Problem # 6. To save space, let us denote

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

We compute

$$\vec{u} \cdot \vec{v}_1 = 1,\ \vec{u} \cdot \vec{v}_2 = 1,\ \vec{v}_1 \cdot \vec{v}_1 = 2,\ \vec{v_2} \cdot \vec{v}_2 = 2,\ \vec{v}_1 \cdot \vec{v}_2 = 1.$$

We look for the orthogonal projection in the form $x\vec{v}_1 + y\vec{v}_2$ such that the difference $\vec{u} - x\vec{v}_1 - y\vec{v}_2$ is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$. This gives us the equations

$$\begin{aligned} (\vec{u} - x\vec{v}_1 - y\vec{v}_2) \cdot \vec{v}_1 &= 0 \\ (\vec{u} - x\vec{v}_1 - y\vec{v}_2) \cdot \vec{v}_2 &= 0, \end{aligned} \qquad \text{that is,} \qquad \begin{aligned} 1 - 2x - y &= 0 \\ 1 - x - 2y &= 0. \end{aligned}$$

Solving the system of equations, we get $x = y = 1/3$, so the orthogonal projection is

$$\frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_2 = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 1/3 \\ 0 \end{bmatrix}.$$

Answer: The orthogonal projection is $\begin{bmatrix} 2/3 \\ 1/3 \\ 1/3 \\ 0 \end{bmatrix}$.