (a) Find the area of R.

   (b) Find the volume obtained by rotating R about the y-axis. Do it with both shells and washers, and verify that the answer is the same.

   (c) Find the volume of the solid whose base is R and whose cross-sections perpendicular to the x-axis are squares.

7. The buoyancy force on a floating object is proportional to the volume of water it displaces. Using this fact, whimsical ecologists plan to study the weights of penguins by floating beachballs on the ocean and enticing penguins to climb on top. They then measure the depth the ball sinks to, and thereby deduce the penguin's weight.



   So given a beachball of radius R that is partially submerged in the water, find a formula for the volume of the ball which is below the water line when its bottom is at depth Y. Check that your formula makes sense for the values $Y = 0$, $Y = R$, and $Y = 2R$.

8. A ball at an initial height h_0 is thrown straight up into the air, with an initial velocity v_0. Gravity causes the ball to accelerate downward at a constant rate, g. (This might be on another planet, so use g rather than $9.8\,\text{m/sec}^2$.)

   (a) Find $v(t)$, the upward velocity of the ball at time t.

   (b) Find $h(t)$, the height of the ball at time t.

   (c) Calculate the quantity $mgh(t) + \frac{1}{2}mv(t)^2$. What do you notice about your answer?

   (d) Use part (c) to calculate the maximum height of the ball. Check using your Math 115 optimization skillz.