Douglass Houghton Workshop, Section 2, Thu 09/25/25

Worksheet Journey into Night

1. Use the definition of the derivative to find $f'(x)$ when $f(x) = \sqrt{x}$. Hint: $(a-b)(a+b) = a^2 - b^2$.

2. Alan has recently joined the UM Archery Club. One day he is shooting at a target on the edge of a cliff, and a momentary distraction causes the arrow to miss the target and fly out over the cliff.

   (a) The arrow's downward acceleration due to gravity is $-32\,\text{ft/sec}^2$. So what is its velocity t seconds after it is dropped? (Keep in mind that the derivative of velocity is acceleration, and the initial velocity is 0.)

   (b) How far has the arrow fallen after t seconds?

   (c) Suppose the arrow takes 3.5 seconds to hit the bottom. How high is the cliff?

   (d) Explain a general method for finding the height of any cliff.

3. Suppose you construct a $1/z$ scale model of the White House, in order to film it blowing up. You will show the film at 24 frames per second. How many frames per second should you *film* so that when you slow the speed down, things will fall at believable speeds?

4. Consider a mirror in the shape of the graph of $y = \pm\sqrt{4x}$.

   (a) Draw the mirror (make it big). What shape is it?

   (b) Draw a light ray travelling leftward along the line $y = -b$, where b is some positive number (making $-b$ negative). At what point P does the ray hit the mirror?

   (c) Find, in terms of b, the slope of the tangent to the mirror at P.

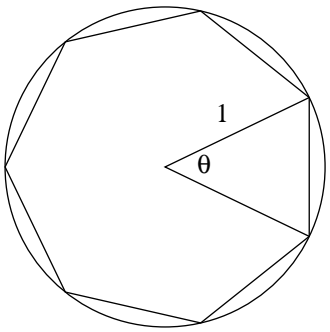
   (d) The *normal* to a curve at a point is the line through that point which is perpendicular to the tangent line. Find the slope of the normal to the mirror at P, and draw both the normal and tangent lines on your graph.

   (e) Suppose a line makes an angle θ with the positive x-axis. What is the slope of the line?

   (f) Let θ be the angle the normal to the mirror at P makes with the light ray $y = -b$. Can you write θ in terms of b? Hint: Use (4d) and (4e).

To be continued...

5. (This problem explains why $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$, but only when θ is measured in radians.) Consider a regular n-sided polygon inscribed in a circle of radius 1.



(a) Let A_n be the area of the polygon. What does A_n approach as n gets large? $\lim_{n\to\infty} A_n =$ ☐

(b) We can compute A_n by dividing the polygon up into triangles which have a vertex at the center. Let θ be the vertex angle (in radians). What is θ in terms of n?

(c) What happens to θ as n gets large?

(d) What is the area of one of the triangles, in terms of θ?

(e) What is A_n in terms of θ?

(f) Substitute into the equation from part (a) so that it includes θ's but not n's. Simplify it as much as you can. Hint: $\sin(2x)=2\sin(x)\cos(x)$.

(g) What would change if we measured θ in degrees instead of radians?

6. Emma produces a range of kitchenware to honor her favorite comic book writers. Her new "Stan Lee" cups have been especially popular. Let $P(m)$ represent her profit, in thousands of dollars, if she produces m thousand cups, and let $F(m)$ represent the number of followers, in thousands, she will have on social media after she produces m thousand cups. Assume that both functions are invertible and differentiable.

(a) Write a complete sentence that gives a practical interpretation of the equation $F^{-1}(200)=8$.

(b) Write a mathematical equation using the functions P, F, and/or their inverses that represents the following statement:

If Emma makes a profit of 30 thousand dollars, she will have 250 thousand followers.

(c) Complete the following sentence to give a practical interpretation of the equation $F'(10)=32$:

If Emma makes 12 thousand cups instead of making 10 thousand cups,
...

(d) Which one statement below that is best supported by the equation $(P^{-1})'(16)=4$:

i. For every 4 thousand cups Emma produces, she makes an extra 16 thousand dollars in profit, roughly.

ii. If Emma has made 15.5 thousand dollars in profit, and would like to make 500 more dollars in profit, she will need to produce about 2 thousand more cups.

iii. If Emma goes from producing 16 thousand cups to producing 17 thousand cups then her profit will increase by about 4 thousand dollars.

iv. If Emma produces an extra 500 cups after producing her first 4 thousand cups, then she will make about an extra 8 thousand dollars in profit.