1. Consider a linear transformation with the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix}$.
Find a basis of the kernel and a basis of the image of the transformation. Describe the kernel and image geometrically (as a line, plane, etc.). Mention the space where the kernel and image lie.

Solution. Reducing the matrix, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The kernel of A consists of all solutions of the system $A\vec{x} = \vec{0}$, that is, of the vectors

$$\begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

where x_3 can be any number. Hence $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is a basis of the kernel and the kernel is a line in $\mathbf{R}^3$. This copy of $\mathbf{R}^3$ is the input space.

To pick a basis of the image, we select the columns of A corresponding to the leading 1s. Thus the vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$ is a basis of the image of A. Hence the image is a plane in $\mathbf{R}^3$. This copy of $\mathbf{R}^3$ is the output space.

Answer. The kernel is a line in $\mathbf{R}^3$ (input space) with a basis consisting of the vector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. The image is a plane in $\mathbf{R}^3$ (output space) with a basis consisting of the vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$.