Douglass Houghton Workshop, Section 1, Wed 04/01/26

Worksheet Never Was So Much Owed By So Many To So Few

1. On April first, Annie & Riley like to play practical jokes on people. Their rule, of course, is that every practical joke done on them demands an equal and opposite retaliation. From year to year they keep a mental "balance sheet" that records how much grief they "owe" or are "owed" by their rivals. Due to various interventions by authority, they have been very good the last few years, and they currently "owe" 100 practical jokes.

   They decide that every year, they will pay off 20% of their "debt", by playing pranks on their friends. At the same time, their friends play 5 pranks on them each year.

   (a) What will the "balance" be at the end of 4/1/2026?

   (b) Fill in the table with the balance B_n at the end of $4/1/(2025+n)$.

n	0	1 (2026)	2 (2027)	3	4	5
B_n	100					

   (c) Find a formula for B_n in terms of n.

   (d) What happens in the long run? (Does the sequence $B_0, B_1, B_2, \ldots$ converge?)

2. Last time we found that latitudes and longitudes can't be averaged to find a midpoint. This left us bitter and disillusioned, but undaunted. If we could just convert to (x, y, z) coordinates, then we'd be in business.

   Write ϕ for latitude and θ for longitude. Define the (x, y, z) coordinate system as:

   - The origin is at the center of the earth.
   - The radius of the earth has length 1.
   - The x-axis goes through the point $(\phi = 0, \theta = 90\text{W})$, near the Galápagos Islands.
   - The y axis goes through the point $(\phi = 0, \theta = 0)$, off the coast of Nigeria.
   - The z axis goes through the North Pole.

   (a) Find z in terms of ϕ and θ. (One of them doesn't matter.)

   (b) Now find x and y. Hint: the plane at latitude ϕ intersects the earth in a circle. Draw it on the board. What is its radius?

3. Write down the Taylor series about $a = 0$ for the following functions, either from memory or by working them out.

   (a) $e^x =$

   (b) $e^{-x} =$

   (c) $\cos(x) =$

   (d) $\sin(x) =$

   (e) $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) =$

   (f) $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) =$

4. It's an interesting idea to start with a sequence of numbers $a_0, a_1, a_2, \ldots$ and try to find a formula for the function with Taylor series $a_0 + a_1 x + a_2 x^2 + \cdots$. Consider the Fibonacci numbers:

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34

where, for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

Suppose $f(x) = F_0 + F_1 x + F_2 x^2 + \cdots$. (It's called the *generating function* for the Fibonacci numbers.)

(a) Write down the first 10 terms of the series for $f(x)$ and $xf(x)$.

(b) What happens when you add those two together? Compare with $f(x)/x$.

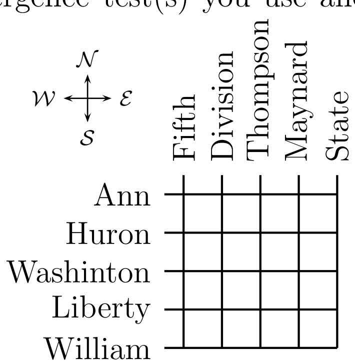
(c) Deduce a simple formula for $f(x)$.

5. (From the Fall, 2013 Math 116 Final Exam) Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{2^n}{3n}(x-5)^n.$$

and rigorously justify your answer, explaining what convergence test(s) you use and how you used them.

6. Last time we found that the number of ways to get from one point to another in the grid at right was an entry in Pascal's Triangle, as long as we only ever go east and south.



(a) Explain how the number of ways to go, say, 3 blocks east and 4 blocks south is related to counting strings of E's and S's.

(b) So how can we count the number of strings of a E's and b S's? You can use the notation $\binom{n}{k}$ for the kth entry in the nth row of Pascal's Triangle, where the top row is row 0 and the left column is column 0.

(c) Now suppose we play 5 games of Roulette, betting on red each time. We know the probability of winning each game is 9/19. What is the probability we win all 5? What's the probability we win 4 and lose 1, in any order? Write out the probabilities for all the possible outcomes, and check that your answers add up to 1. Hint: You can think of a series of wins and losses as a string of W's and L's.