Douglass Houghton Workshop, Section 1, Wed 04/08/26

Worksheet Pain and pleasure, like light and darkness, succeed each other

1. Last time we found a remarkable power series:

$$\frac{x}{1-x-x^2} = x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+21x^8+34x^9+55x^{10}+\cdots = \sum_{n=0}^{\infty} F_n x^n$$

where F_n is the nth Fibonacci number, defined by $F_n = F_{n-1} + F_{n-2}$. That's already a pretty cool result. But if we play our cards right, we can parlay it into a non-recursive formula for the Fibonacci numbers.

(a) If a is a constant, what is the power series for $\frac{1}{1-ax}$ about $x = 0$?

(b) Verify that if $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ then $(1-\alpha x)(1-\beta x) = 1 - x - x^2$.

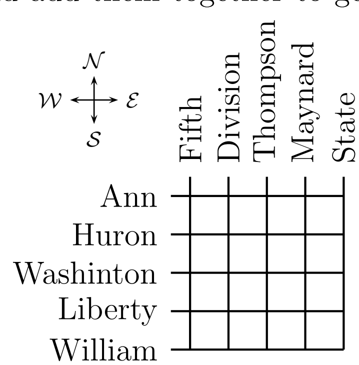
(c) Now suppose we could split the generating function above like this:

$$\frac{x}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

for some constants A and B. Find what A and B must be to make the equation above work for all values of x.

(d) Now find the series for $\frac{A}{1-\alpha x}$ and $\frac{B}{1-\beta x}$, in Σ form, and add them together to get a formula for the Fibonacci numbers.

2. Last time we found that the number of ways to get from one point to another in the grid at right was an entry in Pascal's Triangle, as long as we only ever go east and south.

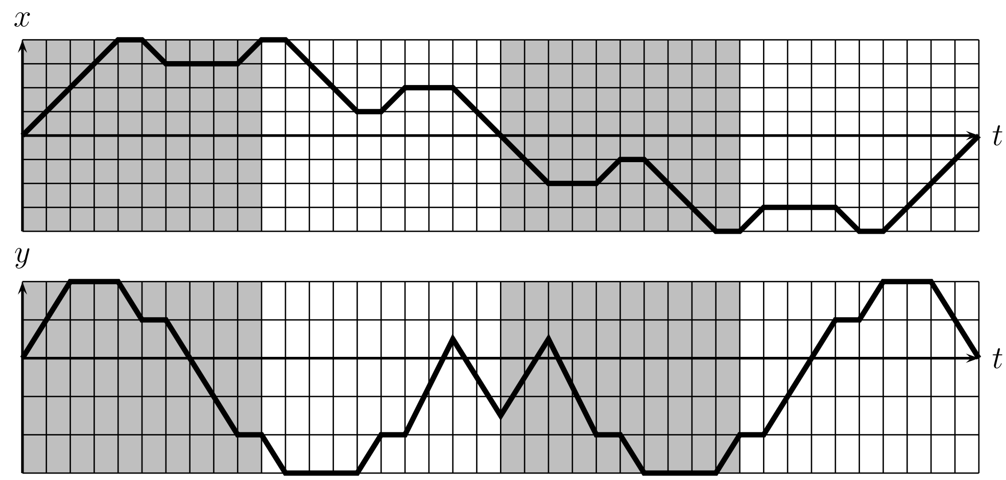


(a) Explain how the number of ways to go, say, 3 blocks east and 4 blocks south is related to counting strings of E's and S's.

(b) So how can we count the number of strings of a E's and b S's? You can use the notation $\binom{n}{k}$ for the kth entry in the nth row of Pascal's Triangle, where the top row is row 0 and the left column is column 0.

(c) Now suppose we play 5 games of Roulette, betting on red each time. We know the probability of winning each game is 9/19. What is the probability we win all 5? What's the probability we win 4 and lose 1, in any order? Write out the probabilities for all the possible outcomes, and check that your answers add up to 1. Hint: You can think of a series of wins and losses as a string of W's and L's.

3. Plot the positions of (x, y) given the graphs of $x(t)$ and $y(t)$ below:



4. (From a Winter, 2014 Math 116 Exam Problem)

   (a) What is the value of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{n!}$?

   (b) What is the value of $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}(-1)^n}{(2n+1)!}$?

   (c) Suppose that $1 + x - \frac{1}{4}x^2 + \frac{1}{10}x^3$ is the third-degree Taylor polynomial for a function $f(x)$. What must the graph of f look like near $x = 0$?

   (d) What is the Taylor series of $2xe^{x^2}$ centered at $x = 0$?

   (e) What is the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x+5)^n 5^{-n}}{n+5}$?

5. Consider an epidemic. On each day t, divide the population into three types:

$$S(t) = \text{The fraction of people who are susceptible to infection,}$$
$$I(t) = \text{The fraction of people currently infected, and}$$
$$R(t) = \text{The fraction of people who have been removed from the pool.}$$

(People are removed from the pool after they have the disease and can no longer become infected.) Suppose that the probability that the disease is transmitted during a single interaction between an infected and susceptible person is .01 (1%).

   (a) Suppose Bob is a susceptible person, and on day t he interacts with n other people, chosen randomly. About how many *infected* people did he interact with?

   (b) What's the probability Bob was NOT infected the first time he met an infected person AND not infected the second time he met an infected person? Assume the interactions are independent.

   (c) What's the probability that Bob wasn't infected by any of his interactions?

   (d) What's the last answer if the probability of transmission is p instead of .01?