Universal Spectra and Tijdeman's Conjecture on Factorization of Cyclic Groups

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ABSTRACT. A spectral set Ω in $\mathbb{R}^n$ is a set of finite Lebesgue measure such that $L^2(\Omega)$ has an orthogonal basis of exponentials $\{e^{2\pi i \langle \lambda, x\rangle} : \lambda \in \Lambda\}$ restricted to Ω. Any such set Λ is called a spectrum for Ω. It is conjectured that every spectral set Ω tiles $\mathbb{R}^n$ by translations. A tiling set $\mathcal{T}$ of translations has a *universal spectrum* Λ if every set Ω that tiles $\mathbb{R}^n$ by $\mathcal{T}$ is a spectral set with spectrum Λ. Recently Lagarias and Wang showed that many periodic tiling sets $\mathcal{T}$ have universal spectra. Their proofs used properties of factorizations of abelian groups, and were valid for all groups for which a strong form of a conjecture of Tijdeman is valid. However Tijdeman's original conjecture is not true in general, as follows from a construction of Szabó [17], and here we give a counterexample to Tijdeman's conjecture for the cyclic group of order 900. This paper formulates a new sufficient condition for a periodic tiling set to have a universal spectrum, and applies it to show that the tiling sets in the given counterexample do possess universal spectra.

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1. Introduction

A *spectral set* Ω in $\mathbb{R}^n$ is a closed set of finite Lebesgue measure such that $L^2(\Omega)$ has an orthogonal basis of exponentials $\{e^{2\pi i \langle \lambda, x\rangle} : \lambda \in \Lambda\}$ restricted to Ω. Any discrete set Λ in $\mathbb{R}^n$ with this property is called a *spectrum* for Ω and (Ω, Λ) is called a *spectral pair.*

In 1974 B. Fuglede [4] studied the problem of finding self-adjoint commuting extensions of the operators $i\frac{\partial}{\partial x_1}, \ldots, i\frac{\partial}{\partial x_n}$ inside $L^2(\Omega)$, and related the existence of such an extension to Ω being a spectral set. He formulated the following conjecture.

Spectral Set Conjecture. *Let Ω be a measurable set of $\mathbb{R}^n$ with finite Lebesgue measure. Then Ω is a spectral set if and only if Ω tiles $\mathbb{R}^n$ by translations.*

Here Ω tiles $\mathbb{R}^n$ with a set $\mathcal{T}$ of translations if $\Omega + \mathcal{T} = \mathbb{R}^n$ and, for $t, t' \in \mathcal{T}$,

$$meas((\Omega + t) \cap (\Omega + t')) = 0 \quad \text{if} \quad t \neq t' .$$

Despite extensive study, this conjecture remains open in all dimensions, in either direction. For work on this problem see [4], [6], [7], [8], [9], [10], [11], [13], [14].

Lagarias and Wang [13] approached the spectral set conjecture in terms of tiling sets. They proved that a large class of tiling sets $\mathcal{T}$ had a *universal spectrum* Λ in the sense that every set Ω that tiles $\mathbb{R}^n$ with the tiling set $\mathcal{T}$ was a spectral set with the spectrum Λ. They considered periodic tiling sets of the form[1] $\mathcal{T} = N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z} + \mathcal{A}$ with $\mathcal{A} \subseteq \mathbb{Z}^n$, for some $N_1, \ldots, N_n \in \mathbb{Z}$. Their results were obtained by reduction to questions about factorizations [2] of abelian groups. (A connection to such factorizations was originally noted in Fuglede [4].) In the one-dimensional case, they formulated the following conjecture.

Universal Spectrum Conjecture. *Let $\mathcal{T} = \mathbb{Z} + \frac{1}{m}A$ where $A \subseteq \mathbb{Z}$ reduced (mod m) admits some factorization $A \oplus B = \mathbb{Z}/m\mathbb{Z}$. Then $\mathcal{T}$ has a universal spectrum of the form $m\mathbb{Z} + \Gamma$, where $\Gamma \subseteq \mathbb{Z}$.*

In support of this conjecture, they proved [13, Theorem 1.2] that if the cyclic group $\mathbb{Z}_m$ has a property called the strong Tijdeman property, then any tile set $\mathcal{T} = \mathbb{Z} + \frac{1}{m}\mathcal{A}$ has a universal spectrum of the form $\Lambda = m\mathbb{Z} + \Gamma$ for some $\Gamma \subseteq \mathbb{Z}$. They showed that the strong Tijdeman property holds for many cyclic groups, and formulated the conjecture that all cyclic groups have the strong Tijdeman property. We do not define the strong Tijdeman property here (see [13]) but observe that a necessary requirement for its truth for $G = \mathbb{Z}/m\mathbb{Z}$ is the truth of the Tijdeman conjecture (mod m) stated in §2. In addition to these results, Petersen and Wang [15, Theorem 4.5] also found other tiling sets $\mathcal{T}$ which have universal spectra.

Recently Coven and Meyerowitz [3] observed that a conjecture equivalent to Tijdeman's Conjecture (mod m) had been made earlier, by Sands [16] in 1977. Sands proved this conjecture holds for all m divisible by at most two distinct primes. Sands' conjecture (*mod*) was disproved by Szabó [17] in 1985, by a direct construction which applies to certain integers m divisible by three or more distinct primes. The smallest counterexample covered by that construction is $m = 2^3 3^3 5^2 = 5400$. In §2 we present a counterexample to this conjecture for $m = 2^2 3^2 5^2 = 900$, found using similar ideas to the construction in Szabó [17]. The counterexamples disprove the Tijdeman Conjecture (mod m) for such m, and this shows that the method for proving the existence of universal spectra given in [13] does not work in general. In consequence, new methods are needed to resolve the universal spectrum conjecture.

In §3 we present a new sufficient condition for the existence of a universal spectrum for a periodic tiling set in $\mathbb{R}^n$ (Theorem 3.1). This criterion is easier to check computationally than a necessary and sufficient condition given in [13]. It seems conceivable that the condition of Theorem 3.1 is actually necessary and sufficient; this remains an unresolved question. If so it might prove useful in resolving the spectral set conjecture for periodic tiling sets.

In §4 we apply the sufficient condition of §3 to show that the tiling sets $\mathcal{T}_A = \mathbb{Z} + \frac{1}{900}A$ and $\mathcal{T}_B = \mathbb{Z} + \frac{1}{900}B$ associated to the counterexample in §2 do have universal spectra. These sets give examples supporting the universal spectrum conjecture which are not covered by the methods of [13] and [15].

2. Counterexample to Tijdeman's Conjecture

A *factorization* (A, B) of the finite cyclic group $G = \mathbb{Z}/m\mathbb{Z}$, written

$$\bar{A} \oplus \bar{B} = G \ , \tag{2.1}$$

is one in which every element $g \in G$ has a unique representation

$$g = \bar{a} + \bar{b}, \quad \bar{a} \in \bar{A} \quad \text{and} \quad \bar{b} \in \bar{B} \ .$$

[1] General periodic tilings can always be reduced to this form by a linear transformation of $\mathbb{R}^n$.

[2] Factorizations of groups are defined in §2.

We write $\bar{A} = \{\bar{a}_0, \ldots, \bar{a}_{n-1}\}$ and $\bar{B} = \{\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{l-1}\}$ with $ln = m$. Let $A = \{a_0, \ldots, a_{n-1}\} \subseteq \mathbb{Z}$ be a lifting of $\bar{A}$ to $\mathbb{Z}$, with $\bar{a}_i = a_i \pmod{m}$, and similarly let B be a lifting of $\bar{B}$. Any factorization (A, B) of $\mathbb{Z}/m\mathbb{Z}$ yields the *direct sum decomposition* of $\mathbb{Z}$, as

$$A \oplus \mathcal{T}_B = \mathbb{Z} \ , \tag{2.2}$$

in which

$$\mathcal{T}_B = m\mathbb{Z} + B \tag{2.3}$$

is a periodic set with period $m\mathbb{Z}$. There has been extensive study of the structure of factorization of finite cyclic groups (more generally finite abelian groups) and of direct sum decomposition (1.2) of the integers $\mathbb{Z}$, the history of which can be found in Tijdeman [18], see also Coven and Meyerowitz [3]. The original conjecture of Tijdeman [18, p. 266] is as follows.

Tijdeman's Conjecture. *If $A \oplus \mathcal{T} = \mathbb{Z}$ with $0 \in A \cap \mathcal{T}$ and A is a finite set with n elements, with g.c.d. $\{a : a \in A\} = 1$, then there exists some prime factor p of n such that all elements of $\mathcal{T}$ are divisible by p.*

Hajós [5] and de Bruijn [1] showed for any direct sum decomposition $A + \mathcal{T} = \mathbb{Z}$ where $|A|$ is finite, the infinite set $\mathcal{T}$ is periodic and necessarily has the form (2.3) and thus corresponds to some factorization (2.1) of a cyclic group $\mathbb{Z}/m\mathbb{Z}$ in which $|A|$ divides m. This permits Tijdeman's conjecture to be reformulated as a series of conjectures for each finite cyclic group $\mathbb{Z}/m\mathbb{Z}$, as follows.

Tijdeman's Conjecture $\pmod{m}$**.** *If $\bar{A} \oplus \bar{B} = \mathbb{Z}/m\mathbb{Z}$ with $\bar{0} \in \bar{A} \cap \bar{B}$ and if $\bar{A}$ lifts to set $A \subseteq \mathbb{Z}$ with $0 \in A$ and g.c.d. $\{a : a \in A\} = 1$, then any lifting $B \subseteq \mathbb{Z}$ with $0 \in B$ has g.c.d. $\{b : b \in B\} \neq 1$.*

Tijdeman showed that this conjecture holds for any m for which $\mathbb{Z}/m\mathbb{Z}$ is a "good" group in the sense of Hajós [5] and de Bruijn [1]. The complete list of cyclic "good" groups are known to be those of order p^n $(n \geq 1)$, pq, pqr, $p^n q$ $(n > 1)$, p^2q^2, p^2qr and $pqrs$, where p, q, r and s are distinct primes, c.f. [18] for references. As indicated in the introduction, this conjecture was actually made earlier by Sands[16], who proved it also holds in the cases $m = p^n q^k$ $(n, k \geq 1)$.

In 1985 Szabó [17] gave a construction which gave counterexamples to Sands' conjecture for certain m divisible by three or more distinct primes. Szabó actually constructs sets $A \subset \mathbb{Z}$ with $0 \in A$ and $\gcd(A) = 1$, which tile the integers, whose members are not uniformly distributed mod k for any $k \geq 2$. Coven and Meyerowitz[3, Lemma 2.5] observe that it follows that any tiling set $C \subset \mathbb{Z}$ for A with $0 \in C$ must have $\gcd(C) = 1$, hence cannot be contained in any subgroup of $\mathbb{Z}/m\mathbb{Z}$, where m is the minimal period of C. The Szabó construction applies to $m = m_1 m_2 ... m_r$ with $r \geq 3$ in which each $m_i = u_i v_i$ with the m_i pairwise relatively prime, u_i is the smallest prime dividing m_i and each $v_i \geq 4$. The smallest m satisfying these conditions is $m = 2^3 3^3 5^2 = 5400$. Here we present a counterexample for $m = 2^2 3^2 5^2 = 900$ which was found using similar ideas.

Theorem 2.1. *Tijdeman's Conjecture* $\pmod{900}$ *is false.*

Proof. We take sets A and B with $|A| = |B| = 30$. The set

$$A = \{0, 36, 72, 108, 144\} \oplus \{0, 100, 200\} \oplus \{0, 225\} \ . \tag{2.4}$$

It has *g.c.d.* $\{a : a \in A\} = 1$, since g.c.d.$\{2^2 3^2, 2^2 5^2, 3^2 5^2\} = 1$. We choose

$$B = \left\{ \begin{array}{cccccccccc} 0 & 30 & 60 & 126 & 180 & 210 & 220 & 240 & 300 & 306 \\ 330 & 360 & 375 & 390 & 480 & 486 & 510 & 520 & 540 & 570 \\ 660 & 666 & 690 & 750 & 780 & 820 & 825 & 840 & 846 & 870 \end{array} \right\} . \tag{2.5}$$

We claim that $\bar{A} \oplus \bar{B} = \mathbb{Z}/900\mathbb{Z}$ is a direct sum. This can be verified by a calculation[3] (a short computer program). However

$$g.c.d.\ \{b : b \in B_0\} = 1,$$

by considering the values 126, 220 and 375. Thus Tijdeman's conjecture $\pmod{900}$ is false. ∎

Remark. Hajós [5] advanced a weaker conjecture concerning direct sum decompositions of a cyclic group G, which is that every factorization $\bar{A} \oplus \bar{B} = G$ is quasiperiodic. We say that a factorization is *quasiperiodic* if one of $\bar{A}$ or $\bar{B}$, say $\bar{B}$, can be partitioned into disjoint subsets $\{\bar{B}_1, \ldots, \bar{B}_m\}$ such that there is a subgroup $H = \{h_1, \ldots, h_m\}$ of G with

$$\bar{A} + \bar{B}_i = \bar{A} + \bar{B}_1 + h_i, \quad 1 \le i \le m .$$

The example $(\bar{A}, \bar{B})$ above for $G = \mathbb{Z}/900\mathbb{Z}$ is quasiperiodic. The choices are $H = \{0, 300, 600\}$ and

$$\begin{array}{rcl} \bar{B}_1 & = & \{\ 0 \quad 126 \quad 180 \quad 306 \quad 360 \quad 486 \quad 540 \quad 666 \quad 820 \quad 846\ \} \\ \bar{B}_2 & = & \{\ 30 \quad 210 \quad 220 \quad 300 \quad 390 \quad 480 \quad 570 \quad 660 \quad 750 \quad 840\ \} \\ \bar{B}_3 & = & \{\ 60 \quad 240 \quad 330 \quad 375 \quad 510 \quad 520 \quad 690 \quad 780 \quad 825 \quad 870\ \} . \end{array}$$

The quasiperiodicity conjecture remains open.

3. Criterion for Universal Spectrum

We formulate a sufficient condition for a universal spectrum for a periodic tiling set in $\mathbb{R}^n$, which is simpler to check than the necessary and sufficient condition given in [13, Theorem 1.1].

Given a finite set $\mathcal{B} \subseteq \mathbb{R}^n$ let $f_{\mathcal{B}}(\lambda)$ denote the exponential polynomial

$$f_{\mathcal{B}}(\lambda) := \sum_{b \in \mathcal{B}} e^{2\pi i <\lambda, b>} \tag{3.1}$$

and let $Z(f_{\mathcal{B}})$ denote its set of real zeros, i.e.

$$Z(f_{\mathcal{B}}) = \{\lambda \in \mathbb{R}^n : f_{\mathcal{B}}(\lambda) = 0\} . \tag{3.2}$$

We recall the following criterion for a set Λ to be a spectrum, taken from [13].

Proposition 3.1. *Let $\Omega = [0, \frac{1}{N_1}] \times \ldots \times [0, \frac{1}{N_n}] + \mathcal{B}$ where $\mathcal{B} \subseteq \frac{1}{N_1}\mathbb{Z} \times \ldots \times \frac{1}{N_n}\mathbb{Z}$ is a finite set. Suppose that $\Gamma \subseteq \mathbb{Z}^n$ is a set of distinct residue classes $(\mathrm{mod}\ N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z})$, i.e. $(\Gamma - \Gamma) \cap (N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z}) = \{\mathbf{0}\}$. Then $\Lambda = (N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z}) + \Gamma$ is a spectrum for Ω if and only if $|\Gamma| = |\mathcal{B}|$ and*

$$\Gamma - \Gamma \subseteq Z(f_{\mathcal{B}}) \cup \{\mathbf{0}\} . \tag{3.3}$$

[3] More generally one can consider the construction in Szabó[17].

Proof. This is [13, Theorem 2.3], after a linear rescaling of Euclidean space $\mathbb{R}^n$ of Ω by a factor $(\frac{1}{N_1}, ..., \frac{1}{N_n})$ and a corresponding dilation of Fourier space by a factor $(N_1, ..., N_n)$. ■

Theorem 3.1. (*Universal Spectrum Criterion*). *Let $\mathcal{T} = \mathbb{Z}^n + \mathcal{A}$ where $\mathcal{A} \subseteq \frac{1}{N_1}\mathbb{Z} \times \ldots \times \frac{1}{N_n}\mathbb{Z}$ is a finite set, and suppose there exists some set Ω such that Ω tiles $\mathbb{R}^n$ by translations using the tiling set $\mathcal{T}$. Consider a set $\Lambda = (N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z}) + \Gamma$ with $\Gamma \subseteq \mathbb{Z}^n$ such that the residue classes Γ (mod $N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z}$) are all distinct, i. e.*

$$(\Gamma - \Gamma) \cap (N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z}) = \{0\} \tag{3.4}$$

Then Λ is a universal spectrum for $\mathcal{T}$ provided that $|\Gamma| = \frac{N_1 N_2 \ldots N_n}{|\mathcal{A}|}$, and

$$(\Gamma - \Gamma) \cap Z(f_{\mathcal{A}}) = \emptyset\ . \tag{3.5}$$

Proof. By Theorem 3.1 of [13] a set Ω tiles $\mathbb{R}^n$ by translations with a periodic tiling set $\mathcal{T}$ if and only if there exists some finite set $\mathcal{B} \subseteq \frac{1}{N_1}\mathbb{Z} \times \ldots \times \frac{1}{N_n}\mathbb{Z}$ giving a factorization

$$\mathcal{A} \oplus \mathcal{B} = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_n} = (\frac{1}{N_1}\mathbb{Z}/\mathbb{Z}) \times \ldots \times (\frac{1}{N_n}\mathbb{Z}/\mathbb{Z}),$$

in which case $\tilde{\Omega} = [0, 1]^n + \mathcal{B}$ also has $\mathcal{T}$ as a tiling set.

By Theorem 1.1 of [13] it suffices to verify that Λ is a spectrum for each set

$$\Omega_{\mathcal{B}} = [0, \frac{1}{N_1}] \times \ldots \times [0, \frac{1}{N_n}] + \mathcal{B}$$

where

$$\mathcal{A} \oplus \mathcal{B} = (\frac{1}{N_1}\mathbb{Z}/\mathbb{Z}) \times \ldots \times (\frac{1}{N_n}\mathbb{Z}/\mathbb{Z})$$

is a direct sum decomposition. This property shows that

$$\Omega_{\mathcal{B}} + \mathcal{A} = [0, \frac{1}{N_1}] \times \ldots \times [0, \frac{1}{N_n}] + \mathcal{B} + \mathcal{A}$$

is a fundamental domain for the $n-$torus $\mathbb{R}^n/\mathbb{Z}^n$. It follows that the Fourier transform

$$\hat{\chi}_{\Omega+\mathcal{A}}(\lambda) = \int_{\Omega+\mathcal{A}} e^{2\pi i <\lambda, x>} dx$$

for $\lambda = k \in \mathbb{Z}^n$ satisfies

$$\hat{\chi}_{\Omega+\mathcal{A}}(k) = \begin{cases} 1 & \text{for} \quad k = \mathbf{0}, \\ 0 & \text{for} \quad k \in \mathbb{Z}^n \setminus \{0\}\ . \end{cases} \tag{3.6}$$

Now

$$\begin{aligned} \hat{\chi}_{\Omega}(\lambda) &= \int_{\Omega} e^{2\pi i <\lambda, x>} dx = \sum_{b \in \mathcal{B}} e^{2\pi i <\lambda, b>} \prod_{j=1}^{n} \int_0^{\frac{1}{N_j}} e^{2\pi i \lambda_j x_j} dx_j \\ &= f_{\mathcal{B}}(\lambda) \prod_{j=1}^{n} \frac{e^{\frac{2\pi i \lambda_j}{N_j}} - 1}{2\pi i \lambda_j}\ , \end{aligned}$$

which gives

$$\hat{\chi}_{\Omega+\mathcal{A}}(\lambda) = f_{\mathcal{A}}(\lambda) f_{\mathcal{B}}(\lambda) \prod_{j=1}^{n} \left(\frac{\sin \frac{\pi\lambda_j}{N_j}}{\pi\lambda_j} \right) e^{\pi i \frac{\lambda_j}{N_j}} \ . \tag{3.7}$$

Comparing (3.6) and (3.7) yields

$$f_{\mathcal{A}}(k) f_{\mathcal{B}}(k) = 0 \quad \text{if} \quad k \in \mathbb{Z}^n \setminus (N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z}) \ . \tag{3.8}$$

Thus we obtain

$$f_{\mathcal{B}}(k) = 0 \quad \text{if} \quad k \notin Z(f_{\mathcal{A}}) \cap (\mathbb{Z}^n \setminus (N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z})) \ . \tag{3.9}$$

By hypothesis $\Lambda = (N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z}) + \Gamma$ with $\Gamma \subseteq \mathbb{Z}^n$, $|\Gamma| = \frac{N_1 \ldots N_n}{|\mathcal{A}|}$,

$$(\Gamma - \Gamma) \cap (N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z}) = \{\mathbf{0}\} \ , \tag{3.10}$$

and $(\Gamma - \Gamma) \cap Z(f_{\mathcal{A}}) = \emptyset$. Thus $|\Gamma| = |\mathcal{B}|$. We claim that

$$\Lambda - \Lambda \subseteq Z(f_{\mathcal{B}}) \cup \{\mathbf{0}\} \ . \tag{3.11}$$

This claim holds since $\Lambda - \Lambda \subseteq \mathbb{Z}^n$, then noting that $Z(f_{\mathcal{B}})$ contains all points of $\mathbb{Z}^n \setminus (N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z})$ not in $Z(f_{\mathcal{A}})$ by (3.9), while (3.10) takes care of points in $N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z}$. Now Proposition 3.1 shows that Ω has Λ as a spectrum, and the theorem follows. ■

Remarks. (1). The main hypothesis in Theorem 3.1 is (3.5), which requires determining the finite set $\Gamma - \Gamma \subseteq \mathbb{Z}^n$ and evaluating $f_{\mathcal{A}}$ at these points. This condition is computationally simpler to check than the criterion of [13, Theorem 1.1], which requires determining all the complementing sets $\mathcal{B}$ to $\mathcal{A}$.

(2). It seems conceivable that the sufficient conditon of Theorem 3.1 might also be a necessary condition for a universal spectrum of the given form $\Lambda = m\mathbb{Z} + \Gamma$ with $\Gamma \subseteq \mathbb{Z}$. To show this one would have to show that for each integer vector $\mathbf{k} \in Z(f_{\mathcal{A}}) \cap (\mathbb{Z}^n \setminus N_1\mathbb{Z} \times \ldots \times N_n\mathbb{Z})$ there exists some set $\mathcal{B}$ with $\mathcal{A} \oplus \mathcal{B} = (\frac{1}{N_1}\mathbb{Z}/\mathbb{Z}) \times \ldots \times (\frac{1}{N_n}\mathbb{Z}/\mathbb{Z})$ such that $\mathbf{k} \notin Z(f_{\mathcal{B}})$.

4. Universal Spectra

We apply the criterion of Theorem 3.1 to show that the the tiling sets $\mathbb{Z} + \frac{1}{900}A$ and $\mathbb{Z} + \frac{1}{900}B$ associated to this counterexample in §2 both have universal spectra.

Theorem 4.1. *Let $\mathcal{T}_A = \mathbb{Z} + \frac{1}{900}A$ with $A = \{0, 36, 72, 108, 144\} \oplus \{0, 100, 200\} \oplus \{0, 225\}$. Then $\mathcal{T}_A$ has the universal spectrum $\Lambda_A = 900\mathbb{Z} + A$.*

Proof. We apply Theorem 3.1 with $n = 1$ and $N_1 = 900$. Then $\mathcal{A} = \frac{1}{900}A$ and $\mathcal{B} = \frac{1}{900}B$ with B given in (2.5) gives a factorization $\mathcal{A} \oplus \mathcal{B} = \frac{1}{900}\mathbb{Z}/\mathbb{Z}$. A calculation gives

$$f_{\mathcal{A}}(\lambda) = \left(\frac{1 - e^{\frac{2\pi i\lambda}{5}}}{1 - e^{\frac{2\pi i\lambda}{25}}} \right) \left(\frac{1 - e^{\frac{2\pi i\lambda}{3}}}{1 - e^{\frac{2\pi i\lambda}{9}}} \right) \left(\frac{1 - e^{\frac{2\pi i\lambda}{2}}}{1 - e^{\frac{2\pi i\lambda}{4}}} \right) \ . \tag{4.1}$$

It follows that the set $Z(f_{\mathcal{A}}) \subseteq \mathbb{Z}$ consists of all integers k such that one or more of the following three conditions hold:

(i) 5 divides k and 25 doesn't divide k.

(ii) 3 divides k and 9 doesn't divide k.

(iii) 2 divides k and 4 doesn't divide k.

The set $\Lambda_A = 900\mathbb{Z} + A$ has $\Lambda \subseteq \mathbb{Z}$ and $(A - A) \cap 900\mathbb{Z} = \{0\}$, since all $a \in A$ have $0 \leq a \leq 900$ and are distinct. Also $|A| = 130 = \frac{900}{|A|}$. To apply Theorem 3.1 it remains to verify

$$(A - A) \cap Z(f_{\mathcal{A}}) = \emptyset \ . \tag{4.2}$$

While $a_i \in A$ for $i = 1, 2$ as

$$a_i = 36k_i + 100l_i + 225m_i$$

with $0 \leq k_i \leq 4$, $0 \leq l_i \leq 2$ and $0 \leq m_i \leq 1$. Then

$$a_1 - a_2 = 36(k_1 - k_2) + 100(l_1 - l_2) + 225(m_1 - m_2)$$

with $|k_1 - k_2| < 5$, $|l_1 - l_2| < 3$ and $|m_1 - m_2| < 2$. Thus if 5 divides $a_1 - a_2$, then 5 divides $k_1 - k_2$ so $k_1 = k_2$, and we conclude 25 divides $a_1 - a_2$. By similar arguments if 3 divides $a_1 - a_2$ then 9 divides $a_1 - a_2$, while if 2 divides $a_1 - a_2$ then 4 divides $a_1 - a_2$. Thus none of (i), (ii), (iii) hold, and (4.2) follows. ■

Remark. The proof of Theorem 4.1 easily generalizes to the sets A as appearing in the general construction of Szabó [17]: All such tiling sets $\mathbb{Z} + \frac{1}{m}A$ have a universal spectrum.

Theorem 4.2. *Let $\mathcal{T}_B = \mathbb{Z} + \frac{1}{900}B$ with B given by (2.5). Then $\mathcal{T}_B$ has the universal spectrum $\Lambda_B = 900\mathbb{Z} + B$.*

Proof. We do not have a conceptual proof of this result; however the conditions of Theorem 3.1 can be verified by a direct calculation (on the computer.) A key fact is that $Z(f_{\mathcal{B}}) \cap \mathbb{Z} \subseteq \mathbb{Z} \setminus 900\mathbb{Z}$ and that the complement of the set $\mathbb{Z}(f_{\mathcal{B}}) \cap \mathbb{Z}$ in $\mathbb{Z}$ is exactly[4] $Z(f_{\mathcal{A}}) \cup 900\mathbb{Z}$. Because this fact holds, one can prove Λ_B is a universal spectrum for $\mathcal{T}_B$ by checking that

$$B - B \subseteq Z(f_{\mathcal{A}}) \cap \{0\} \ .$$

Since $Z(f_{\mathcal{A}})$ is given by conditions (i)–(iii) in the proof of Theorem 4.1, it suffices to verify that every nonzero element of $B - B$ satisfies one of (i)–(iii). This can be done by hand. ■

Remarks. In Theorem 4.1 and Theorem 4.2 the universal spectrum Λ exhibited is a scaled version of the tiling set $\mathcal{T}$. This fact is special to these examples, and cannot hold in general. Any tiling set $\mathcal{T} = \mathbb{Z} + \frac{1}{m}\mathcal{C}$ with the property that $\Lambda = m\mathbb{Z} + \mathcal{C}$ is a universal spectrum must have $m = |\mathcal{C}|^2$, by Proposition 3.1.

(2). The proof of Theorem 4.1 easily generalizes to apply to the sets A appearing in the general construction of Szabó [17]: All such tiling sets $\mathbb{Z} + \frac{1}{m}A$ have a universal spectrum.

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[4] To verify this, by (3.10) it suffices to check that $f_{\mathcal{B}}(k) \neq 0$ whenever $k \in Z(f_{\mathcal{A}})$.

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