Douglass Houghton Workshop, Section 1, Mon 02/23/26

Worksheet If You Tickle Us, Do We Not Laugh?

1. We've done a few integrals with sines and cosines. Fill in the table on the right with the values of $\int_{-\pi}^{\pi} f(x)g(x)\,dx$, where f is the row and g is the column.

$f \setminus g$	1	$\sin(nx)$	$\cos(nx)$
1			
$\sin(mx)$			
$\cos(mx)$			

2. Let $h(x) = 5 + 2\cos(x) + \sin(x) - 5\cos(2x) + 3\sin(2x)$.

   (a) Use your calculator to compute:

$$\int_{-\pi}^{\pi} h(x)\,dx =$$

$$\int_{-\pi}^{\pi} h(x)\cos(x)\,dx =$$

$$\int_{-\pi}^{\pi} h(x)\sin(x)\,dx =$$

$$\int_{-\pi}^{\pi} h(x)\cos(2x)\,dx =$$

$$\int_{-\pi}^{\pi} h(x)\sin(2x)\,dx =$$

   (b) Explain the results using the table above.

3. Predict what the integrals in (2a) above will be if we change $h(x)$ to

$$h(x) = 2 + 3\cos(x) - 7\sin(x) - 4\cos(2x) + \sin(2x).$$

4. Generalize: What will those integrals be if

$$(\star) \qquad h(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x).$$

5. Suppose that you have a function with the form of Equation ($\star$), but you don't know the coefficients. You can, however, find integrals like the ones above.

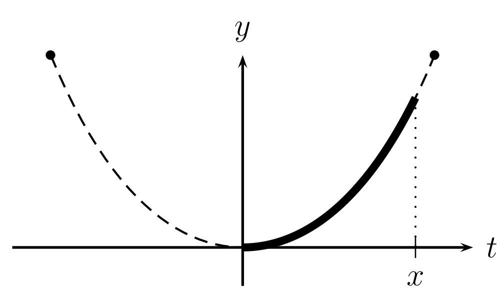
   (a) How can you find a_1, the coefficient of $\cos(x)$?

   (b) How can you find a_n and b_n?

6. We've made some progress finding the shape of a hanging chain. If the shape is given by $F(x)$, then by considering forces and arc length we've shown that

$$F''(x) = \frac{\delta g}{T_0}\sqrt{1 + F'(x)^2}$$

where T_0 is the tension at the bottom of the chain, δ is the mass density of the chain, and g is acceleration due to gravity (all constants). Where to go from here? We'd like to find a formula for $F(x)$.



(a) Hmmm. No Fs, only F's. And lots of constants. Let $y = F'(x)$, and put all the constants together into one constant. That should make it look better.

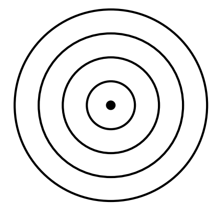
(b) What is y when x is 0? Now you have an initial value to go with your differential equation.

(c) Separate the variables and solve the differential equation.

7. Prove whether the following improper integrals converge or diverge.

(a) $\displaystyle\int_3^\infty \frac{\ln(x)}{x^2}\,dx$ (b) $\displaystyle\int_0^\infty \frac{3}{4x^2 + 5\sqrt{x}}\,dx$

8. Consider a game of "continuous darts." The board is circular, as you expect, with radius 1. The goal is to get as close to the middle as possible. If a dart lands a distance r from the bullseye, its score is $1 - r$. (So every number between 0 and 1 is a possible score.)



A novice player throws a dart which lands randomly somewhere on the board. That means that for any region R on the board,

$$\boxed{\text{Prob(dart lands in } R) = \frac{\text{area of } R}{\text{area of board}}}.$$

(a) Fill in the table with the probabilities that the dart scores **below** the given value.

x	0	1/4	1/2	3/4	1
Prob(score $< x$)					

(b) Let x be any number. Find the probability that the score is less than x.

(c) Find the median score.

9. (This problem appeared on a Winter, 2009 Math 116 exam) The quantity

$$\int_1^\infty \frac{dx}{\sqrt{(a^2 + x)(b^2 + x)(c^2 + x)}}$$

roughly models the resistance that football-shaped plankton encounter when falling through water. Note that $a = 1$, $b = 2$, and $c = 3$ are constants that describe the dimensions of the plankton. Find a value of M for which

$$\int_1^M \frac{dx}{\sqrt{(a^2 + x)(b^2 + x)(c^2 + x)}}$$

differs from the original model of resistance by at most 0.001. Hint: make use of the integral and the comparison test.