

Lecture 1

The SUSY Algebra

Outline of Course

- **Introduction to Supersymmetry**: algebra, field theory representations, superspace etc.

Covered in the first few lectures, but not in great depth. Students are encouraged to work through additional material, as needed.

- **SUSY phenomenology**: the MSSM and SUSY breaking.

Central to approach in textbook, but we cover it **minimally**.

- **Modern developments**: nonperturbative methods. **This is emphasis.**

Effective Lagrangians, symmetry breaking, anomalies, ... : key concepts of advanced QFT that are well illustrated by SUSY examples.

- Some (relatively) **recent applications**: SUSY breaking in Beyond the Standard Model scenarios.

Textbook: "Modern Supersymmetry" by John Terning

Outline of Lecture 1

- Appetizer: radiative corrections to the Higgs mass.

The quadratic divergence in the SM and its cancellation in SUSY theory.

- The SUSY Algebra.

An anti-commutator mixing spacetime and internal symmetry.

- Spontaneous SUSY breaking: a first look.

The vacuum energy: to vanish or not to vanish, that is the question.

- Representations of SUSY.

SUSY generalizations of the simplest QFTs.

Reading: Terner 1.1-1.3

Unreasonable effectiveness of the SM

- Key ingredient in Standard Model: the Higgs particle.
- Required properties: a vacuum expectation value (VEV) of 246 GeV and mass broadly of the same order.
- Radiative corrections: there are quadratically divergent contributions to the mass.
- Interpretation in effective QFT: the Higgs mass is naturally of the same order as unknown high energy physics so **the low energy description is inconsistent** (it is not a systematic approximation).
- Proposed resolution: a symmetry (SUSY) requires couplings such that quadratic divergences cancel. Thus **the low energy description is consistent**, if the high energy theory is supersymmetric.

The Higgs Mechanism

Yukawa coupling between the Higgs field (H^0) and a top quark:

$$\mathcal{L}_{\text{Yukawa}} = -\frac{y_t}{\sqrt{2}} H^0 \overline{t}_L t_R + h.c.$$

The Higgs field acquires a VEV (somehow: details are unimportant) :

$$H^0 = \langle H^0 \rangle + h^0 = v + h^0$$

so that the **top acquires a mass** (this is the Higgs mechanism):

$$m_t = \frac{y_t v}{\sqrt{2}}$$

Corrollary: other couplings between the Higgs excitation h_0 and the top follows from the same Yukawa coupling.

The Quadratic Divergence

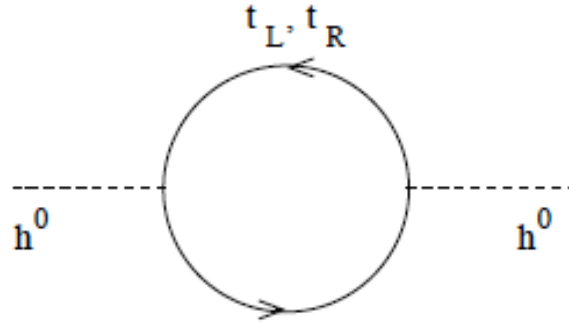


Figure 1: The top loop contribution to the Higgs mass term.

Correction to h_0 mass due to a top loop:

$$\begin{aligned}
 -i\delta m_h^2|_{\text{top}} &= (-1)N_c \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[\frac{-iy_t}{\sqrt{2}} \frac{i}{\not{k} - m_t} \left(\frac{-iy_t^*}{\sqrt{2}} \right) \frac{i}{\not{k} - m_t} \right] \\
 &= -2N_c |y_t|^2 \int \frac{d^4 k}{(2\pi)^4} \frac{k^2 + m_t^2}{(k^2 - m_t^2)^2} \\
 &= \frac{iN_c |y_t|^2}{8\pi^2} \int_0^{\Lambda^2} dk_E^2 \frac{k_E^2 (k_E^2 - m_t^2)}{(k_E^2 + m_t^2)^2}
 \end{aligned}$$

Details:

- Traces over γ -matrices: $\text{Tr}(\not{k} + m_t)^2 = 4(k^2 + m_t^2)$
- Euclidean continuation: $k_0 \rightarrow ik_4$, $k^2 \rightarrow -k_E^2$
- $d^4k_E = 2\pi^2 k_E^3 dk_E = \pi^2 k_E^2 dk_E^2$

Evaluation of integral (using $x = k_E^2 + m_t^2$):

$$\begin{aligned}\delta m_h^2|_{\text{top}} &= -\frac{N_c |y_t|^2}{8\pi^2} \int_{m_t^2}^{\Lambda^2} dx \left(1 - \frac{3m_t^2}{x} + \frac{2m_t^4}{x^2} \right) \\ &= -\frac{N_c |y_t|^2}{8\pi^2} \left[\Lambda^2 - 3m_t^2 \ln \left(\frac{\Lambda^2 + m_t^2}{m_t^2} \right) + \dots \right]\end{aligned}$$

Notation: ... are terms that are finite as $\Lambda \rightarrow \infty$.

Lesson from computation: virtual top quarks give **quadratically and logarithmically divergent** contributions to the Higgs mass.

Interpretation (reminder): we consider Λ the large (but finite!) scale where high energy physics omitted in the low energy description becomes important. Thus: the quadratic “divergence” shows that **the unknown high energy physics is the natural scale for the Higgs mass**.

UV Completion: a Toy Model

Hypothesis: there are also N light scalar particles ϕ_L, ϕ_R and they couple to the Higgs as:

$$\mathcal{L}_{\text{scalar}} = -\frac{\lambda}{2}(h^0)^2(|\phi_L|^2 + |\phi_R|^2) - h^0(\mu_L|\phi_L|^2 + \mu_R|\phi_R|^2) - m_L^2|\phi_L|^2 - m_R^2|\phi_R|^2$$

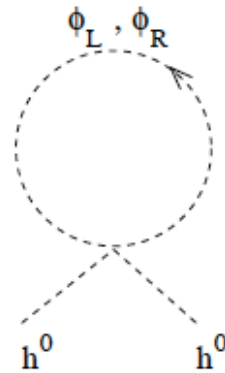


Figure 2: Scalar boson contribution to the Higgs mass term via the quartic coupling.

Contribution to the Higgs mass from the **quartic coupling to virtual new scalars**:

$$-i\delta m_h^2|_2 = -i\lambda N \int \frac{d^4 k}{(2\pi)^4} \left[\frac{i}{k^2 - m_L^2} + \frac{i}{k^2 - m_R^2} \right]$$

$$\Rightarrow \delta m_h^2|_2 = \frac{\lambda N}{16\pi^2} \left[2\Lambda^2 - m_L^2 \ln \left(\frac{\Lambda^2 + m_L^2}{m_L^2} \right) - m_R^2 \ln \left(\frac{\Lambda^2 + m_R^2}{m_R^2} \right) + \dots \right].$$

Cancellation: the new contribution cancels the Λ^2 divergence from top quarks, if it happens that $N = N_c$ and $\lambda = |y_t|^2$.

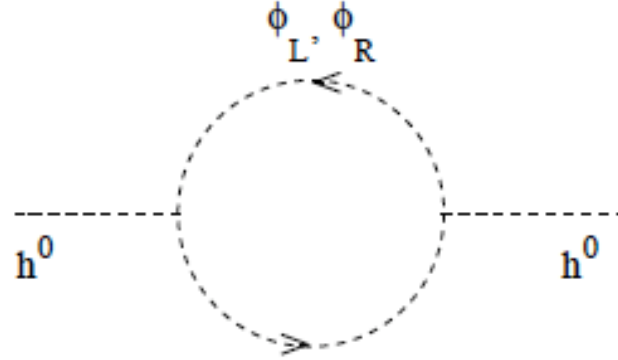


Figure 3: Scalar boson contribution to the Higgs mass term via the trilinear coupling.

Contribution to the Higgs mass from the **cubic coupling to virtual new scalars**:

$$\begin{aligned}
 -i\delta m_h^2|_3 &= N \int \frac{d^4 k}{(2\pi)^4} \left[\left(-i\mu_L \frac{i}{k^2 - m_L^2} \right)^2 + \left(-i\mu_R \frac{i}{k^2 - m_R^2} \right)^2 \right] \\
 \Rightarrow \delta m_h^2|_3 &= -\frac{N}{16\pi^2} \left[\mu_L^2 \ln \left(\frac{\Lambda^2 + m_L^2}{m_L^2} \right) + \mu_R^2 \ln \left(\frac{\Lambda^2 + m_R^2}{m_R^2} \right) + \dots \right].
 \end{aligned}$$

Cancellation: If $m_t = m_L = m_R$ and $\mu_L^2 = \mu_R^2 = 2\lambda m_t^2$, then $\log \Lambda$ are canceled as well.

Conclusion: the toy model of the high energy completion of the SM cancels all the divergences from the top quark, for specific values of the couplings between the Higgs and the new scalars.

Status

- The standard model requires a Higgs particle, and it contributes quadratic divergences to the Higgs mass.
- Interpretation: new physics **at the same scale as the standard model** is important. So it is mandatory that we model such new physics and include it.
- A minimal model of the additional physics: two new scalars included in the low energy theory, with specific couplings to Higgs. This model is simple and acceptable, in that no further new physics must be included in the low energy theory.

The model is ad hoc in that masses and couplings of the new particles are postulated to have specific values.

- The point: **SUSY guarantees these relations** and so offers a principled (based on symmetry) model of the new physics.

SUSY algebra

The novelty: the SUSY algebra has generators that are **spinors under Lorentz invariance**.

Notation for SUSY generators: complex, anticommuting Weyl spinors Q_α and their complex conjugates $Q_{\dot{\alpha}}^\dagger$.

Convention: Weyl spinors are just left-handed ($\alpha = 1, 2$), but their complex conjugates are right-handed.

Fundamental SUSY anti-commutator (with P_μ the four-momentum):

$$\{Q_\alpha, Q_{\dot{\alpha}}^\dagger\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu,$$

Pauli-matrices:

$$\sigma_{\alpha\dot{\alpha}}^\mu = (1, \sigma^i), \quad \bar{\sigma}^{\mu\dot{\alpha}\alpha} = (1, -\sigma^i)$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

SUSY is an **internal symmetry** in that it is independent on spacetime position. In Fourier space, this is expressed by the commutator

$$[P_\mu, Q_\alpha] = [P_\mu, Q_\alpha^\dagger] = 0$$

It is useful to keep track of “SUSY-ness” by the **R-symmetry generator R**:

$$[Q_\alpha, R] = Q_\alpha \quad [Q_\alpha^\dagger, R] = -Q_\alpha^\dagger$$

In short: Q_α decreases the R -quantum number by 1, while $Q_\alpha^\dagger$ increases it by 1.

The SUSY generator is a spacetime spinor so due to the spin-statistics theorem its action **turns fermions into bosons**, and *vice versa*. Formally:

$$\{(-1)^{\mathbf{F}}, Q_\alpha\} = 0$$

where

$$\begin{aligned} (-1)^{\mathbf{F}} |\text{boson}\rangle &= +1 |\text{boson}\rangle \\ (-1)^{\mathbf{F}} |\text{fermion}\rangle &= -1 |\text{fermion}\rangle \end{aligned}$$

Fermion-Boson degeneracy

All particles in a representation of SUSY (=a supermultiplet) have the same mass (because $P_\mu P^\mu = M^2$ commutes with Q_α).

Gauge generators also commute with supercharges, so all particles in a supermultiplet have the same gauge charges.

The Hamiltonian is part of the SUSY algebra:

$$H = P^0 = \frac{1}{4}(Q_1 Q_1^\dagger + Q_2 Q_2^\dagger + Q_1^\dagger Q_1 + Q_2^\dagger Q_2)$$

Consequence for spectrum (derived using completeness $\sum_i |i\rangle\langle i| = 1$ in any given supermultiplet):

$$\begin{aligned}
\sum_i \langle i | (-1)^{\mathbf{F}} P^0 | i \rangle &= \frac{1}{4} \left(\sum_i \langle i | (-1)^{\mathbf{F}} Q Q^\dagger | i \rangle + \sum_i \langle i | (-1)^{\mathbf{F}} Q^\dagger Q | i \rangle \right) \\
&= \frac{1}{4} \left(\sum_i \langle i | (-1)^{\mathbf{F}} Q Q^\dagger | i \rangle + \sum_{ij} \langle i | (-1)^{\mathbf{F}} Q^\dagger | j \rangle \langle j | Q | i \rangle \right) \\
&= \frac{1}{4} \left(\sum_i \langle i | (-1)^{\mathbf{F}} Q Q^\dagger | i \rangle + \sum_{ij} \langle j | Q | i \rangle \langle i | (-1)^{\mathbf{F}} Q^\dagger | j \rangle \right) \\
&= \frac{1}{4} \left(\sum_i \langle i | (-1)^{\mathbf{F}} Q Q^\dagger | i \rangle + \sum_j \langle j | Q (-1)^{\mathbf{F}} Q^\dagger | j \rangle \right) \\
&= \frac{1}{4} \left(\sum_i \langle i | (-1)^{\mathbf{F}} Q Q^\dagger | i \rangle - \sum_j \langle j | (-1)^{\mathbf{F}} Q Q^\dagger | j \rangle \right) \\
&= 0.
\end{aligned}$$

Conclusion: in each supermultiplet, there is the same **number of fermions and bosons**.

$$N_F = N_B$$

The Vacuum

The **vacuum is the ground state of the theory**. It is denoted $|0\rangle$.

In a SUSY theory the vacuum energy (density) is non-negative

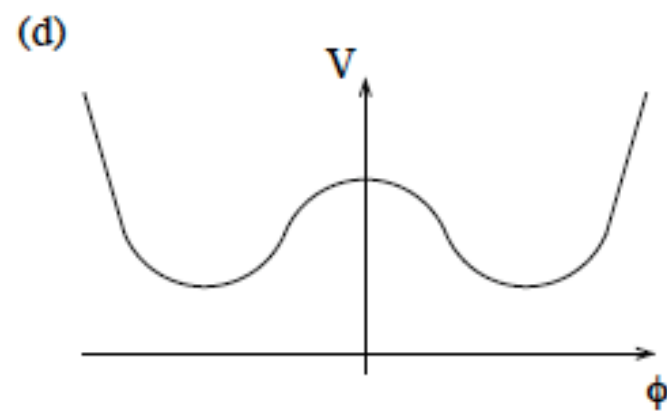
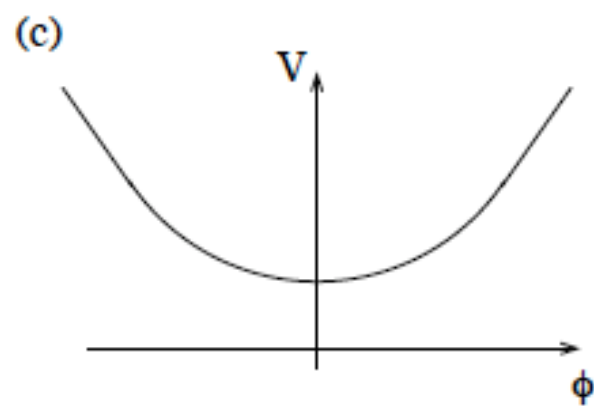
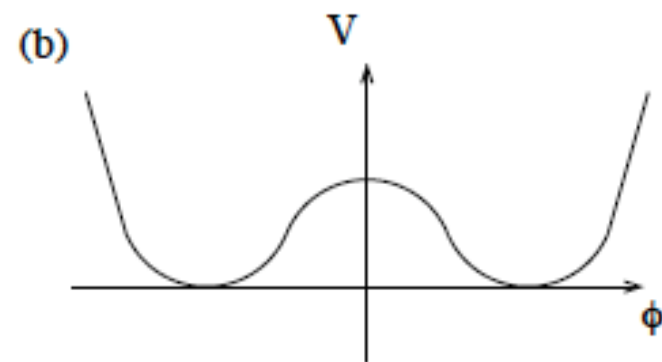
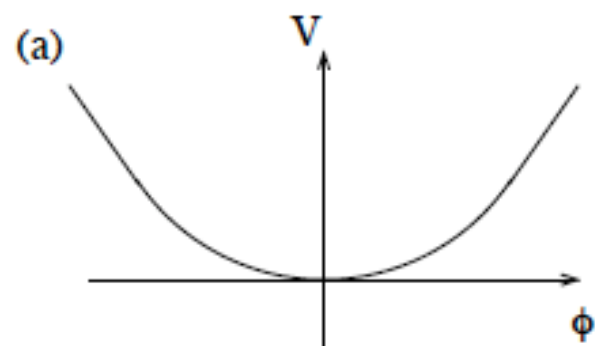
$$\langle 0|H|0\rangle = \frac{1}{4}\langle 0|(Q_1Q_1^\dagger + Q_2Q_2^\dagger + Q_1^\dagger Q_1 + Q_2^\dagger Q_2)|0\rangle \geq 0$$

The vacuum energy vanishes exactly when the ground state is invariant under SUSY

$$\langle 0|H|0\rangle \quad \Leftrightarrow \quad Q_\alpha|0\rangle = Q_\alpha^\dagger|0\rangle = 0$$

Conversely, SUSY is broken exactly when the ground state energy is positive

$$\langle 0|H|0\rangle \neq 0 \quad \Leftrightarrow \quad Q_\alpha|0\rangle \neq 0 \text{ or } Q_\alpha^\dagger|0\rangle \neq 0$$



SUSY representations

Goal: determine the particle spectrum in simple supermultiplets.

Reminder: representations of Relativistic QFT are labeled by mass, spin, and a component of spin: $|m, s, s_3\rangle$.

The fundamental anti-commutator for a massive particle in the rest frame: $p_\mu = (m, \vec{0})$:

$$\begin{aligned}\{Q_\alpha, Q_{\dot{\alpha}}^\dagger\} &= 2m \delta_{\alpha\dot{\alpha}} \\ \{Q_\alpha, Q_\beta\} &= 0 \\ \{Q_{\dot{\alpha}}^\dagger, Q_{\dot{\beta}}^\dagger\} &= 0\end{aligned}$$

Intuition: Q_α are fermionic annihilation oscillators (for $\alpha = 1, 2$), and $Q_{\dot{\alpha}}^\dagger$ are the corresponding creation operators.

Remark: there is no absolute distinction so the opposite assignment of creation/annihilation works as well.

Implementation: starting from any state $|m, s', s'_3\rangle$, define the **Clifford vacuum**:

$$|\Omega_s\rangle = Q_1 Q_2 |m, s', s'_3\rangle$$

By construction, the Clifford vacuum is annihilated by the annihilation operators

$$Q_1 |\Omega_s\rangle = Q_2 |\Omega_s\rangle = 0$$

Starting from the Clifford vacuum, we construct the general **massive supermultiplet**:

$$\begin{aligned} &|\Omega_s\rangle \\ &Q_1^\dagger |\Omega_s\rangle, Q_2^\dagger |\Omega_s\rangle \\ &Q_1^\dagger Q_2^\dagger |\Omega_s\rangle \end{aligned}$$

There are several relevant values for s , the spin of the Clifford vacuum $|\Omega_s\rangle$.

The **massive “chiral” multiplet** ($s = 0$):

state	s_3
$ \Omega_0\rangle$	0
$Q_1^\dagger \Omega_0\rangle, Q_2^\dagger \Omega_0\rangle$	$\pm\frac{1}{2}$
$Q_1^\dagger Q_2^\dagger \Omega_0\rangle$	0

Spectrum: a complex boson and a massive two component fermion (=a Majorana fermion).

The **massive vector multiplet** ($s = \frac{1}{2}$):

state	s_3
$ \Omega_{\frac{1}{2}}\rangle$	$\pm\frac{1}{2}$
$Q_1^\dagger \Omega_{\frac{1}{2}}\rangle, Q_2^\dagger \Omega_{\frac{1}{2}}\rangle$	0, 1, 0, -1
$Q_1^\dagger Q_2^\dagger \Omega_{\frac{1}{2}}\rangle$	$\pm\frac{1}{2}$

Spectrum: a massive vector field, a scalar field, and two Majorana fermions.

Massless particles

Reminder: in the special case of **vanishing** mass, representations of relativistic QFT are labeled by energy and helicity: $|E, \lambda\rangle$.

The fundamental SUSY anticommutator in the canonical massless Lorentz frame (where $p_\mu = (E, 0, 0, -E)$):

$$\begin{aligned}\{Q_1, Q_1^\dagger\} &= 4E \\ \{Q_2, Q_2^\dagger\} &= 0 \\ \{Q_\alpha, Q_\beta\} &= 0 \\ \{Q_{\dot{\alpha}}^\dagger, Q_{\dot{\beta}}^\dagger\} &= 0\end{aligned}$$

Massless supermultiplets are “shortened” because $Q_2|\Omega_\lambda\rangle$ has vanishing norm:

$$\langle\Omega_\lambda|Q_2Q_2^\dagger|\Omega_\lambda\rangle + \langle\Omega_\lambda|Q_2^\dagger Q_2|\Omega_\lambda\rangle = 0 \quad \Rightarrow \quad \langle\Omega_\lambda|Q_2^\dagger Q_2|\Omega_\lambda\rangle = 0$$

For the purpose of counting states, we simply take $Q_2|\Omega_\lambda\rangle = 0$.

Starting from an arbitrary state $|E, \lambda'\rangle$, we can thus construct the Clifford vacuum as the “highest weight” state:

$$|\Omega_\lambda\rangle = Q_1|E, \lambda'\rangle$$

It is obviously annihilated by the lone annihilation operator:

$$Q_1|\Omega_\lambda\rangle = 0$$

General massless supermultiplet

The general massless super-multiplet is simply

state	helicity
$ \Omega_\lambda\rangle$	λ
$Q_1^\dagger \Omega_\lambda\rangle$	$\lambda + \frac{1}{2}$

The “shortening” refers to the massless multiplets having fewer states, which in turn comes about because one of the candidate creation operators ($Q_2^\dagger$) in fact annihilates the state.

The massless representation is so small that it generally conflicts with CPT invariance (and so of no interest in QFT).

To avoid this, the spectrum must include the CPT conjugate supermultiplet:

state	helicity
$ \Omega_{-\lambda-\frac{1}{2}}\rangle$	$-\lambda - \frac{1}{2}$
$Q_1^\dagger \Omega_{-\lambda-\frac{1}{2}}\rangle$	$-\lambda$

The massless chiral multiplet

The special case where the Clifford vacuum has vanishing helicity $\lambda = 0$:

state	helicity
$ \Omega_0\rangle$	0
$Q_1^\dagger \Omega_0\rangle$	$\frac{1}{2}$

Include CPT conjugate states:

state	helicity
$ \Omega_{-\frac{1}{2}}\rangle$	$-\frac{1}{2}$
$Q_1^\dagger \Omega_{-\frac{1}{2}}\rangle$	0

The spectrum of the massless chiral supermultiplet: a complex scalar and a two component fermion (a Weyl fermion).

The massless vector multiplet

The special case where the Clifford vacuum has helicity $\lambda = \frac{1}{2}$:

state	helicity
$ \Omega_{\frac{1}{2}}\rangle$	$\frac{1}{2}$
$Q_1^\dagger \Omega_{\frac{1}{2}}\rangle$	1

and its CPT conjugate:

state	helicity
$ \Omega_{-1}\rangle$	-1
$Q_1^\dagger \Omega_{-1}\rangle$	$-\frac{1}{2}$

The spectrum of the massless vector supermultiplet: a massless vector particle and a two component fermion (a Weyl fermion).

Superpartners: Terminology

Matter supermultiplets (chiral multiplets):

fermion $\leftrightarrow$ sfermion

quark $\leftrightarrow$ squark

Gauge supermultiplets (vector multiplets):

gauge boson $\leftrightarrow$ gaugino

gluon $\leftrightarrow$ gluino

Outlook

This lecture: simple algebraic manipulations.

Challenge: present field theories that realize the representations of the superalgebra that we have found.

More challenges: include interactions that still respect the superalgebra.

Then analyze the resulting **quantum** theory.