Douglass Houghton Workshop, Section 1, Mon 09/22/25

Worksheet How Noble in Reason, How Infinite in Faculty!

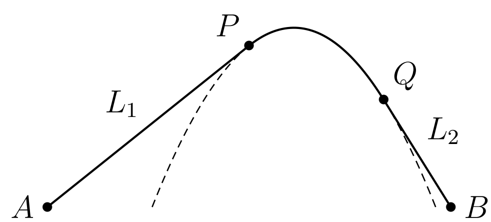
1. Why is it necessary to define the derivative in terms of a limit? Draw a picture that describes how the derivative is the limit of the slopes of some lines.

2. The *power rule for derivatives* says that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. Use the definition of the derivative to prove it for the case where n is a positive integer. Hint: Pascal's triangle.

3. Suppose you are asked to design the first ascent and drop for a new roller coaster at Cedar Point. (You get to name it, too!) By studying photographs of your favorite coasters, you decide to make the slope of the ascent 0.8 and the slope of the drop -1.6. You decide to connect these two straight inclines $y = L_1(x)$ and $y = L_2(x)$ with part of a parabola $y = f(x) = ax^2 + bx + c$, where x and $f(x)$ are measured in feet. For the track to be smooth there can't be abrupt changes in direction, so you want the linear segments L_1 and L_2 to be tangent to the parabola at the transition points P and Q. To simplify the equations, you decide to put the origin at P.

   (a) Name your coaster.

   (b) Suppose the horizontal distance between P and Q is 100 feet. Write equations in a, b, and c that will ensure that the track is smooth at the transition points.

   (c) Solve the equations in (3b) for a, b, and c to find a formula for $f(x)$.

   (d) Plot L_1, f, and L_2 to verify graphically that the transitions are smooth.

   (e) Find the difference in elevation between P and Q. How wide is the coaster at the height of Q?

   (f) Suppose the base of the hill (the distance from A to B in the picture) is 300 feet long. How high is the hill? HINT: Let y be the vertical distance between Q and A. Find what y should be in order to make the width be 300.

4. (Winter, 2010) Suppose $W(h)$ is an invertible function which tells us how many gallons of water an h-foot tall oak tree uses on a hot summer day.

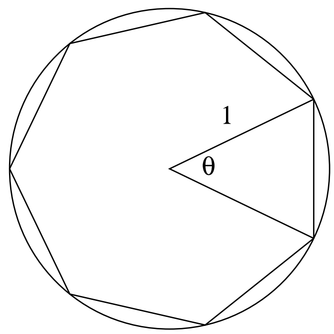
   (a) Give practical interpretations of $W(50)$ and $W^{-1}(40)$.

   (b) Suppose that an average oak tree is A feet tall and uses G gallons of water on a hot summer day. Answer in terms of W, A, and G:

      i. A farmer has 25 oak trees, and each one is 10 feet taller than an average oak tree. How much water will they use on a hot summer day?

      ii. The farmer also has some oak trees which use 5 fewer gallons of water on a hot summer day than an average oak tree does. How tall are his trees?

5. (This problem explains why $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, but only when θ is measured in radians.) Consider a regular n-sided polygon inscribed in a circle of radius 1.



(a) Let A_n be the area of the polygon. What does A_n approach as n gets large? $\lim_{n \to \infty} A_n = \square$

(b) We can compute A_n by dividing the polygon up into triangles which have a vertex at the center. Let θ be the vertex angle (in radians). What is θ in terms of n?

(c) What happens to θ as n gets large?

(d) What is the area of one of the triangles, in terms of θ?

(e) What is A_n in terms of θ?

(f) Substitute into the equation from part (a) so that it includes θ's but not n's. Simplify it as much as you can. Hint: $\sin(2x) = 2\sin(x)\cos(x)$.

(g) What would change if we measured θ in degrees instead of radians?

6. Let $f(x) = x^2$ and $g(x) = \frac{1}{3}x^3$.

(a) Sketch the graphs of both functions on the same grid, for $x \in [-1, 3]$.

(b) The vertical line $x = x_0$ intersects the two graphs at $(x_0, f(x_0))$ and $(x_0, g(x_0))$. For which vertical lines are the tangents at those points parallel? Try to guess the number of solutions by looking at the graph. Then calculate.

(c) On which horizontal lines do the graphs have parallel tangents?