Douglass Houghton Workshop, Section 1, Wed 02/18/26

Worksheet How far that little candle throws its beams!

So shines a good deed in a naughty world.

1. Recall that:

$$\begin{aligned}\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y)\\ \sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y)\\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y)\\ \cos(x-y) &= \cos(x)\cos(y) + \sin(x)\sin(y)\end{aligned}$$

   (a) Use the first two identities to get $\sin(x)\cos(y)$ in terms of $\sin(x+y)$ and $\sin(x-y)$.

   (b) Use the next two identities to get $\cos(x)\cos(y)$ in terms of $\cos(x+y)$ and $\cos(x-y)$.

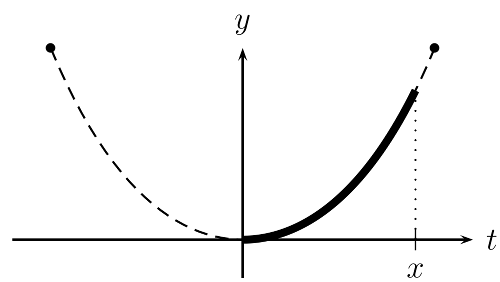
   (c) Do the same for $\sin(x)\sin(y)$.

2. Using the results of the last problem, and assuming m and n are positive integers, find

   (a) $\displaystyle\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx$ (b) $\displaystyle\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx$

3. In our quest to determine the shape of a hanging chain, we have found that the forces on a portion of the chain obey a certain relationship: if $m(x)$ is the mass of the chain between the middle and position x, T_0 is the tension in the chain at the bottom, and $y = F(x)$ is the shape of the chain, then in order to make the forces balance we must have:

$$\frac{m(x)g}{T_0} = F'(x).$$



   (a) How could you calculate $m(x)$ if you knew $F(x)$?

   (b) Some of that we know how to do. Use it to modify the equation above. Feel free to combine unknown constants into one, and simplify as much as possible.

4. A spaceship seeks to reach a height H above the surface of the earth. The force of gravity at any time is

$$\boxed{F_g = G\frac{Mm}{r^2}}$$

G = The universal gravitational constant
M = The mass of the earth
m = The mass of the spaceship
r = The distance from the spaceship to the center of the earth.

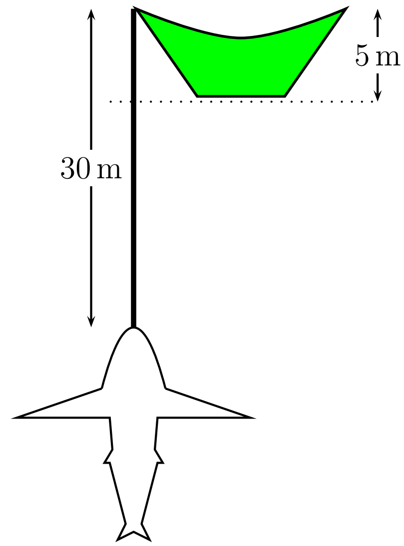
   (a) How much work will it take to raise the spaceship from the surface of the earth to a point H meters above the surface? Use R for the radius of the earth. Don't assume gravity is constant as the ship moves upward!

   (b) How much work would it take to push the spaceship entriely beyond the reach of Earth's gravity? (Let $H \to \infty$.)

(c) If the ship is travelling at velocity v, it will have kinetic energy $\frac{1}{2}mv^2$. That energy will be converted into work to move the ship upward. What speed must the ship be going near the surface to leave the earth's gravity well? This is the earth's *escape velocity*.

(d) Look up the values of G, M, and R, and get a numerical answer in miles per second.

5. Sofi, working as a marine scientist, is reeling in a large shark she caught onto her boat. The edge of her boat lies 5 meters above the water as shown in the figure below. The total length of the sharking line is 30 meters. The shark weighs 500 newtons in water, and her sharking line weighs 30 newtons per meter out of water, and 10 newtons per meter in water. The figure below depicts this situation - the sharking line is the thick dark line and the boat is shaded. Write an expression which gives the work Sofi does pulling the shark's snout to the surface of the water.



6. We have a new tool for evaluating limits, called *L'Hôpital's Rule*. It allows us to resolve some limits we couldn't do last fall. In particular, we can prove our favorite result, about Michael Phelps's towel.

(a) Recall our model of toweling: water spreads out evenly over Michael and the towel. So if Michael has area $1\,\text{m}^2$ and starts with wetness W, after using a towel of size T his new wetness will be

$$W\left(\frac{1}{1+T}\right).$$

Suppose instead he divides his towel of size T into n parts. How wet will he be if he starts with 1 liter of water on him?

(b) What happens if he divides the towel into more and more pieces? Let

$$L = \lim_{n\to\infty}(\text{the formula you found in part (6a)}).$$

Take the ln of both sides of the equation above. It's OK to move the ln inside the limit, because ln is a continuous function.

(c) Let $p = 1/n$. As $n \to \infty$, $p \to 0$. So rewrite your limit with p's instead of n's.

(d) In order to use L'Hôpital's Rule, you need to have something of the form $0/0$ or ∞/∞. So get you limit in that form, and resolve it.