Douglass Houghton Workshop, Section 2, Thu 10/09/25

Worksheet Magnificent

1. (This problem appeared on a Winter, 2009 Math 115 Exam) Suppose a is a positive (non-zero) constant, and consider the function

$$f(x) = \frac{1}{3}x^3 - 4a^2x.$$

Determine all maxima and minima of f in the interval $[-3a, 5a]$. For each, specify whether it is global or local.

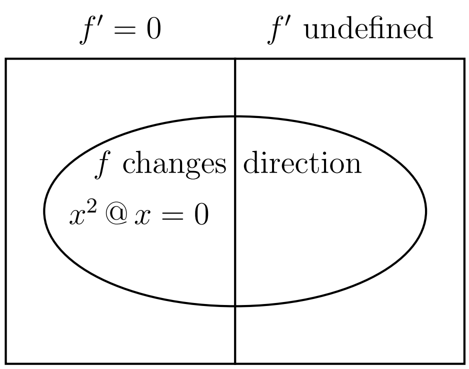
2. Suppose $h(x)$ is a continuous function defined for all real numbers x. The derivative and second derivative of $h(x)$ are given by

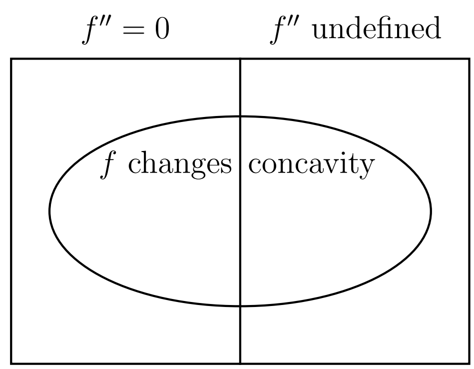
$$h'(x) = \frac{2x}{3(x^2-1)^{2/3}} \qquad \text{and} \qquad h''(x) = -\frac{2(x^2+3)}{9(x^2-1)^{5/3}}.$$

   (a) Find the all critical points and local extrema of $h(x)$. Use calculus to classify the critical points and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema.

   (b) Find all inflection points of $h(x)$, and justify that you've found them all.

3. The diagrams below each have 4 regions, representing different ways a function can behave at a point. In each region write an example of a function and a point that meets the criteria. For example, in the intersection of "$f' = 0$" and "f changes direction", we have x^2 @ $x = 0$, because the derivative of x^2 is indeed 0 at $x = 0$, and the function switches from decreasing to increasing there.





4. (This problem appeared on a Fall, 2005 Math 115 Final Exam) Using techniques from calculus, find the dimensions which will maximize the surface area of a circular cylinder whose height h and radius r, each in centimeters, are related by

$$h = 8 - \frac{r^2}{3}.$$

5. Molecules absorb far-infrared radiation because its excites their rotation. The absorption coefficient a of a given liquid varies with the frequency ω of the radiation according to

$$a(\omega) = \frac{10}{\omega^2 - 2c\omega + 125}$$

where c is some constant $(0 \leq c \leq 11)$.

   (a) For what value of the frequency ω is the absorption a maximum?

   (b) Graph $a(\omega)$ for $c = 11$. How would you describe the shape of this graph?

   [Note: with appropriate parameters this function describes the shapes of the lines in many kinds of spectroscopy].

6. (Adapted from a Fall, 2016 Math 116 exam problem) As a future software engineer, Drew spends many hours every day writing code. Let $w(t)$ be a function that models the number of lines of code that Drew writes in a day if he works t hours that day. Drew works at least one hour and at most 18 hours each day. A formula for $w(t)$ is given by

$$w(t) = \begin{cases} -2t^2 + 28t & \text{if } 1 \leq t \leq 3 \\ -0.5t^2 + 9t + 43.5 & \text{if } 3 < t \leq 18. \end{cases}$$

   (a) Find the values of t that minimize and maximize $w(t)$ on the interval $[1, 18]$.

   (b) What is the largest number of lines of code that Drew can expect to write in a day according to this model?

7. Let's find the derivative of $f(x) = \ln(x)$.

   (a) Write down the limit definition of the derivative of $f(x) = \ln(x)$.

   (b) Use the rules of logs to combine terms so that you have something that looks like

$$\lim_{h \to 0} \ln(\text{something})$$

   (c) Let's limit ourselves to the right-hand limit, $h \to 0^+$. (The other side is a little different, but no harder, so we'll skip it.) Let $n = \frac{1}{h}$. Then as $h \to 0^+$, $n \to \infty$. So modify your expression so it's all in terms of n instead of h.

   (d) Now move the limit inside the ln, which is OK since ln is continuous.

   (e) This is starting to look like our old friend, the Michael Phelps limit:

$$\lim_{n \to \infty} \left(1 + \frac{T}{n}\right)^n = e^T$$

   Use that to finish the proof and find the derivative of $\ln(x)$.