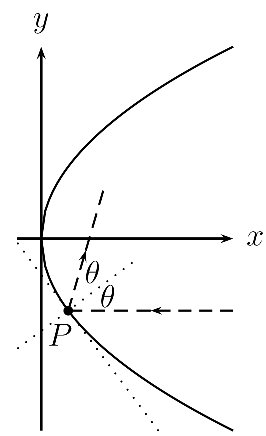
Douglass Houghton Workshop, Section 1, Wed 10/08/25

Worksheet Love All, Trust a Few, Do Wrong to None

1. We're thinking about a parabolic mirror in the shape of the graph of $y = \pm\sqrt{4x}$. So far we've found:



   - A light ray $y = -b$ hits the mirror at $P = (b^2/4, -b)$.
   - The slope of the tangent at that point is $-2/b$.
   - The normal line at the same point has slope $b/2$.
   - When a line makes an angle α with the x-axis, it has slope $\tan\alpha$.
   - So if we call the angle between the normal line and the horizontal θ, then $tan(\theta) = b/2$.
   - Then the angle of the reflected ray to the horizontal is 2θ.
   - Also, $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$.

   (a) Find the slope of the reflected ray.

   (b) Write an equation for the reflected ray.

   (c) Where does the reflected ray intersect the x-axis? What is surprising about this answer?

   (d) Graph several rays, with their reflections.

   (e) What's cool about this type of mirror?

2. (This problem appeared on a Winter, 2009 Math 115 Exam) Suppose a is a positive (non-zero) constant, and consider the function

$$f(x) = \frac{1}{3}x^3 - 4a^2x.$$

   Determine all maxima and minima of f in the interval $[-3a, 5a]$. For each, specify whether it is global or local.

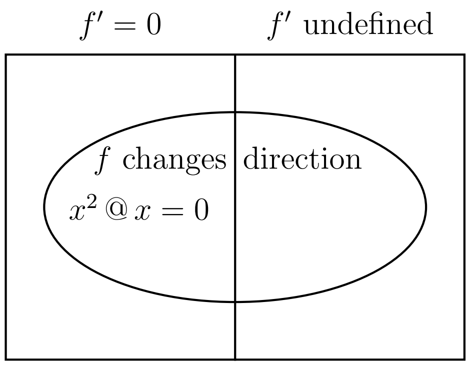
3. Suppose $h(x)$ is a continuous function defined for all real numbers x. The derivative and second derivative of $h(x)$ are given by

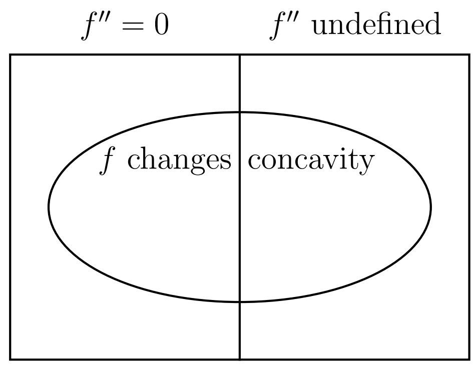
$$h'(x) = \frac{2x}{3(x^2-1)^{2/3}} \qquad \text{and} \qquad h''(x) = -\frac{2(x^2+3)}{9(x^2-1)^{5/3}}.$$

   (a) Find the all critical points and local extrema of $h(x)$. Use calculus to classify the critical points and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema.

   (b) Find all inflection points of $h(x)$, and justify that you've found them all.

4. The diagrams below each have 4 regions, representing different ways a function can behave at a point. In each region write an example of a function and a point that meets the criteria. For example, in the intersection of "$f' = 0$" and "f changes direction", we have $x^2 \,@\, x = 0$, because the derivative of x^2 is indeed 0 at $x = 0$, and the function switches from decreasing to increasing there.





5. (This problem appeared on a Fall, 2005 Math 115 Final Exam) Using techniques from calculus, find the dimensions which will maximize the surface area of a circular cylinder whose height h and radius r, each in centimeters, are related by

$$h = 8 - \frac{r^2}{3}.$$

6. (This problem appeared on the Fall, 2008 Math 115 Final Exam) At the Michigan-Ohio State basketball game this year, the Michigan Band discovers that the amount of time it spends playing "Hail to the Victors" has a direct impact on the number of points our team scores. If the band plays for x minutes, then the Wolverines will score

$$W(x) = -.48x^2 + 7.2x + 63$$

points. Assume that the band can play for a maximum of 10 minutes.

(a) How long should the band play to maximize the number of points Michigan scores?

(b) The band affects how many points Ohio State scores as well. x minutes of playing results in the Buckeyes scoring

$$B(x) = -x^2 + 8x + 84$$

points. Find the number of minutes the band should play to maximize the margin of victory for Michigan.

(c) What will be the score of the game for the case you found in part (b)?