Repetitive Delone Sets and Quasicrystals

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Abstract

This paper studies the problem of characterizing the simplest aperiodic discrete point sets, using invariants based on topological dynamics. A Delone set of finite type is a Delone set X such that $X - X$ is locally finite. Such sets are characterized by their patch-counting function $N_X(T)$ of radius T being finite for all T. We formulate conjectures relating slow growth of the patch-counting function $N_X(T)$ to the set X having a nontrivial translation symmetry.

A Delone set X of finite type is repetitive if there is a function $M_X(T)$ such that every closed ball of radius $M_X(T) + T$ contains a complete copy of each kind of patch of radius T that occurs in X. This is equivalent to the minimality of an associated topological dynamical system with $\mathbb{R}^n$-action. There is a lower bound for $M_X(T)$ in terms of $N_X(T)$, namely $M_X(T) \geq c(N_X(T))^{1/n}$ for some positive constant c depending on the Delone set constants r, R, but there is no general upper bound for $M_X(T)$ purely in terms of $N_X(T)$. The complexity of a repetitive Delone set X is measured by the growth rate of its repetitivity function $M_X(T)$. For example, the function $M_X(T)$ is bounded if and only if X is a periodic crystal. A set X is *linearly repetitive* if $M_X(T) = O(T)$ as $T \to \infty$ and is *densely repetitive* if $M_X(T) = O(N_X(T))^{1/n}$ as $T \to \infty$. We show that linearly repetitive sets and densely repetitive sets have strict uniform patch frequencies, i.e. the associated topological dynamical system is strictly ergodic. It follows that such sets are diffractive, in the sense of having a well-defined diffraction measure. In the reverse direction, we construct a repetitive Delone set X in $\mathbb{R}^n$ which has $M_X(T) = O(T(\log T)^{2/n}(\log\log\log T)^{4/n})$, but does not have uniform patch frequencies. Aperiodic linearly repetitive sets have many claims to be the simplest class of aperiodic sets, and we propose considering them as a notion of "perfectly ordered quasicrystal".

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1. Introduction

The most ordered discrete sets in $\mathbb{R}^n$ are those with crystalline order, which have a full rank lattice of global translation symmetries. Such sets in $\mathbb{R}^3$ provide good models for the atomic structure of crystals. Quasicrystalline materials are materials whose x-ray diffraction spectra have sharp spots, indicating some long-range order in the atomic structure, but which lack a lattice of periods — typically they exhibit symmetries which are impossible for any fully periodic arrangement of the atoms. A few quasicrystalline materials are thermodynamically stable with apparent zero entropy, and can be grown as "perfectly ordered quasicrystals" whose long-range atomic order seems to be as good as that of crystals, even though this atomic order is apparently aperiodic. One such example is the Al-Pd-Mn icosahedral qusicrystalline phase, described in [15]. Most quasicrystalline materials are less ordered, and are thought to have positive entropy and be entropically stabilized. The atomic structure of these materials is not known precisely, although there is now extensive theory and there are various proposed models which give reasonable agreement with data, see [20], [40], [64]. The less ordered quasicrystalline materials are modelled with random tilings, for which see [17] and [56].

The general problem motivating this paper is that of characterizing the "simplest" locally finite point sets X in $\mathbb{R}^n$ that are aperiodic. A *locally finite set* is a set with only finitely many points in any bounded region, and an *aperiodic*[2] set is one with no translation symmetries. There is an enormous variety of aperiodic sets which have some long range order, see 'The Aperiodic Zoo' in Senechal [64]. Does there exist a nice characterization of the "simplest" aperiodic sets, which might be used to define a notion of "perfectly ordered quasicrystal"? We study this question for Delone sets, and our notions of simplicity involve constraints on the patches in the set and invariants from topological dynamics.

Our definitions are inclusive and allow sets with translation symmetries. In other words, we view a fully periodic set, or "crystal", as a special kind of "quasicrystal".

[2] Some authors call such sets *nonperiodic*.

Definition 1.1. A *Delone set*, or *(r,R)-set*, is a discrete set X in $\mathbb{R}^n$ that has the two properties:

(i) *Uniform Discreteness.* There is an $r > 0$ such that each open ball of radius r in $\mathbb{R}^n$ contains at most one point of X.

(ii) *Relative Density.* There is an $R > 0$ such that each closed ball of radius R in $\mathbb{R}^n$ contains at least one point of X.

The classical measure of long range order of a Delone set is its set of global translation symmetries or, more generally, its set of global symmetries under Euclidean motions.

Definition 1.2. The *period lattice* of a Delone set X in $\mathbb{R}^n$ is the lattice of translation symmetries Λ_X of X, given by

$$\Lambda_X := \{\mathbf{t} \in \mathbb{R}^n : X + \mathbf{t} = X\} \ . \tag{1.1}$$

It is a free abelian group, with rank satisfying $0 \le \operatorname{rank}(\Lambda_X) \le n$.

The most regular Delone sets are those with a full set of translation symmetries, i.e. $\operatorname{rank}(\Lambda_X) = n$. We call such an X an *ideal crystal.* As noted earlier, a Delone set X is aperiodic if $\operatorname{rank}(\Lambda_X) = 0$, i.e. $\Lambda_X = \{\mathbf{0}\}$.

In this paper we consider sets that need not have any long range order in this classical sense, but which do have a weaker sort of order described by restrictions on their "local" structure. In [25], [26] the first author introduced several classes of Delone sets X determined by local restrictions on their set of interpoint vectors $X - X$.

Definition 1.3. (i) A Delone set X in $\mathbb{R}^n$ is *finitely generated* if the additive group $[X - X]$ in $\mathbb{R}^n$ generated by the set $X - X$ of interpoint vectors is finitely generated.

(ii) A Delone set X is a *Delone set of finite type* if $X - X$ is locally finite, i.e. the intersection of $X - X$ with any bounded set is finite.

(iii) A Delone set X is a *Meyer set* if $X - X$ is a Delone set.

These concepts form a hierarchy, because any Delone set of finite type is finitely generated, and any Meyer set is a Delone set of finite type. Almost all sets studied in quasicrystallography are Delone sets of finite type, see [25], and this is the class of sets that we study in this paper. The narrower class of Meyer sets was introduced earlier as giving useful models for quasicrystalline structures, see [38]–[41].

We consider measures of complexity of a Delone set X in $\mathbb{R}^n$ based on its "local" structure. The *T-patch* of X centered at a point $\mathbf{x} \in X$ is

$$\mathcal{P}_X(\mathbf{x};T) := X \cap B(\mathbf{x};T) \ , \tag{1.2}$$

in which $B(\mathbf{x};T)$ denotes the closed ball of a radius T centered at $\mathbf{x}$. The *T-atlas* $\mathcal{A}_X(T)$ of X consists of all T-patches of X translated to the origin:

$$\mathcal{A}_X(T) := \{\mathcal{P}_X(\mathbf{x};T) - \mathbf{x} : \mathbf{x} \in X\} \ . \tag{1.3}$$

The *atlas* $\mathcal{A}_X$ of X is the union of all T-atlases for $T > 0$. The structure of the atlas gives the following complexity measure.

Definition 1.4. The *patch-counting function*, $N_X(T)$, counts the number of T-patches in X up to translation-equivalence, i.e.

$$N_X(T) := |\mathcal{A}_X(T)| \ .$$

For most Delone sets the patch-counting function is infinite for large T. The Delone sets whose patch-counting functions are finite for all values of T are exactly the Delone sets of finite type, and for such sets the growth rate of the patch-counting function $N_X(T)$ provides a quantitative measure of the complexity of X.

The patch-counting function can be viewed as a generalization of the word-counting function or *permutation number* $p(T)$ that is studied in the combinatorics of words, and also generalizes the *(symbolic) complexity function* studied in symbolic dynamics, as in Ferenczi [11]. We give a more detailed overview of these areas at the end of this section.

In [25] it was shown that $N_X(T)$ satisfies the upper bound

$$N_X(T) \leq \exp(c_0 T^n) \quad \text{for all } T > 0 \ , \tag{1.4}$$

with a positive constant $c_0 = c_0(X)$. One can further show that any Delone set of finite type has a finite *configurational entropy*

$$H_c(X) := \lim_{T \to \infty} \frac{1}{T^n} \log N_X(T) \ . \tag{1.5}$$

(The existence of the limit can be established by a subadditivity argument.) At the other extreme, the slowest possible growth rate for $N_X(T)$ is to be eventually constant, which occurs when X is an ideal crystal. In this case the eventual value of $N_X(T)$ is equal to the number of points of X in a fundamental domain for the lattice of translations of X. It appears that sufficiently slow growth of the patch counting function $N_X(T)$ forces a set X to have global translation symmetries. In this direction we formulate the following conjecture.

Conjecture 1.1 *Any aperiodic Delone set X in $\mathbb{R}^n$ satisfies*

$$\limsup_{T \to \infty} \frac{N_X(T)}{T^n} > 0. \tag{1.6}$$

In §2 we formulate stonger conjectures, and give examples showing that one cannot replace lim sup by lim inf in this conjecture, when the dimension $n \geq 3$.

A weak type of translational order for a Delone set is given by the notion of repetitivity, and carries with it an associated complexity measure.

Definition 1.5. A Delone set X is *repetitive*[3] if for each $T > 0$ there is a finite number $M > 0$ such that every closed ball B of radius M contains the center of a translate of every possible T-patch in X. That is, for every T-patch $\mathcal{P}$ in X, B contains a point of X which is the center of a T-patch of X that is a translate of $\mathcal{P}$. (This T-patch may extend outside the ball B.) The *repetitivity function* $M_X(T)$ is the least such radius M for each $T > 0$.

[3] There is a parallel concept for tilings, where various terms are used: "almost periodic" in [57, 66], "tilings with local isomorphism" in [16, 55], and "repetitive" in [64]. In symbolic dynamical systems the term is "uniformly recurrent".

Any repetitive Delone set is necessarily a Delone set of finite type, since the definition implies that $N_X(T)$ is finite for all T. The repetitivity function $M_X(T)$ of such a set provides an additional quantitative measure of its complexity.

A slightly different measure of repetitivity, which we denote by $M'_X(T)$, is the least radius M such that every ball of radius M contains a translate of every T-patch $\mathcal{P}$ of X. Clearly

$$M'_X(T) = M_X(T) + T. \tag{1.7}$$

Both $M_X(T)$ and $M'_X(T)$ are increasing functions of T. Our reasons for choosing $M_X(T)$ instead of $M'_X(T)$ as the fundamental repetitivity function are the following.

(i) $M_X(T)$ is the exact analogue of the *recurrence function* in symbolic dynamics, which was introduced in 1938 by Morse and Hedlund [42, 43].

(ii) $M_X(T)$ shares with $N_X(T)$ the property that ideal crystals can be characterized by $M_X(T)$ being bounded for all T.

(iii) $M_X(T)$ has a natural description as the largest of the covering radii of the sets of centers of the different translation-equivalent T-patches of X.

The complexity measures of a Delone set X that we study in this paper are more generally properties of the ensemble of sets that agree with X everywhere locally.

Definition 1.6 The *local isomorphism class*[4] $[[X]]$ of X (under translations) consists of all sets Y that have the same atlas as X, i.e.

$$[[X]] := \{Y : \mathcal{A}_Y = \mathcal{A}_X\} . \tag{1.8}$$

Two sets Y and Y' in the same local isomorphism class cannot be distinguished by examining finite regions, because each finite region of one appears in the other after a suitable translation. In measuring the complexity of sets X it seems natural to consider all elements of a local isomorphism class $[[X]]$ to be of equal complexity. Many geometric quantities associated to a Delone set X in $\mathbb{R}^n$ are actually invariants of the local isomorphism class $[[X]]$, including the properties of being a finitely generated Delone set, a Delone set of finite type, or a Meyer set.

Local isomorphism classes of Delone sets have a natural complexity ordering, given by the partial order $\leq_{\mathcal{T}}$ defined by[5]

$$[[X']] \leq_{\mathcal{T}} [[X]] \quad \text{if } \mathcal{A}_{X'} \subseteq \mathcal{A}_X . \tag{1.9}$$

If f is a function whose arguments are discrete point sets in $\mathbb{R}^n$ and whose values lie in a partially ordered set, we call f *monotonic* if either

$$[[X']] \leq_{\mathcal{T}} [[X]] \Rightarrow f(X') \leq f(X) \quad \text{or} \quad [[X']] \leq_{\mathcal{T}} [[X]] \Rightarrow f(X') \geq f(X) ,$$

i.e. the ordering of $f(\cdot)$ is compatible with the partial ordering above. In the former case f is *monotonic increasing* and in the latter *monotonic decreasing*. Monotonic functions are automatically local isomorphism class invariants. We allow such functions to take values in the extended positive real numbers $\mathbb{R}^+ \cup \{0, \infty\}$.

[4]The term "local indistinguishability class" has also been proposed for this concept.

[5]The suffix $\mathcal{T}$ is to indicate that the partial order derives from inclusion of translation atlases. In [26] this partial order is considered along with a partial order $\leq_{\mathcal{I}}$ derived from inclusion of isometry atlases.

Lemma 1.1 *The following functions of a discrete point set X in $\mathbb{R}^n$ are monotonic and, in particular, are local isomorphism invariants.*

(i) The Delone set parameters r (monotonic decreasing) and R (monotonic decreasing).

(ii) The interpoint vector set $X - X$ (monotonic increasing; the partial order is inclusion).

(iii) The patch-counting function $N_X(T)$ (monotonic increasing).

(iv) The repetitivity function $M_X(T)$ (monotonic increasing).

(v) The period lattice Λ_X (monotonic decreasing; the partial order is inclusion).

Proof. A function f is monotonic increasing if it is the supremum of some function g defined on the atlas $\mathcal{A}_X$ and is monotonic decreasing if it is the infimum of some such function g. It is straightforward to verify that the functions in this theorem all fall into one of these two categories. $\square$

This complexity ordering leads to a notion of "simplest" Delone sets X as those for which $[[X]]$ is a minimal element in the partial order on local isomorphism classes. We call such sets *translation-minimal Delone sets.* In §3 we review several equivalent notions of minimality, following [26]. For Delone sets of finite type the notion of translation-minimality is shown to be equivalent both to repetitivity and to minimality of an associated topological dynamical system with an $\mathbb{R}^n$-action. The topological dynamical system $(\mathcal{X}_X, \mathbb{R}^n)$ associated to X is defined analogously to tiling dynamical systems, for which see [55]–[58] and [66]. In the case of a Delone set of finite type which is translation-minimal, one has $[[X]] = \mathcal{X}_X$, while in general $[[X]] \subseteq \mathcal{X}_X$. In connection with models for quasicrystals, Radin [53] and Radin and Wolff [55] have argued that minimality of such a dynamical system is analogous to being in a "ground state". Since the diffraction spectra of quasicrystals indicate long range translational order, translation-minimality seems a natural condition to require of any structure modelling a thermodyamically stable quasicrystalline material with zero entropy. It is however a less restrictive condition, since there exist translation-minimal sets with positive configurational entropy, cf. Furstenberg [12, Theorem III.2].

We note that the dynamical systems $(\mathcal{X}_X, \mathbb{R}^n)$ studied in §3 include systems isomorphic to any symbolic dynamical system with a $\mathbb{Z}^n$-action. These can be encoded as Delone sets $X \subset \mathbb{Z}^n$, with translations restricted to $\mathbb{Z}^n$. To encode systems with a finite number of symbols, we encode the symbols as specific patterns of points in a "block" consisting of a cubical integer grid of a fixed size. One can find such encodings that ensure the Delone set property and allow the location of the edges of "blocks" to be recognized.

We consider translation-minimality to be a necessary requirement for the "simplest" aperiodic sets. The main object of this paper is to establish that, for a translation-minimal Delone set of finite type, slow growth of the repetitivity function $M_X(T)$ puts significant restrictions on its structure, as exemplified by Theorem 6.1. This theorem implies, in particular, that if the repetitivity function of such a set grows at most linearly in T then the set is "diffractive" in the sense defined in §6.

In §4 we give the lower bound

$$M_X(T) \geq r(N_X(T)^{1/n} - 1) \ . \tag{1.10}$$

for the growth rate of the repetitivity function $M_X(T)$ in terms of the patch-counting function $N_X(T)$, and in Corollary 4.2b we show that there is no general upper bound, even when X is restricted to the class of Meyer sets and $N_X(T) = O(T^n)$.

The main part of the paper, §5–§8, concerns repetitive Delone sets that have the slowest possible growth rate of $M_X(T)$ in one sense or another.

Definition 1.7. A repetitive Delone set of finite type is *densely repetitive* if

$$M_X(T) = O((N_X(T))^{1/n}) \quad \text{as} \quad T \to \infty \ . \tag{1.11}$$

By (1.10) this is the slowest possible growth rate of $M_X(T)$ relative to the number of patches $N_X(T)$. We also formulate the following definition bounding the growth rate of $M_X(T)$ in terms of T.

Definition 1.8. A repetitive Delone set of finite type is *linearly repetitive* if

$$M_X(T) = O(T) \quad \text{as} \quad T \to \infty \ . \tag{1.12}$$

Aside from ideal crystals, linearly repetitive sets have the slowest possible growth rate of $M_X(T)$ in absolute terms, as we indicate in §8. However, the precise relationship between the concepts of linear and dense repetitivity is not clear at present, and the theory would simplify if the following were true.

Conjecture 1.2a *All aperiodic linearly repetitive sets are densely repetitive.*

In §8 we show that an equivalent form of this conjecture is:

Conjecture 1.2b *Any aperiodic linearly repetitive set X in $\mathbb{R}^n$ satisfies*

$$\liminf_{T\to\infty} \frac{N_X(T)}{T^n} > 0. \tag{1.13}$$

Added in proof: Since writing this, Conjectures 1.2a and 1.2b have been proved by Lenz [29].

Many constructions in the quasicrystal literature produce linearly repetitive Delone sets. Linear repetitivity of the Penrose tiling is shown in Grünbaum and Shephard [16, p.563]. More generally, self-similar tiling constructions with the unique composition property yield linearly repetitive sets, see Solomyak [67, Lemma 2.3].

In §5–§8 we prove results about the structure of densely repetitive sets and linearly repetitive sets. In §5 we show that a certain class of measure-like functions determined by the local structure of X, which we call "weight distributions", are approximately a constant multiple of volume when X is densely repetitive. In §6 this result is used to establish the main result of the paper, which is that densely repetitive sets and linearly repetitive sets have strict uniform patch frequencies. (This notion is defined in §6.) This result is equivalent to asserting that the dynamical system with $\mathbb{R}^n$-action associated to X is strictly ergodic; it thus uses a hypothesis in topological dynamics to draw a conclusion in metrical dynamics. The result implies that any densely or linearly repetitive set X has a uniquely defined diffraction measure in the sense of

Hof [18, 19]. Thus X has long-range order under translations in the sense of [25]. We conclude §6 with an example showing that the hypotheses of this result are nearly best possible: there exists a repetitive Delone set X in $\mathbb{R}^n$ with repetitivity function satisfying

$$M_X(T) = O(T(\log T)^{2/n}(\log\log\log T)^{4/n}) \ , \tag{1.14}$$

which does not have uniform patch frequencies.

A finitely generated Delone set X has an associated *address map* $\phi : [X] \to \mathbb{Z}^s$ which is obtained by choosing a fixed basis of the finitely generated abelian group $[X]$ spanned by the vectors in X. Address maps were introduced in [25] as a way to study the regularity of the structure of such Delone sets. Delone sets X of finite type were characterized as those finitely generated Delone sets that satisfy the Lipschitz-type condition that there exists a constant C such that

$$\|\phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)\| \le C\|\mathbf{x}_1 - \mathbf{x}_2\|, \quad \text{for } \mathbf{x}_1, \mathbf{x}_2 \in X, \tag{1.15}$$

where the norms are Euclidean norms on $\mathbb{R}^s$ and $\mathbb{R}^n$ respectively. Meyer sets were characterized by the existence of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^s$ and a constant C' such that

$$\|\phi(\mathbf{x}) - L(\mathbf{x})\| \le C' \ , \quad \text{for } \mathbf{x} \in X. \tag{1.16}$$

In §7 we show that densely repetitive sets satisfy an intermediate property: for each densely repetitive set X there exists a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^s$ such that

$$||\phi(\mathbf{x}) - L(\mathbf{x})|| = o(||\mathbf{x}||), \quad \text{for } \mathbf{x} \in X. \tag{1.17}$$

For linearly repetitive sets X we obtain the stronger result that there exists a constant $\delta = \delta(X) > 0$ such that

$$\|\phi(\mathbf{x}) - L(\mathbf{x})\| = O(\|\mathbf{x}\|^{1-\delta}) \quad \text{for } \mathbf{x} \in X. \tag{1.18}$$

We complement this by showing that there are linearly repetitive sets X for which there is a sequence $\{\mathbf{x}_i\} \subset X$ with $\|\mathbf{x}_i\| \to \infty$ and

$$||\phi(\mathbf{x}_i) - L(\mathbf{x}_i)|| > ||\mathbf{x}_i||^{1-\delta} \ ,$$

for some $\delta < 1$.

In §8 we discuss aperiodic linearly repetitive sets as candidates for the title of simplest aperiodic sets and propose this class of sets as a possible notion of "perfectly ordered quasicrystal". We exhibit the large class of one-dimensional linearly repetitive sets associated to Beatty sequences of the real numbers $\alpha \in [0, 1]$ whose continued fraction expansions have bounded partial quotients. Finally, we raise the question to what extent does the property of being a linearly repetitive set X put algebraic restrictions on the image of the linear mapping L in (1.17).

We conclude this introduction with a brief review of related results in the symbolic dynamics literature. The analogue of the patch-counting function $N_X(T)$ is the *block growth function*, which in higher-dimensional cases is sometimes called the *(symbolic) complexity function* $c(R)$, which counts the number of rectangular blocks of shape R. It has long been known that in the one dimensional case the bound $c(m) \le m$ for a single value of m implies periodicity. Coven and Hedlund [6] and Coven [5] characterize all symbolic sequences with minimal aperiodic complexity $c(m) = m + 1$. Among other studies relating dynamical properties to one-dimensional complexity are Allouche [1] and Ferenczi [11]. A review of results concerning two-dimensional rectangular symbolic complexity appears in Berthé and Vuillon [3]. Various positive and negative results relating low symbolic complexity growth to periodicity in two and higher dimensions appear in Sander and Tijdeman [59, 60]. The analogue of the repetitivity function $M_X(T)$ in symbolic dynamics might be called the *(uniform) recurrence function,* see Furstenberg [13, p. 27], and has been little studied. Boshernitzan [4] gives a result concerning slow growth of this function in the one-dimensional case.

2. Growth of the Patch-Counting Function and Periodicity

We study the question to what extent does slow growth of the patch-counting function $N_X(T)$ of a Delone set of finite type force it to have global translation symmetries. In §1 we conjectured that $\limsup N_X(T)/T^n > 0$ for aperiodic Delone sets of dimension n. Here we formulate stronger conjectures and give limited evidence for them. In the opposite direction, for dimensions $n \geq 3$ we construct aperiodic Delone sets of finite type that have $\liminf N_X(T)/T^n$ equal to zero.

Sufficiently slow growth of $N_X(T)$ is known to imply the existence of a full rank lattice of translation symmetries.

Theorem 2.1 *If a Delone set X with constants (r, R) satisfies*

$$N_X(T) < \left\lfloor \frac{T}{R} \right\rfloor \quad \textit{for some } T \tag{2.1}$$

then it is an ideal crystal, i.e. it has a full rank lattice Λ_X of translation symmetries.

A proof of this is given in [27], where it is shown to be best possible, in the sense that for every $\epsilon > 0$ there are sets X in $\mathbb{R}^n$ with

$$N_X(T) < \left(\frac{1}{R} + \epsilon\right) T \quad \text{for all } T > T_0$$

that are not ideal crystals.

A version of Theorem 2.1 holds even when $N_X(T)$ is replaced with the function $N_X^*(T)$ that counts patches up to isometry, cf. Dolbilin et al. [8, Theorem 1.3]. They showed that if $N_X^*(T) < c(n, r, R)\ T$ with

$$c(n, r, R)^{-1} = 2(n^2 + 1) R \log_2\left(\frac{2R}{r} + 2\right), \tag{2.2}$$

then X is an ideal crystal. If we count patches under isometry, there is no analogue of this result allowing a faster growth rate that forces the existence of a lower rank lattices of periods. Dolbilin and Pleasants [9] show that for each $n \geq 3$ and $\epsilon > 0$ there is an aperiodic Delone set X with $N_X^*(T) < T^{1+\epsilon}$. The Delone sets X in these counterexamples are not of finite type; their construction uses twists through irrational multiples of π to break a rank $n-1$ periodicity and make $N_X(T)$ infinite for every T without significantly increasing $N_X^*(T)$. Whether such counterexamples can occur in dimension $n = 2$ is an open question.

For the function $N_X(T)$ counting patches under translations, we believe there do exist weaker conditions on the growth rate of $N_X(T)$ which will force the existence of some translation symmetries. Such conditions are not known. For definiteness we formulate here two conjectures in arbitrary dimension, in the strongest form that remains compatible with known counterexamples. The first is given in the contrapositive form.

Conjecture 2.1 *For any dimension $n \geq 1$ and Delone constants (r, R) there is a positive constant $c = c(n, r, R)$ such that any Delone set X in $\mathbb{R}^n$ with constants (r, R) that is aperiodic satisfies*

$$\limsup_{T\to\infty} \frac{N_X(T)}{T^n} \geq c(n, r, R) \ . \tag{2.3}$$

This conjecture strengthens Conjecture 1.1 in requiring the constant c to depend only on n, r and R. If true, this result would be best possible in the sense that there exist aperiodic Delone sets X of finite type in $\mathbb{R}^n$ with

$$N_X(T) \leq (3T)^n \quad \text{for all } T \geq 1 \ ,$$

for example $X := \mathbb{Z}^n - \mathbf{0}$.

Conjecture 2.1 is the special case $j = 1$ of the following more specific conjecture that relates slow growth of $N_X(T)$ to the rank of the period lattice of X.

Conjecture 2.2 *Let the dimension n be given. For each j with $1 \leq j \leq n$ there is a positive constant $c_j = c_j(n, r, R)$ such that any Delone set X in $\mathbb{R}^n$ with constants (r, R) that satisfies*

$$\frac{N_X(T)}{T^{n+1-j}} < c_j \quad \textit{for all } T > T_0(X)$$

has at least j linearly independent periods, i.e. $\mathrm{rank}(\Lambda_X) \geq j$.

Theorem 2.1 establishes the case $j = n$ of Conjecture 2.2 (which includes the case $n = 1$ of Conjecture 2.1) in a very strong form. All other cases of these conjectures are open.

We now show that Conjecture 2.2 above cannot be strengthened to obtain the conclusion from a single value of T, when $j \leq n - 2$ and $n \geq 3$. This contrasts with the case $j = n$, where Theorem 2.1 shows that the conclusion follows from a single value of T satisfying (2.1). In fact not even having the hypothesis for a sequence of values of T tending to infinity is sufficient, because we show that it is possible for $N_X(T)/T^{n+1-j}$ to fluctuate widely as $T \to \infty$. We prove this below by modifying a construction of Sander and Tijdeman [59], who gave an example in $\mathbb{R}^3$ consisting of $\mathbb{Z}^3$ with two widely separated skew lines of lattice points removed.

Theorem 2.2 *In dimension $n \geq 3$ there exist aperiodic repetitive Meyer sets X such that*

$$\liminf_{T\to\infty} \frac{N_X(T)}{T^{\lceil (n+1)/2 \rceil + \epsilon}} = 0 \tag{2.4}$$

for every $\epsilon > 0$.

Remark. The sets we construct satisfy

$$\limsup_{T\to\infty} \frac{N_X(T)}{T^n} > 0 \ ,$$

as Conjecture 1.1 would require.

Proof. The examples we shall give will be inductively constructed subsets of the lattice of integer points in $\mathbb{R}^n$. We describe in detail the case $n = 3$, but the argument generalizes easily to space of arbitrary dimension. We start from an idea of Sander and Tijdeman [59] that if two skew lines of integer points are removed from the integer lattice the resulting Delone set has $N(T) \leq CT^2$ when T is less than the distance between the lines but $N(T) > cT^3$ when T is significantly greater than the distance between the lines. The other ingredient is to notice that if the configuration of skew lines is repeated periodically then the order of $N(T)$ as a function of T declines as T becomes large in comparison to the fundamental region of the periodicity. At the next stage we alter the fundamental regions that lie along skew lines at a larger scale. (The simplest alteration is to replace the lines of points that were removed from these lines of regions.) This has the effect of again achieving $N(T) > cT^3$ when T is significantly greater than the distance between the new skew lines. We now repeat this configuration periodically with a huge fundamental region to slow down the growth of $N(T)$ again, and so on. In this way the size of $N(T)$ can be induced to fluctuate between cT^3 and not much bigger than T^2 as T tends to infinity.

To make this explicit, choose a positive integer a_1 and remove from $\mathbb{Z}^3$ the three systems of lines of lattice points parallel to the axes

$$S_x = \{\mathbf{x} : \mathbf{x} \equiv (0, a_1, -a_1) \pmod{(1, 4a_1, 4a_1)}\}$$

and the corresponding systems S_y and S_z with the coordinates cyclically permuted. (Two systems of lines would be enough — we take three for the sake of symmetry.) Denote by X_1 the resulting subset of $\mathbb{Z}^3$ and call the lines that have been removed X_1*-lines.* Then X_1 has period $4a_1$ in each of the coordinate directions. Each X_1-line is the center line of an infinite right square prism of width $2a_1$ and the interiors of these prisms are disjoint (though the prisms of non-parallel X_1-lines may touch), so the minimum distance between X_1-lines is $2a_1$. Hence

$$N_{X_1}(T) \leq 12T^2 \quad \text{for} \quad T \leq a_1,$$

since any ball of radius T meets at most one of the deleted lines and $4T^2$ is a trivial upper bound for $N_X(T)$ when X is $\mathbb{Z}^3$ with a line of lattice points parallel to a coordinate axis removed.

Now choose $a_2 = (4b_2 + 1)a_1$ for some positive integer b_2 and let X_2 be the set derived from X_1 by replacing the subsystem

$$\{\mathbf{x} : \mathbf{x} \equiv (0, a_2, -a_2) \pmod{(1, 4a_2, 4a_2)}\}$$

and the corresponding subsystems of S_y and S_z with the coordinates cyclically permuted. (Since $a_2 \equiv a_1 \pmod{4a_1}$, these are subsystems of the X_1-lines.) The periodicity of X_1 is broken on X_2 only by the presence of certain lines, the X_2*-lines*, whose minimum distance apart is $2a_2$, and consequently

$$N_{X_2}(T) \leq 48a_1T^2 \quad \text{for} \quad T \leq a_2$$

since a ball of radius T meets at most one X_2-line (hence a factor $12T^2$ on the right) and the subset of X_2 in the cylinder of radius $2T$ centered on an X_2-line has period $4a_1$ in the direction of the line (hence a factor $4a_1$). The set X_2 itself has period $4a_2$ in each coordinate direction.

This procedure can be iterated. At the next step we choose $a_3 = (4b_3 + 1)a_2$ for some positive integer b_3 and form X_3 by removing again points $\mathbf{x}$ with

$$\mathbf{x} \equiv (0, a_3, -a_3) \pmod{(1, 4a_3, 4a_3)}$$

(and their images under cyclic permutations of the axes). For X_3 we have

$$N_{X_3}(T) \leq 48a_2T^2 \quad \text{for} \quad T \leq a_3 \ .$$

In general we have,

$$N_{X_j}(T) \leq 48a_{j-1}T^2 \quad \text{for} \quad T \leq a_j \ . \tag{2.5}$$

Since the alterations needed to obtain each set from the previous one occur further and further from the origin, the sequence of sets $X_1, X_2, X_3, \ldots$ converges to a set X, which is a Meyer set with Delone constants $r = 1/2$ and $R = \sqrt{5}/2$.

We next note that

$$N_X(T) = N_{X_j}(T) \quad \text{for} \quad T \leq a_j/2 \ . \tag{2.6}$$

This is because each T-patch $\mathcal{P}$ of X meets at most one X_j-line. The integer points on this line are either all in X or all in the complement of X. If their status is the same in X as in X_j then $\mathcal{P}$ is also a T-patch of X_j. If the status in X of the X_j-line is the reverse of its status in X_j then we can find a $\mathbf{y} \equiv \mathbf{x} \pmod{4a_{j-1}}$ whose distance from all X_j-lines is $\geq T$, and the T-patch of $\mathbf{y}$ in X_j is a translate of the T-patch of $\mathbf{x}$ in X. We also see from this argument that X is repetitive with $M_X(T) \leq 4\sqrt{3}a_j$ (the diameter of the fundamental region of X_j) for $T \leq a_j/2$. Now

$$N_X(T) \leq 48a_{j-1}T^2 \quad \text{when} \quad T = a_j/2 \ ,$$

so choosing $a_j > 2e^{a_{j-1}}$ gives

$$N_X(T) \leq 48T^2 \log T \quad \text{for} \quad T = a_j/2$$

and the sequence $\{a_j/2\}$ tends to infinity. If instead we take $T = 2a_j$ and consider T-patches with centers in the ball of radius $a_j - 1$ about the midpoint $\mathbf{m} = (-a_j, a_j, 0)$ of the perpendicular transversal of the closest X_j-lines to the origin in the x and y directions (the distance between these lines being $2a_j$) then each T-patch will meet only these two X_j-lines in these directions, which will be identifiable by the break in $4a_{j-1}$ periodicity thus allowing the position of $\mathbf{m}$ within the patch to be identified. These T-patches are therefore all distinct. Hence $N_X(T)$ is of order at least $(\pi/6)T^3$ for these values of T, confirming the remark preceding this proof. In the opposite direction, it follows from (2.5) and (2.6) that $N_X(T) \leq 96T^3$ for all T.

The function $N_X(T)$ grows in spurts: near the a_j's it increases by a factor nearly T while T is increasing only by a factor 4, but between the a_j's its growth is steady.

Finally, suppose that $\mathbf{p} = (p, q, r)$ is a period of X. Since the planes $x = a$ that have some lines parallel to the z-axis deleted all have $a \equiv a_1 \pmod{4a_1}$, p is a multiple of $4a_1$. Since the planes $x = a$ that have a proportion $< 1/4a_1$ of the lines parallel to the z-axis deleted all have $a \equiv a_2 \pmod{4a_2}$, p is a multiple of $4a_2$. Since the planes $x = a$ that have a proportion $< 1/4a_1$

but $> 1/4a_1 - 1/4a_2$ of the lines parallel to the z-axis deleted all have $a \equiv a_3 \pmod{4a_3}$, p is a multiple of $4a_3$. Continuing like this, we see that $p = 0$. Similarly, $q = 0$ and $r = 0$, so X is aperiodic.

A similar construction works for a general dimension $n \geq 3$ with the systems of non-intersecting lines replaced by systems of non-intersecting $\lfloor (n-1)/2 \rfloor$-dimensional affine subspaces. $\square$

Remark. Theorem 2.2 implies that the hypothesis of linear repetitivity is needed in Conjecture 1.2b, for $n \geq 3$; repetitivity alone is not sufficient for the conclusion.

3. Minimality

This section formulates various notions of minimality and shows their equivalence. We associate to any Delone set X a topological dynamical system $(\mathcal{X}_X, \mathbb{R}^n)$ with an $\mathbb{R}^n$-action, and show that for Delone sets of finite type the concepts of translation-minimality, minimality in the sense of topological dynamics, and repetitivity are equivalent. These results were discussed in [26, 68], but the topology on the space $\mathcal{X}_X$ given there is not quite right; a topology with the required properties is given in Lenz and Stollmann [30], and we follow their definition below. One wants a topology on the collection of all closed sets in $\mathbb{R}^n$ in which translation of sets is continuous and the collection of all closed sets is compact.

We describe a topology on the collection $\mathcal{F}(\mathbb{R}^n)$ of closed sets in $\mathbb{R}^n$. The *natural topology* on the collection of closed sets $\mathcal{F}(\mathbb{R}^n)$ in $\mathbb{R}^n$ is defined by a neighborhood basis. Given an integer $k \geq 1$, and closed sets F_1, F_2 define the modified distance function

$$d_k(F_1, F_2) := \inf\{\delta : \ F_1 \cap B(\mathbf{0}; k) \subset F_2 + B(\mathbf{0}; \delta), F_2 \cap B(\mathbf{0}; k) \subset F_1 + B(\mathbf{0}; \delta) \vee \delta = 1\}.$$

The function $d_k(F_1, F_2)$ does not satisfy the triangle inequality, but it is monotone in the cutoff parameter k, i.e. $d_k(F_1, F_2) \leq d_{k+1}(F_1, F_2)$, and it also satisfies

$$d_k(F_1, F_2) \leq d_H(F_1 \cap B(\mathbf{0}; k), F_2 \cap B(\mathbf{0}; k)),$$

where $d_H(K_1, K_2)$ is the *Hausdorff distance*, defined for compact sets K_1, K_2 by

$$d_H(K_1, K_2) = \max\left(\max_{\mathbf{x}_1 \in K_1} \min_{\mathbf{x}_2 \in K_2} ||\mathbf{x}_1 - \mathbf{x}_2||\,,\ \max_{\mathbf{x}_2 \in K_2} \min_{\mathbf{x}_1 \in K_1} ||\mathbf{x}_1 - \mathbf{x}_2|| \right).$$

The natural topology is defined by taking as a basis of neighborhoods of a closed set F all the sets

$$N_{k,\epsilon}(F) := \{F' \in \mathcal{F}(\mathbb{R}^n) : d_k(F, F') < \epsilon\}$$

for $k \in \mathbb{Z}^+, \epsilon > 0$. In this topology two closed sets are "close" if they approximately coincide in a large ball around the origin. This topology is metrizable but not in any canonical way, see [30]. It induces a topology on the set of all (r, R) sets by restriction. (The "metrics" proposed in [26, 68] for the topology on Delone sets are not metrics; they violate the triangle inequality.)

In particular, a sequence of Delone sets $\{X_m : \ m \geq 1\}$ which are all (r, R)-sets converges in this topology to a limit set X, necessarily also a Delone set with parameters (r, R), if and only if the convergence is "pointwise" in the sense that each point $\mathbf{x} \in X$ is a limit of a sequence of points $\mathbf{x}_m \in X_m$, and for arbitrarily large radius T and arbitrarily small $\epsilon > 0$, for all sufficiently large m all points in X_m within distance T of the origin are within ϵ of some point in X.

Definition 3.1. A *Delone dynamical system* $(\mathcal{X}, \mathbb{R}^n)$ is a collection of Delone sets $\mathcal{X}$ in $\mathbb{R}^n$ that is closed in the natural topology and also closed under translations, i.e.

$$X \in \mathcal{X} \Rightarrow X + \mathbf{t} \in \mathcal{X} \quad \text{for all } \mathbf{t} \in \mathbb{R}^n \ . \tag{3.1}$$

It is a topological dynamical system with the $\mathbb{R}^n$-action given by translations.

A Delone set X in $\mathbb{R}^n$ generates a Delone dynamical system $(\mathcal{X}_X, \mathbb{R}^n)$, where $\mathcal{X}_X$ is the closure in the natural topology of the $\mathbb{R}^n$-orbit of X which is

$$\mathcal{O}_X := \{X + \mathbf{t} : \mathbf{t} \in \mathbb{R}^n\} \ . \tag{3.2}$$

Clearly $\mathcal{O}_X \subseteq [[X]] \subseteq \mathcal{X}_X$. This construction of $\mathcal{X}_X$ is analogous to those developed for tiling dynamical systems, for which see Radin [54], Radin and Wolff [55], Robinson [57, 58] and Solomyak [66, Section 2]. It gives a wide class of interesting dynamical systems, essentially including symbolic dynamical systems as a special case[6] when the points of X form a Delone subset of the lattice $\mathbb{Z}^n$.

Definition 3.2. A Delone dynamical system $(\mathcal{X}, \mathbb{R}^n)$ is *minimal* if $\mathcal{X}_X = \mathcal{X}$ for each $X \in \mathcal{X}$. Equivalently, for each $X \in \mathcal{X}$ the orbit $\mathcal{O}_X$ is dense in $\mathcal{X}$.

Theorem 3.1 *Let X be a Delone set of finite type in $\mathbb{R}^n$. Then:*

(i) $\mathcal{X}_X$ is a compact set in the natural topology.

(ii) $Y \in \mathcal{X}_X \Leftrightarrow [[Y]] \leq_{\mathcal{T}} [[X]]$.

(iii) There exists a $Y \in \mathcal{X}_X$ such that $(\mathcal{X}_Y, \mathbb{R}^n)$ is minimal.

Proof. (i) It suffices shows that every sequence of translates of a set X has a subsequence that converges to a limit Delone set "pointwise" in the sense given above. Convergence in this sense follows by a standard argument in topological dynamics, using König's Infinity Lemma and the fact that X has finitely many translation-inequivalent patches of radius T, cf. Radin and Wolff [55], Robinson [57, 58], Solomyak [66, Lemma 2.1]. (Alternatively, one can use the fact that it $\mathcal{X}_X$ is a closed subset of the set $\mathcal{F}(\mathbb{R}^n)$ of all closed sets in $\mathbb{R}^n$ with the natural topology, which is compact by Lenz and Stollmann [30, Theorem 1.1].)

(ii) It is clear that if $[[Y]] \leq_{\mathcal{T}} [[X]]$ then $Y \in \mathcal{X}_X$. For the converse we have to show that every T-patch $\mathcal{P}_Y(\mathbf{y}, T)$ of Y is translation equivalent to a T-patch of X. Since X is of finite type there are only finitely many vectors $\mathbf{v} \in X - X$ with $||\mathbf{v}|| \leq T$. Let δ be the minimum distance between them and put $k = \max(||\mathbf{y}|| + T + \delta/2, 2/\delta)$. If $X + \mathbf{t}$ is in the neighborhood $N_{k,\delta/2}(Y)$ of of Y in the natural topology then, within a radius $T + \delta/2$ of $\mathbf{y}$, each point of either $X + \mathbf{t}$ or Y is within a distance $\delta/2$ of a point of the other. In partiular, since the interpoint distance set of $Y - Y$ is contained in that of $X - X$, all the interpoint distances of the corresponding points of Y and of X inside radius $T + \delta/2$ must match, by the definition of δ. It follows that $\mathcal{P}_Y(\mathbf{y}, T)$ is translation equivalent to $\mathcal{P}_X(\mathbf{x}, T)$.

(iii) This follows by a Zorn's lemma argument, using the compactness of $\mathcal{X}_X$, see for example Furstenberg [13, p. 29]. $\square$

[6] One can encode symbols by viewing the points of the symbol space as k-cubes in the lattice $k\mathbb{Z}^n$ and allowing different configurations of points in the k-cubes, as mentioned in §1.

Theorem 3.2 *Let X be a Delone set of finite type. The following conditions are equivalent.*
(i) $[[X]]$ is translation-minimal.
(ii) The topological dynamical system $(\mathcal{X}_X, \mathbb{R}^n)$ is minimal. That is, $\mathcal{X}_X = [[X]]$.
(iii) X is repetitive.

Proof. This is a form of Gottschalk's Theorem, see for example [48, p. 136] where it is proved with the group C_∞ in place of $\mathbb{R}^n$.

(i) $\Leftrightarrow$ (ii). This is immediate from (ii) of Theorem 3.1.

(i) $\Rightarrow$ (iii). We prove the contrapositive. Suppose X is not repetitive. Then there is a T-patch $\mathcal{P}$ of X for which there exists a sequence of patches $\{\mathcal{P}(\mathbf{x}_i; T_i) : i \geq 1\}$, with $T_i \to \infty$ as $i \to \infty$, none of which contain a translate of $\mathcal{P}$. Since $\mathcal{X}_X$ is compact we can extract a subsequence of the sequence of sets $\{X - \mathbf{x}_i\}$ which converges to a limit set $Y \in \mathcal{X}_X$ in the natural topology. Now Y contains no translate of $\mathcal{P}$, hence $[[Y]] <_{\mathcal{T}} [[X]]$ and X is not translation-minimal.

(iii) $\Rightarrow$ (i). Suppose that X is repetitive and $[[Y]] \leq_{\mathcal{T}} [[X]]$. Now Y contains at least one $(M_X(T)+T)$-patch, which necessarily is a patch of X, so it contains a translate of every T-patch of X. This holds for all T, so $\mathcal{A}_X \subseteq \mathcal{A}_Y$. Hence $[[X]] = [[Y]]$ and X is translation-minimal. $\square$

Remark. The compactness of $[[X]]$ is important in modelling electronic properties of quasicrystals using C^*-algebras, see Bellissard [2].

4. Repetitivity Function versus Patch-Counting Function

In this section we study minimal Delone sets X of finite type and investigate to what extent the repetitivity function $M_X(T)$ is related to the patch-counting function $N_X(T)$. We first derive a lower bound for $M_X(T)$ in terms of $N_X(T)$.

Theorem 4.1 *Let X be a Delone set of finite type that is repetitive. If X is an (r,R)-set, then*

$$M_X(T) \geq r(N_X(T)^{1/n} - 1) \ . \tag{4.1}$$

Proof. The closed ball $B(\mathbf{0}; M_X(T))$ contains points $\mathbf{x}_i$ of X whose T-patches comprise all $N_X(T)$ translation equivalence classes. Since X is an (r,R)-set, the balls of radius r around these points $\mathbf{x}$ are disjoint. They are all contained in the ball with center $\mathbf{0}$ and radius $M_X(T) + r$, hence

$$\mathrm{vol}_n(B(\mathbf{0}; M_X(T) + r) \geq \sum_{i=1}^{N_X(T)} \mathrm{vol}_n(B(\mathbf{x}_i; r)) \ .$$

This yields

$$(M_X(T) + r)^n \geq r^n N_X(T) \ ,$$

from which (4.1) follows. $\square$

It is known that there exist translation-minimal sets X in one dimension that are of positive entropy. That is, for these X there is a $c > 0$ such that

$$N_X(T) \geq e^{cT} \ , \quad \text{for all } T > 0 \ .$$

Furthermore there even exist such sets X that have strict uniform patch frequencies, as defined in §6, see Pleasants [50].

In the remainder of this section we show by construction that there is no upper bound on how fast the repetitivity function can increase, even if X is required to be a cut-and-project set in $\mathbb{R}^n$ with $N_X(T) = O(T^n)$. Cut-and-project sets are a special subclass of Meyer sets which have been extensively studied as models of quasicrystals, see [25, 39, 61, 62, 49]. The construction will be based on a result in one-dimensional symbolic dynamics.

Definition 4.1. Let $B \in \{0,1\}^{\mathbb{Z}}$ be a bi-infinite 0-1 sequence. The *recurrence function* $\tilde{M}_B$ of B is defined by $\tilde{M}_B(\ell)$ ($\ell = 1, 2, \dots$) being equal to the maximum difference in index between the leftmost symbols of two consecutive occurrences of any word $w \in \{0,1\}^{\ell}$ of length ℓ that occurs at least once in B. (Here $\tilde{M}_B(\ell) = +\infty$ if some word w of length ℓ occurs exactly once in B.)

Definition 4.2 A *Beatty sequence*

$$B_\alpha = \{b_k\} \in \{0,1\}^{\mathbb{Z}}$$

is a bi-infinite 0-1 sequence associated to a real number $\alpha \in [0,1]$ by

$$b_k := \lfloor (k+1)\alpha \rfloor - \lfloor k\alpha \rfloor \ , \quad \text{for } k \in \mathbb{Z} \ . \tag{4.2}$$

All Beatty sequences for irrational α have $\tilde{M}_{B_\alpha}(k)$ finite for all $k \geq 1$, see [43]. We call a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ a *growth function* if it is continuous and nondecreasing.

Theorem 4.2 *Given any growth function g there exists an irrational $\alpha \in [0,1]$ whose Beatty sequence $B = B_\alpha$ has recurrence function*

$$\tilde{M}_B(\ell) > g(\ell) \ , \quad \textit{for infinitely many } \ell \geq 1. \tag{4.3}$$

Proof. Morse and Hedlund [43, Theorem 9.1 and Lemma 10.3.], show that the recurrence function of B is explicitly given by

$$\tilde{M}_B(\ell) = q_k + q_{k+1} \quad \text{for } q_k \leq \ell < q_{k+1} \ , \tag{4.4}$$

where p_k/q_k is the k-th convergent of the regular continued fraction expansion of α. If we take α to be the number whose continued fraction expansion is $[0, a_1, a_2, a_3, \dots]$, where the partial quotients $a_1, a_2, \dots$ are chosen successively to satisfy $a_{k+1} \geq g(q_k)$ for $k \geq 0$, then we have

$$\tilde{M}_B(q_k) = q_k + q_{k+1} = q_k + a_{k+1}q_k + q_{k-1} > a_{k+1} \geq g(q_k)$$

for $k \geq 0$. □

Corollary 4.2a. *Given any growth function $g(T)$ there exists a translation-minimal one-dimensional cut-and-project set X projected from $\mathbb{Z}^2$ using $[0,1)$ as a window whose repetitivity function satisfies*

$$M_X(T_i) > g(T_i) \, , i = 1, 2, \dots \, , \tag{4.5}$$

for a sequence of $T_i \to \infty$.

Proof. For any real number $\alpha \in (0,1)$ the one-dimensional cut-and-project set X obtained by projecting the integer lattice in $\mathbb{R}^2$ orthogonally on to a line of slope α, using the interval $[0,1)$ of the y-axis as the window, is identical to the sequence of intervals obtained by replacing the symbols 0 and 1 in the Beatty sequence $B = B_\alpha$ of α by intervals of lengths s and ℓ, where s and ℓ depend on α and satisfy $1/\sqrt{2} < s < \ell < \sqrt{2}$. Consequently

$$M_X(T) > \tilde{M}_{B_\alpha}(\lfloor T/\sqrt{2} \rfloor). \quad \square$$

Corollary 4.2b. *Given any growth function $g(T)$ there is a translation-minimal aperiodic Meyer set Y in $\mathbb{R}^n$ with patch-counting function $N_Y(T) = O(T^n)$ whose repetitivity function satisfies*

$$M_Y(T_i) > g(T_i)\,, \quad i = 1, 2, \dots\,, \tag{4.6}$$

for a sequence of $T_i \to \infty$.

Proof. All Beatty sequences $B = B_\alpha$ with irrational α are aperiodic and have

$$N_B(k) = k + 1 \quad \text{for all } k = 1, 2, \dots\,.$$

Thus the cut-and-project sets X constructed in Corollary 4.2a are aperiodic and have $N_X(T) = O(T)$. Take $Y = X^n \subseteq \mathbb{R}^n$ to be the direct product of n such sets. Since every cut-and-project set X is a Meyer set ([40, Corollary 5.6]) and since direct products of Meyer sets are Meyer sets, Y is a Meyer set. Clearly $N_Y(T) = O(T^n)$ and $M_Y(T) \geq M_X(T)$, so satisfies (4.6). $\square$

The Meyer sets constructed in Corollary 4.2b are cut-and-project sets from $\mathbb{R}^{2n}$ to $\mathbb{R}^n$ with the window being the hypercube $[0,1)^n$.

5. Locally Defined Functions on Repetitive Sets

In this section we show that the properties of linear repetitivity and dense repetitivity of a Delone set X in $\mathbb{R}^n$ cause a large class of functions defined locally with reference to X to have uniquely determined average values on $\mathbb{R}^n$. These general results will be applied in §6 to patch frequencies (with implications for autocorrelation measures and diffraction) and in §7 to estimating path lengths in address space in the course of showing that, for both densely repetitive and linearly repetitive sets, the address map can be reasonably well approximated by a linear function.

We shall consider functions defined on "boxes" (that is, direct products of coordinate intervals) in $\mathbb{R}^n$ and express our results in terms of the volume, $\mathrm{vol}_n(B)$, surface area, $\sigma(B)$, and width, $\omega(B)$, of a box B.

Definition 5.1. A *box* in $\mathbb{R}^n$ is a subset of the form

$$B = \{(x_1, x_2, \dots, x_n) \mid a_i \leq x_i \leq b_i \ (i = 1, \dots, n)\},$$

where $a_i < b_i \in \mathbb{R}$ for each i. If $l_i = b_i - a_i$ $(i = 1 \dots n)$ then the *volume*, *surface area* and *width* of B are given by

$$\mathrm{vol}_n(B) = \prod_{i-1}^{n} l_i\,, \qquad \sigma(B) = 2\,\mathrm{vol}_n(B) \sum_{i=1}^{n} \frac{1}{l_i}\,, \qquad \omega(B) = \min_{1 \leq i \leq n} l_i\,.$$

The class of functions we deal with is the following.

Definition 5.2. A *local weight distribution* on a Delone set X in $\mathbb{R}^n$ is a function $\mathbf{w}$ with values in $\mathbb{R}^s$ whose domain is the set of all boxes B in $\mathbb{R}^n$ with $\omega(B) > U_0$ (for some $U_0 \geq 0$) and which has the following properties:

(a) $\mathbf{w}(B) = O(\mathrm{vol}_n(B))$,

(b) $\mathbf{w}$ is "approximately translation invariant" on X, in the sense that if

$$(B + \mathbf{t}) \cap X = (B \cap X) + \mathbf{t}$$

for some $\mathbf{t} \in \mathbb{R}^n$ then

$$\mathbf{w}(B + \mathbf{t}) = \mathbf{w}(B) + O(\sigma(B)) ,$$

and

(c) $\mathbf{w}$ is "approximately additive", in the sense that if $B = B_1 \cup B_2 \cup \ldots \cup B_k$ is a partition of B into non-overlapping smaller boxes then

$$\mathbf{w}(B) = \sum_{i=1}^{k} \mathbf{w}(B_i) + O\left(\sum_{i=1}^{k} \sigma(B_i)\right) .$$

In this definition the O-constants depend only on $\mathbf{w}$.

Let $\mathcal{B}(U)$ denote the set of "squarish" boxes all of whose side lengths l_i lie in the range $U \leq l_i \leq 2U$. Given a real-valued local weight distribution w on a Delone set X we define, for $U > U_0$, the upper and lower local densities

$$f^+(U) = \sup_{B \in \mathcal{B}(U)} w(B)/\mathrm{vol}_n(B) \quad \text{and} \quad f^-(U) = \inf_{B \in \mathcal{B}(U)} w(B)/\mathrm{vol}_n(B) . \tag{5.1}$$

Lemma 5.1 *If w is a real-valued weight distribution on a Delone set X and the upper and lower densities f^+, f^- are as in (5.1) then $f^+(U)$ and $f^-(U)$ tend to limits as $U \to \infty$.*

Proof. We show that f^+ is an "approximately decreasing" function of U (and f^- is "approximately increasing").

If $W \geq U$ then any box $B \in \mathcal{B}(W)$ can be subdivided into boxes B_i of $\mathcal{B}(U)$. (This is clearly possible in the case $n = 1$ of an interval on the line, and in higher dimensional cases a subdivision that is a direct product of 1-dimensional subdivisions can be used.) Since $\sum \mathrm{vol}_n(B_i) = \mathrm{vol}_n(B)$ the number of boxes $B_i \in \mathcal{B}(U)$ needed to cover B is $O(W^n/U^n)$. Also $\sigma(B_i) = O(U^{n-1})$ for each of them. So (c) of Definition 5.2 (on division by $\mathrm{vol}_n(B)$) gives

$$f^+(W) < f^+(U) + \frac{C_1}{U} , \tag{5.2}$$

for some constant C_1.

In view of (a) of Definition 5.2 $\liminf f^+$ is finite. For all $\epsilon > 0$ there is a $U > 1/\epsilon$ with

$$f^+(U) < \liminf_{U \to \infty} f^+(U) + \epsilon ,$$

so by (5.2)

$$f^+(W) < \liminf_{U \to \infty} f^+(U) + (1 + C_1)\epsilon \quad \text{for all } W > U,$$

and letting $W \to \infty$ gives

$$\limsup_{W\to\infty} f^+(W) \le \liminf_{U\to\infty} f^+(U) + (1 + C_1)\epsilon \ .$$

Since ϵ can be chosen arbitrarily small we have

$$\limsup f^+ \le \liminf f^+,$$

showing that $f^+(U)$ tends to a limit as $U \to \infty$.

The proof that $f^-(U)$ tends to a limit is similar. $\square$

Theorem 5.1 *If X is a linearly repetitive Delone set then there is a constant $\delta = \delta(X) > 0$ such that every local weight distribution $\mathbf{w}$ on X has an average value $\mathbf{f}$ in $\mathbb{R}^s$ satisfying*

$$\frac{\mathbf{w}(B)}{\mathrm{vol}_n(B)} = \mathbf{f} + O(\omega(B)^{-\delta}) \tag{5.3}$$

as $\omega(B)$ tends to infinity. (The O-constant may depend on $\mathbf{w}$ as well as X.)

Proof. There is no loss of generality in assuming that $\mathbf{w} = w$ is real-valued, since a vector-valued $\mathbf{w}$ can be treated by applying the result for real-valued w to each coordinate. For real-valued w the existence of the constant field $\mathbf{f} = f$ is equivalent to the two limits of Lemma 5.1 being the same. We denote these limits by f^+ and f^-.

Throughout the proof we shall always take $U \ge U_1 = \max(U_0, R, 1)$. For such a U we can find a box $B_1' \in \mathcal{B}(U)$ with

$$\frac{w(B_1')}{\mathrm{vol}_n(B_1')} < f^-(U) + \frac{1}{U} \ .$$

Now B_1' is contained in some V-patch of X, where $V = (2\sqrt{n}+1)U > 2\sqrt{n}\,U + R$, and by linear repetitivity every ball of radius $C'V$ contains a point of X whose V-patch is a translate of this one, where C' is the implied constant in (1.12). Consequently if $W = 2(C'V + V + U) = C_2U$, where $C_2 = 2(C' + 1)(2\sqrt{n} + 1) + 2$ then every box $B \in \mathcal{B}(W)$ contains a box B_1 which is translation equivalent to B_1' on X and whose distance from the boundary of B is $\ge U$. In a similar way as in the proof of Lemma 5.1 we can subdivide B into boxes of $\mathcal{B}(U)$, one of which is B_1. Also

$$\mathrm{vol}_n(B_1) \ge U^n = C_2^{-n} W^n \ge c\,\mathrm{vol}_n(B)$$

with $c = (2C_2)^{-n} > 0$. So by the reasoning that led to (5.2) we obtain

$$f^+(W) < c\left(f^-(U) + \frac{1}{U}\right) + (1 - c)f^+(U) + \frac{C_1}{U} \tag{5.4}$$

and letting $U \to \infty$ gives

$$cf^+ \le cf^-,$$

showing that $f^+ = f^- = f$, say.

To establish the convergence rate we first repeat the above argument to obtain

$$f^-(W) > cf^+(U) + (1 - c)f^-(U) - \frac{C_1 + 1}{U} \tag{5.5}$$

then subtract (5.5) from (5.4) to obtain

$$\Delta(W) < (1-2c)\Delta(U) + \frac{2(C_1+1)}{U} \ , \tag{5.6}$$

where $\Delta(U) = f^+(U) - f^-(U) \geq 0$.

We now show that

$$\Delta(U) < C_3 U^{-\delta}, \text{ for } U \geq U_1 \ ,$$

where

$$\delta = \log(1-c)^{-1}/\log C_2 \tag{5.7}$$

and C_3 is chosen large enough to ensure that $\Delta(U) < C_3 U^{-\delta}$ for $U_1 \leq U \leq C_2 U_1$ and $cC_3 > 2(C_1+1)$. Assume, as an induction hypothesis, that

$$\Delta(U) < C_3 U^{-\delta} \text{ for } U_1 \leq U \leq C_2^k U_1 \ .$$

This holds for $k=1$. Given $W \leq C_2^{k+1} U_1$ we have

$$\begin{aligned} \Delta(W) &< (1-2c)\Delta(W/C_2) + \frac{2(C_1+1)C_2}{W} \ , \quad \text{by (5.6)}, \\ &< C_3(1-c)W^{-\delta}C_2^{\delta}, \quad \text{by the induction hypothesis}, \\ &= C_3 W^{-\delta}, \quad \text{by (5.7)}. \end{aligned}$$

This completes the induction and shows that $\Delta(U) < C_3 U^{-\delta}$ for all $U > U_1$.

Now letting $W \to \infty$ in (5.2) (and its analogue for f^-) gives

$$f^-(U) - \frac{C_1}{U} < f < f^+(U) + \frac{C_1}{U}$$

so

$$\left| \frac{w(B)}{\mathrm{vol}_n(B)} - f \right| \leq \Delta(U) + \frac{C_1}{U} = O(U^{-\delta}) \tag{5.8}$$

for $B \in \mathcal{B}(U)$, since $\delta < 1$.

Finally, an arbitrary box B with $\omega(B) > U_1$ can be subdivided into $O(\mathrm{vol}_n(B)/\omega(B)^n)$ boxes of $\mathcal{B}(\omega(B))$ with total surface area $O(\mathrm{vol}_n(B)/\omega(B))$, so (5.8) and (c) of Definition 5.2 give

$$\frac{w(B)}{\mathrm{vol}_n(B)} = f + O(\omega(B)^{-\delta}) \ ,$$

completing the proof. □

Theorem 5.2 *If X is a densely repetitive Delone set and $\mathbf{w}$ is a local weight distribution on X then there is an $\mathbf{f}$ in $\mathbb{R}^s$ with*

$$\frac{\mathbf{w}(B)}{\mathrm{vol}_n(B)} = \mathbf{f} + o(\omega(B)) \ ,$$

as $\omega(B)$ tends to infinity.

Proof. Again there is no loss of generality in assuming that $\mathbf{w} = w$ is real-valued.

Given $U > U_0$ we have defined $f^+(U)$ and $f^-(U)$. We now also define $f^0(U)$ as follows. Let $\mathcal{P}_X(\mathbf{x}_1, \sqrt{n}\,U), \ldots, \mathcal{P}_X(\mathbf{x}_N, \sqrt{n}\,U)$ be a set of $N = N(\sqrt{n}\,U)$ translation inequivalent $\sqrt{n}\,U$-patches of X and let

$$w_i = w_i(U) = w(\mathcal{C}(\mathbf{x}_i, U)) \quad (i = 1, \ldots, N) \ ,$$

where $\mathcal{C}(\mathbf{x}_i, U)$ is the cube of side $2U$ with center $\mathbf{x}_i$ (which is contained in $\mathcal{P}_X(\mathbf{x}_i, \sqrt{n}\,U)$). We take $f^0(U)$ to be the median of the set of numbers

$$\{w_i/(2U)^n : i = 1, \ldots, N(\sqrt{n}\,U)\} \ ,$$

that is, the middle number when the set is in size order (when N is odd) or the average of the two middle numbers (when N is even). There is an ambiguity in this definition as it stands, because two points $\mathbf{x}_i$, $\mathbf{x}_i'$ with translation equivalent $\sqrt{n}\,U$-patches may nevertheless give different values of w_i. However, by (b) of Definition 5.2 the values differ by at most a constant multiple of U^{n-1}, and consequently $f^0(U)$ is defined to within a constant multiple of $1/U$ and its behavior as U tends to infinity is unaffected by the ambiguity. The definition could be made unambiguous by adopting some specific rule for choosing the points $\mathbf{x}_1, \ldots, \mathbf{x}_N$.

We now choose a sequence $U_1, U_2, \ldots$, tending to infinity, with $U_1 > U_0$ and $M_X(\sqrt{n}\,U_k) \geq \sqrt{n}\,U_k$ for $i = 1, 2, \ldots$. (If there is no such infinite sequence then X is linearly repetitive and the result is a consequence of Theorem 5.1.) Put $W_k = 7M_X(\sqrt{n}\,U_k)$ and consider an arbitrary box $B \in \mathcal{B}(W_k)$. To get a non-trivial bound for $w(B)$ it is no longer enough to find a single sub-box B_1 of B in $\mathcal{B}(U_k)$ with $w(B_1)$ small, since when the growth of M_X is faster than linear this no longer fills a positive proportion of B. We need a large number of well-spaced sub-boxes with small w. By the definition of $f^0(U_k)$ there are $\geq N/2 = N(\sqrt{n}\,U_k)/2$ translation inequivalent $\sqrt{n}\,U_k$-patches of X such that if $\mathcal{P}$ is a patch translation equivalent to any one of them and $\mathcal{C}$ is the coordinate cube of side $2U_k$ inscribed in $\mathcal{P}$ then $w(\mathcal{C})/(2U_k)^n \leq f^0(U_k) + C_4/U_k$ for some C_4. By the definition of M_X every cube of side $2M_X(\sqrt{n}\,U_k) + \sqrt{n}\,U_k$ contains at least one patch translation equivalent to each of these. Hence B contains $\geq N/2$ subcubes $\mathcal{C}(\mathbf{x}, U_k)$ of side $2U_k$ for each of which $w/(2U_k)^n \leq f^0(U_k) + C_4/U_k$ and none of which comes within a distance $2\sqrt{n}\,U_k$ of the boundary of B. We require a set of subcubes that do not overlap, and in fact it will be convenient to have them a distance at least $4\sqrt{n}\,U_k$ apart. Since each of our subcubes $\mathcal{C}(\mathbf{x}, U_k)$ has a point $\mathbf{x}$ of X at its center and a ball of radius $6\sqrt{n}\,U_k$ contains at most $((6\sqrt{n}\,U_k + r)/r)^n$ points of X, each subcube $\mathcal{C}(\mathbf{x}, U_k)$ we pick excludes at most $((6\sqrt{n}\,U_k + r)/r)^n - 1$ others. Consequently we can choose at least $\frac{1}{2} r^n N/(7\sqrt{n}\,U_k)^n \geq \frac{1}{2}(rM/7\sqrt{n}\,C'' U_k)^n$ of these subcubes of B that are at least a distance $4\sqrt{n}\,U_k$ apart, where C'' is the implied constant in (1.11) and $M = M_X(\sqrt{n}\,U_k)$.

We next verify that B has a partition into boxes of $\mathcal{B}(U_i)$ which includes all these subcubes. This can be done by thinking of $\mathbb{R}^n$ as composed of a fixed background grid of packed cubes of side U_k. Each chosen cube $\mathcal{C}(\mathbf{x}, U_k)$ meets a set of grid cubes forming a box-shaped block, which we enlarge by the inclusion of an outer single-layer shell of grid cubes disjoint from $\mathcal{C}(\mathbf{x}, U_k)$. Because of the separation of the chosen cubes their enlarged blocks of grid cubes do not overlap. As in previous proofs, this enlarged block can be partitioned into boxes of $\mathcal{B}(U_k)$ one of which is $\mathcal{C}(\mathbf{x}, U_k)$. Similarly, we consider the two-layer shell of grid cubes consisting of those that meet the boundary of B together with an inner single-layer lining shell. Again, this two-layer shell can be partitioned into boxes of $\mathcal{B}(U_k)$ and does not overlap the shells of the chosen cubes because of the distance of these shells from the boundary of B. So B can be partitioned into the cubes $\mathcal{C}(\mathbf{x}, U_k)$ (of side $2U_k$), the boxes of $\mathcal{B}(U_k)$ just described and the remaining grid cubes (of side U_k).

Since $\mathrm{vol}_n(B) \leq (14M)^n$ and the total volume of the cubes $\mathcal{C}(\mathbf{x}, U_k)$ is $\geq \frac{1}{2}(2rM/7\sqrt{n}\,C'')^n$, (c) of Definition 5.2, on division by $\mathrm{vol}_n(B)$, gives

$$f^+(W_k) < c\left(f^0(U_k) + \frac{C_4}{U_k}\right) + (1-c)f^+(U_k) + \frac{C_1}{U_k}\ ,$$

where $c = \frac{1}{2}(r/49\sqrt{n}\,C'')^n$. On letting $k \to \infty$ we obtain

$$\liminf_{U\to\infty} f^0(U) \geq f^+.$$

A similar argument, choosing cubes $\mathcal{C} \subseteq B$ with $w/(2U_k)^n \geq f^0(U_k) - C_4/U_k$, leads to

$$\limsup_{U\to\infty} f^0(U) \leq f^-.$$

Hence $f^+ = f^-$. The proof is completed as in the last two paragraphs of the proof of Theorem 5.1. □

We end this section by showing that linear repetitivity is not the exact cut-off point for the universal existence of unique averages for local weight distributions. It is possible to trade some of the error term in Theorem 5.1 for a weakening of the hypothesis and show that every marginally superlinearly repetitive Delone set has this property.

Theorem 5.3 *If X is a Delone set in $\mathbb{R}^n$ satisfying*

$$M_X(T) = O(T(\log T)^{1/n})$$

as T tends to infinity and $\mathbf{w}$ is a local weight distribution on X then there is an $\mathbf{f}$ in $\mathbb{R}^s$ with

$$\frac{\mathbf{w}(B)}{\mathrm{vol}_n(B)} = \mathbf{f} + o(\omega(B))\ ,$$

as $\omega(B)$ tends to infinity.

Proof. We first note that the derivation of (5.4), (5.5) and (5.6) in the proof of Theorem 5.1 remains valid when C' is a function of T instead of being constant. If we assume that

$$M_X(T) < CT(\log T)^{1/n} \quad \text{for } T \geq U',$$

where we can suppose that $U' \geq \max(U_0, 2\sqrt{n}+1)$, and take $W = C_5U(\log U)^{1/n}$ with $C_5 = 4(2\sqrt{n}+1)C + 2$ then (5.6) becomes

$$\Delta(W) < \left(1 - \frac{2c_1}{\log U}\right)\Delta(U) + \frac{2(C_1+1)}{U},$$

with $c_1 = (2C_5)^{-n}$ and C_1 is as before. If we put $E(T) = \max(\Delta(T), T^{-1/2})$ then we can find a U_1 such that

$$E(W) < \left(1 - \frac{c_1}{\log U}\right)E(U) \quad \text{for } U \geq U_1. \tag{5.9}$$

We can also take $U_1 \geq \max(C_5, 4)$.

We now define an increasing sequence $U_1 < U_2 < \ldots$ by

$$U_{k+1} = C_5 U_k (\log U_k)^{1/n} .$$

Clearly $U_k \to \infty$ as $k \to \infty$. To bound U_k above we use an induction argument to show that

$$\log U_k < U_1 k \log(U_1 k) \log\log(U_1 k) \ .$$

For the induction step we have

$$\begin{aligned} \log U_{k+1} &= \log U_k + \frac{1}{n} \log\log U_k + \log C_5 \\ &< U_1 k \log(U_1 k) \log\log(U_1 k) + \log(U_1 k) + \log\log(U_1 k) + \log\log\log(U_1 k) + \log C_5 \ , \\ &\quad\ \text{by the induction hypothesis,} \\ &< U_1 (k+1) \log(U_1 k) \log\log(U_1 k) \ , \quad \text{because } U_1 \geq 4 \text{ and } U_1 \geq C_5, \\ &< U_1 (k+1) \log(U_1 (k+1)) \log\log(U_1 (k+1)) \ . \end{aligned}$$

Now, for any $U \geq U_1$, let K be the largest integer with $U_K \leq U$. Then repeated application of (5.9) gives

$$E(U) < C_6 \prod_{k=2}^{K} \left(1 - \frac{c_1}{U_1 k \log(U_1 k) \log\log(U_1 k)} \right) ,$$

where C_6 is an upper bound for $E(U)$ in the range $U_1 \leq U < U_2$. Since the product on the right diverges to 0 as K tends to infinity, $E(U)$ (and hence $\Delta(U)$) tends to 0 as U tends to infinity. The proof is now completed as in the last two paragraphs of the proof of Theorem 5.1. $\square$

6. Repetitive Sets and Uniform Patch Frequencies

One form of long range order in a discrete set X is the property that, for each T, all of its T-patches are evenly distributed in the sense of having limiting frequencies. We show that densely repetitive sets and linearly repetitive sets have this property.

Definition 6.1. Given a T-patch $\mathcal{P}$ of X translated to $\mathbf{0}$, and a bounded region $D \subset \mathbb{R}^n$ define

$$n_{\mathcal{P}}(D) := |\{\mathbf{x} \in D : \mathcal{P} + \mathbf{x} \subseteq X\}| \ .$$

That is, $n_{\mathcal{P}}(D)$ counts the number of points of X in D whose T-patches are translation equivalent to $\mathcal{P}$.

Definition 6.2. A Delone set X has *uniform patch frequencies* if every patch $\mathcal{P}$ has a uniform limiting frequency $f(\mathcal{P})$ in the sense that

$$\frac{n_{\mathcal{P}}(B(\mathbf{t}; U))}{\kappa_n U^n} = f(\mathcal{P}) + o(1) \tag{6.1}$$

uniformly in $\mathbf{t}$ as $U \to \infty$, where κ_n is the volume of the unit ball in $\mathbb{R}^n$ (so that $\kappa_n U^n = \mathrm{vol}_n(B(\mathbf{x}; U))$). It has *strict uniform patch frequencies* if, in addition, $f(\mathcal{P})$ is nonzero for every patch $\mathcal{P}$ that occurs in X.

A topological dynamical system on a compact space Ω with an $\mathbb{R}^n$-action is *uniquely ergodic* if it has a unique probability measure on Borel sets that is invariant under the $\mathbb{R}^n$-action. The property of having uniform patch frequencies is equivalent to the dynamical system $([[X]], \mathbb{R}^n)$ being uniquely ergodic, see Solomyak [66, Theorem 3.3][7]. The extra property that the frequency of every patch that occurs in X is positive implies that $([[X]], \mathbb{R}^n)$ is minimal, hence strictly ergodic. Lunnon and Pleasants [34] take the property of having strict uniform patch frequencies as part of the definition of being strongly quasicrystallographic.

The main result of this paper is as follows.

Theorem 6.1 *If X is a Delone set in $\mathbb{R}^n$ that is either linearly repetitive or densely repetitive, then X has strict uniform patch frequencies. In the case that X is linearly repetitive the error term in (6.1) has the form $O(U^{-\delta})$, uniformly in $\mathbf{t}$, where $\delta > 0$ depends only on X but the O-constant may depend on the patch $\mathcal{P}$.*

Proof We note that $n_{\mathcal{P}}$ is a local weight distribution on X. It clearly satisfies (a) of Definition 5.2 and (c) (with no error term necessary). The error term in (b) arises from points $\mathbf{x}$ in $B + \mathbf{t}$ whose T-patches extend outside $B + \mathbf{t}$. Such points are confined to a border region of $B + \mathbf{t}$ with thickness T whose volume is $\leq T\sigma(B)$, and provided $\omega(B) \geq r$ the number of them is $O(\sigma(B))$.

Now Theorems 5.1 and 5.2 ensure the existence of a limiting frequency $f(\mathcal{P})$ such that

$$\frac{n_{\mathcal{P}}(B)}{\mathrm{vol}_n(B)} = f(\mathcal{P}) + \epsilon(B) \tag{6.2}$$

with $\epsilon(B) \to 0$ as $\omega(B) \to \infty$ in both the linearly and densely repetitive cases. In the linearly repetitive case $\epsilon(B) = O(\omega(B)^{-\delta_1})$, where δ_1 is the δ of Theorem 5.1. To see that $f(\mathcal{P}) > 0$ note that for every large integer N every cube of side $2M_X(T)N$ can be subdivided into N^n cubes of side $2M_X(T)$ so contains at least N^n distinct points of X whose T-patches are translation equivalent to $\mathcal{P}$. Letting $N \to \infty$ now gives $f(\mathcal{P}) \geq (2M_X(T))^{-n} > 0$.

To complete the proof we need to make the transition from boxes to balls. Given a ball $B(\mathbf{t}; U)$ we can cover it with $k = O(U^{n/2})$ cubes $C_1, \ldots, C_k$ of side $\sqrt{U}$ of which at most $O(U^{(n-1)/2})$ are not contained in $B(\mathbf{t}; U)$. So

$$\mathrm{vol}_n(B(\mathbf{t}; U)) = kU^{n/2} - O(U^{n-1/2})$$

and the error in estimating $n_{\mathcal{P}}(B(\mathbf{t}; U))$ as $\sum n_{\mathcal{P}}(C_i)$ is also $O(U^{n-1/2})$. These approximations to the volume and patch count of $B(\mathbf{t}; U)$ lead to

$$\begin{aligned} \frac{n_{\mathcal{P}}(B(\mathbf{t}; U))}{\mathrm{vol}_n(B(\mathbf{t}; U))} &= \frac{1}{kU^{n/2}} \sum_{i=1}^{k} n_{\mathcal{P}}(C_i) + O(U^{-1/2}) \\ &= f(\mathcal{P}) + \frac{1}{k} \sum_{i=1}^{k} \epsilon(C_i) + O(U^{-1/2}) \end{aligned}$$

by (6.2). The last two terms on the right tend to 0 as $U \to \infty$ in both the linearly and densely repetitive cases and in the linearly repetitive case they are $O(U^{-\delta})$, with $\delta = \delta_1/2$. □

[7]Solomyak's result is stated for tilings, but the proof carries over to Delone sets of finite type. In his terminology "finite number of local patterns" means "finite type" and "local isomorphism property" means "repetitive."

The major consequence of Theorem 6.1 is that X is diffractive in the sense of Hof [18], as we now explain.

Definition 6.3. An *autocorrelation measure* (in the sense of Hof [18, 19]) of a discrete set X in $\mathbb{R}^n$ is any limit measure in the vague topology, as $T \to \infty$, of the set of discrete measures μ_T defined for $T > 0$ by

$$\mu_T := \frac{1}{\kappa_n T^n} \sum_{\|\mathbf{x}_1\|,\|\mathbf{x}_2\|<T} \delta_{\mathbf{x}_1 - \mathbf{x}_2} \ . \tag{6.3}$$

In this definition we view a measure μ as a linear functional on the space $\mathcal{K}$ of complex-valued continuous functions f with compact support on $\mathbb{R}^n$ which have the additional property that for any compact set E there is a constant a_E (depending on μ) such that

$$|\mu(f)| \le a_E ||f|| \quad \text{if } \mathrm{supp}(f) \subseteq E \ ,$$

where $||f|| = \sup\{|f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\}$, and we say that a sequence of measures $\{\mu_k\}$ *converges to* μ *in the vague topology* if

$$\lim_{k\to\infty} \mu_k(f) = \mu(f) \ , \quad \text{for all } f \in \mathcal{K} \ .$$

Any Delone set X has at least one autocorrelation measure[8] and typically (when X has large scale irregularities) has many autocorrelation measures.

Definition 6.4. A Delone set X is *diffractive* if it has a unique autocorrelation measure γ_X. The *diffraction measure* of X is the Fourier transform $\hat{\gamma}_X$ of the autocorrelation measure γ_X.

The diffraction measure $\hat{\gamma}$ is a mathematical analogue of the far-field x-ray diffraction spectrum of X, see Hof [18, 19]. It is necessarily a positive measure. Note that this definition of a Delone set being diffractive does not require that its diffraction measure contain any pure point component.

The existence of uniform patch frequencies implies that X has a unique autocorrelation measure $\gamma = \gamma_X$ and hence that it is diffractive in the sense above. It also implies that X has unique k-point correlation measures for all $k \ge 1$.

Theorem 6.1 now yields:

Corollary 6.1 *If X is a linearly or densely repetitive Delone set in $\mathbb{R}^n$ then it has a unique autocorrelation measure γ_X. This measure γ_X is a pure discrete measure supported on $X - X$. In particular X is diffractive.*

Proof. The existence and uniqueness of γ_X follows from Example 2.1 in Hof [18]. Now $X - X$ is a discrete set, since X is a Delone set of finite type, hence all measures μ_T in (6.3) are supported on $X - X$, and γ_X inherits this property. □

Theorem 6.1 also applies to symbolic dynamical systems given by a two-sided shift Σ on a finite alphabet $\mathcal{A} = \{0, 1, ..., m-1\}$. Elements of such a shift can be encoded as Delone sets of finite type on the line $\mathbb{R}$, by assigning m intervals of different fixed lengths to the symbols,

[8] This follows by the remark before [18, Prop. 2.2] because the measure $\mu_X := \sum_{\mathbf{x}\in X} \delta_{\mathbf{x}}$ is translation-bounded, as defined in [18].

and the shift is minimal with a recurrence function that grows at most linearly in the word length ℓ if and only if the associated Delone sets are linearly repetitive. Clearly such linearly repetitive shifts have the number of words of length ℓ bounded above by $C\ell$ for some constant C (by the analogue for shifts of Theorem 4.1). Theorem 6.1 implies that linearly repetitive shifts are uniquely ergodic. Boshernitzan [4] proves a related result: Any minimal shift whose word count function is bounded above by $C\ell$ has finitely many ergodic invariant measures, the number of them bounded in terms of C and the number of symbols m.

We conclude this section by showing that there are Delone sets without uniform patch frequencies whose repetitivity functions grow only slightly faster than linear. In particular, we show that Theorem 5.3 is not far from best possible.

Lemma 6.1 *Given positive integers n and N, there exists a sequence $\{a_k\}$ of positive integers with the following properties:*

(i) $a_k > N$ for all k,

(ii) $k \log k\, (\log\log k)^2 < a_k < 2k\log k\,(\log\log k)^2$ for all sufficiently large k,

(iii) a_k is an even n-th power,

(iv) the sequence $\{\rho_k\}$ of real numbers in $[0,1]$ defined recursively by

$$\begin{aligned} \rho_0 &= 1\,, \\ \rho_k &= \frac{N}{2a_k}\rho_{k-1} + \left(1 - \frac{N}{2a_k}\right)(1-\rho_{k-1}) \qquad k = 1,2,\ldots \end{aligned} \tag{6.4}$$

does not tend to a limit.

Proof. The recurrence (6.4) can be rewritten as

$$\rho_k - \frac{1}{2} = -\left(1 - \frac{N}{a_k}\right)\left(\rho_{k-1} - \frac{1}{2}\right),$$

whence

$$\rho_k = \frac{1}{2} + \frac{(-1)^k}{2}\prod_{j=1}^{k}\left(1 - \frac{N}{a_j}\right).$$

If we choose the sequence $\{a_k\}$ so that $a_k > N$ for each k and the infinite product

$$\prod_{k=1}^{\infty}\left(1 - \frac{N}{a_j}\right)$$

does not diverge to zero then $\liminf \rho_k < \frac{1}{2}$ and $\limsup \rho_k > \frac{1}{2}$. Any choice of a_k's satisfying (i), (ii) and (iii) of the lemma (which are clearly compatible) fulfills these requirements. □

Theorem 6.2 *The n-dimensional cubic lattice $\mathbb{Z}^n$ can be two-colored in such a way that as $T \to \infty$ the proportion of white lattice points in the cube with center $\mathbf{0}$ and side length T does not tend to a limit and*

$$M(T) = O(T(\log T)^{2/n}(\log\log\log T)^{4/n})\ .$$

Here $M(T)$ is the natural extension to the two-colored lattice of the repetitivity function for Delone sets of finite type.

Proof. As a preliminary, let $\{a_k\}$ be a sequence of positive integers satisfying Lemma 6.1 with $N = 2^{2^n+n}$, and let $\{\rho_k\}$ be the corresponding sequence of real numbers.

It is convenient to regard the coloring as tiling $\mathbb{R}^n$ with black and white unit cubes centered at the points of the cubic lattice.

First take 2^{2^n} cubes of side 2, each composed of 2^n unit cubes and each representing a different one of the 2^{2^n} possible vertex types at the central vertex, and arrange them within the cube $\mathcal{C}_1$ of side length $s_1 = a_1^{1/n}$ that lies in the corner of the section of $\mathbb{R}^n$ where all coordinates are positive. This can be done because s_1 is even and $(s_1/2)^n > 2^{2^n}$, and it is possible to arrange that a white unit cube is placed adjacent to the origin. Fill the remainder of $\mathcal{C}_1$ with black unit cubes. The proportion of white cubes in $\mathcal{C}_1$ is $N/2a_1 = \rho_1$. Next repeat this construction with copies of $\mathcal{C}_1$ in place of white unit cubes, copies of the color-reversal $\overline{\mathcal{C}}_1$ of $\mathcal{C}_1$ in place of black unit cubes and a_2 in place of a_1. The result is that the cube $\mathcal{C}_2$ of side length $s_2 = (a_1a_2)^{1/n}$ at the corner of the positive sector of $\mathbb{R}^n$ is tiled with copies of $\mathcal{C}_1$ and $\overline{\mathcal{C}}_1$ in such a way as to include all "vertex types" of them, to have a proportion ρ_2 of white unit cubes and to have a copy of $\mathcal{C}_1$ adjacent to the origin. This construction can be iterated indefinitely to give a tiling of the positive sector of $\mathbb{R}^n$ by black and white unit cubes that, for each k, can be decomposed into copies of $\mathcal{C}_k$ and $\overline{\mathcal{C}}_k$, where $\mathcal{C}_k$ is the cube of side length $s_k = (a_1a_2\cdots a_k)^{1/n}$ in the corner of the sector. Finally we extend the tiling to the whole of $\mathbb{R}^n$ by reflection in the coordinate hyperplanes. The proportion of white unit cubes in $\mathcal{C}_k$ is ρ_k, and since this does not tend to a limit the proportion of white unit cubes of the tiling in a large central cube does not tend to a limit.

To bound $M(T)$ for the tiling, let k be the smallest integer with $s_k \geq 2T$. Then, for any $\mathbf{x} \in \mathbb{Z}^n$, $B(\mathbf{x}, T)$ is contained in a cube of side $2s_k$ made up of 2^n copies of cubes of types $\mathcal{C}_k$ and $\overline{\mathcal{C}}_k$ with a common vertex. Since this vertex type occurs in every copy of $\mathcal{C}_{k+1}$ and $\overline{\mathcal{C}}_{k+1}$ at the next level, there is a point with the same T-patch as $\mathbf{x}$ within every ball of radius $2\sqrt{n}\, s_{k+1}$. By our choice of k, $s_{k-1} < 2T$, so

$$\begin{aligned} M(T) \leq 2\sqrt{n}\, s_{k+1} &= 2\sqrt{n}\,(a_ka_{k+1})^{1/n} s_{k-1} \\ &< 4\sqrt{n}\,(a_ka_{k+1})^{1/n} T \\ &= O(T(k\log k(\log\log k)^2)^{2/n}) \ , \end{aligned} \tag{6.5}$$

by the right hand inequality of Lemma 6.1 (ii). Also by the left hand inequality of Lemma 6.1 (ii)

$$\log s_{k-1} > \frac{1}{n}\log((k-1)!) - O(1) > \frac{1}{n}\, k\log k - O(k)$$

and hence

$$k = O(\log T/\log\log T) \ .$$

So (6.5) gives

$$M(T) = O(T(\log T)^{2/n}(\log\log\log T)^{4/n}). \quad \Box$$

Corollary 6.2 *There exists a Meyer set X in $\mathbb{R}^n$ with*

$$M_X(T) = O(T(\log T)^{2/n}(\log\log\log T)^{4/n})$$

that does not have uniform patch frequencies.

Proof. Replace each black lattice point $\mathbf{x}$, in the coloring given by Theorem 6.2, by the pair of points $\mathbf{x} \pm (1,1,\dots,1)/3$, leaving the white lattice points unchanged. The result is a Meyer set X in which the original white points can be identified by their $\sqrt{n}$-patches. An argument similar to that at the end of the proof of Theorem 6.1 shows that if this patch had a uniform frequency f in X then the frequency of its occurrence in the cube with center $\mathbf{0}$ and side length T would tend to f as $T \to \infty$. Since this is not the case, X does not have uniform patch frequencies. Also $M_X(T) \le M(T + \sqrt{n}/3)$, where the function on the right is the repetitivity function of the two-colored lattice of Theorem 6.2. $\square$

Remarks. The following three remarks clarify some features of Theorem 6.2 and its proof.

1. There are innumerable ways of coding the two colors by placement of points. Perhaps a more elegant one is simply to translate all the black points by $(1,1,\dots,1)/2$, but the justification of this requires consideration of $s_1\sqrt{n}$-patches, instead of $\sqrt{n}$-patches, because of the impossibility of distinguishing between black and white points at smaller scales.

2. The exponent of $\log T$ in Theorem 6.2 is $2/n$ instead of $1/n$ because rounding effects oblige us essentially to deal with cubes at levels differing by 2 intead of at adjacent levels only. One might think, therefore, that the result of Corollary 6.2 would hold with an exponent $1/n$ in place of $2/n$. This is not so, however, because Theorem 5.3 can easily be strengthened to show that every Delone set X with $M_X(T) = O(T(\log T \log\log T)^{1/n})$ has a uniform average for every local weight distribution. The exponent $4/n$ of $\log\log\log T$ can, of course, be reduced to $(2+\epsilon)/n$ for any $\epsilon > 0$.

3. The upper and lower frequencies of white lattice points in cubes with center $\mathbf{0}$ are not local isomorphism invariants. For example, if in the proof of Theorem 6.2 we extend our tiling of the positive sector using color-reversing reflections in the coordinate hyperplanes then we obtain a coloring of $\mathbb{Z}^n$ locally isomorphic to the coloring in the theorem that has a proportion $1/2$ of white tiles in all boxes centered at the origin.

7. Repetitive Sets and the Address Map

The address map introduced in [25] provides a measure of the regularity of structure of finitely generated Delone sets X.

Definition 7.1. Let X be a finitely generated Delone set in $\mathbb{R}^n$, and suppose that the additive group $[X]$ generated by X in $\mathbb{R}^n$ is of finite rank $s \ge n$. An *address map* $\phi : [X] \to \mathbb{Z}^s$ is an isomorphism obtained by picking a fixed basis $\{\mathbf{y}_1,\dots,\mathbf{y}_s\}$ of $[X]$. Every $\mathbf{y} \in [X]$ has a unique decomposition $\mathbf{y} = n_1\mathbf{y}_1 + \dots + n_s\mathbf{y}_s$, with all $n_i \in \mathbb{Z}$, and

$$\phi(\mathbf{y}) := (n_1, n_2, \dots, n_s)\ . \tag{7.1}$$

is the address map associated to this basis.

Address maps with different bases differ by multiplication on the left by an element of $GL(s,\mathbb{Z})$. An address map is linear on $[X]$, which typically is dense in $\mathbb{R}^n$, but does not generally extend to a linear function on $\mathbb{R}^n$.

In this section we show that the address maps of densely repetitive and linearly repetitive sets satisfy (1.17) and (1.18) respectively.

Theorem 7.1 *Let X be a Delone set of finite type with $\mathbf{0} \in X$ and* $\operatorname{rank}(X) = \operatorname{rank}(X-X) = s$. *Fix an address map $\phi : [X] \to \mathbb{Z}^s$. If X is either linearly repetitive or densely repetitive, then there is a linear function $L : \mathbb{R}^n \to \mathbb{R}^s$ such that*

$$||\phi(\mathbf{x}) - L(\mathbf{x})|| = o(||\mathbf{x}||), \quad \textit{for } \mathbf{x} \in X \ . \tag{7.2}$$

In the case that X is linearly repetitive the error term can be improved to $O(||\mathbf{x}||^{1-\delta})$, for some $\delta = \delta(X) > 0$.

Remark. The linear function L of this theorem is uniquely determined by X, since if L^* is another such linear function then

$$||L(\mathbf{x}) - L^*(\mathbf{x})|| = o(||\mathbf{x}||) \quad \text{for } \mathbf{x} \in X \ , \tag{7.3}$$

giving $L \equiv L^*$ since X is relatively dense.

Proof. We derive L from the address map ϕ by a limiting process. Given $\mathbf{t} \in \mathbb{R}^n$ let $\mathbf{x}[\mathbf{t}]$ denote some point in X with

$$||\mathbf{x}[\mathbf{t}] - \mathbf{t}|| \leq R \ . \tag{7.4}$$

Since there are finitely many points of X within a radius R of $\mathbf{t}$ it is clearly possible to adopt a convention for deciding, in all circumstances, which one to pick, so resolving the ambiguity of this definition. Alternatively, we could note that the effects of different choices are always absorbed by the error terms.

We shall define L as a "smoothing" of $\phi \circ \mathbf{x}$ by

$$L(\mathbf{v}) := \lim_{U \to \infty} \frac{1}{U} \phi(\mathbf{x}[U\mathbf{v}]) \tag{7.5}$$

for $\mathbf{v} \in \mathbb{R}^n$. In showing that this limit exists it is convenient to prove a stronger uniformly translation invariant result, namely that the limit

$$L(\mathbf{v}) = \lim_{U \to \infty} \frac{1}{U} \mathbf{p}(\mathbf{t}, U\mathbf{v} + \mathbf{t}) \tag{7.6}$$

exists, for each $\mathbf{v} \in \mathbb{R}^n$, uniformly for $\mathbf{t} \in \mathbb{R}^n$ and is independent of $\mathbf{t}$, where

$$\mathbf{p}(\mathbf{t}_1, \mathbf{t}_2) = \phi(\mathbf{x}[\mathbf{t}_2]) - \phi(\mathbf{x}[\mathbf{t}_1]) \ .$$

represents the displacement in address space due to traversing a path from $\mathbf{t}_1$ to $\mathbf{t}_2$ in object space. (So (7.5) is the "centered" case $\mathbf{t} = \mathbf{0}$). It is sufficient to prove that the limit (7.6) exists when $\mathbf{v}$ belongs to the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ since if $\mathbf{v} = \sum_{k=1}^n v_k \mathbf{e}_k$ and we put $\mathbf{v}_i := \sum_{k=i}^n v_k \mathbf{e}_k$ then

$$\mathbf{p}(\mathbf{t}, U\mathbf{v} + \mathbf{t}) = \sum_{i=1}^n \mathbf{p}(U\mathbf{v}_{i+1} + \mathbf{t}, Uv_i\mathbf{e}_i + U\mathbf{v}_{i+1} + \mathbf{t}) \ . \tag{7.7}$$

Moreover, if we can show that

$$\frac{1}{U} \mathbf{p}(\mathbf{t}, U\mathbf{e}_i + \mathbf{t}) = L(\mathbf{e}_i) + O(U^{-\delta}) \tag{7.8}$$

uniformly in $\mathbf{t}$ then we obtain (7.2) with error term $O(||\mathbf{x}||^{1-\delta})$ by taking $\mathbf{v} = \mathbf{x}$, $\mathbf{t} = \mathbf{0}$ and $U = 1$ in (7.7).

The proof of (7.6) is in two stages. In the first stage we replace the single path by a bunch of parallel "sample paths" filling a box and use Theorems 5.1 and 5.2 to prove the existence of an average proportional displacement in address space. In the second stage we apply this to a narrow box and use the Lipschitz property to show that the average displacement is a good approximation to the actual displacement due to traversing a single path.

For the first stage, let

$$B = \{(t_1, t_2, \ldots, t_n) \ : \ a_k \le t_k \le b_k \quad (k = 1, \ldots, n)\} = [a_i, b_i] \times B^{(i)}$$

be a box in $\mathbb{R}^n$, where $B^{(i)}$ is the $(n-1)$-dimensional box

$$B^{(i)} = \{(t_1, \ldots t_{i-1}, t_{i+1}, \ldots, t_n) \ : \ a_k \le t_k \le b_k \quad (k = 1, \ldots, i-1, i+1, \ldots n)\} \ ,$$

let $G^{(i)} = B^{(i)} \cap \mathbb{Z}^{n-1}$ be the grid of points of the $(n-1)$-dimensional integer lattice that are in $B^{(i)}$ and let m_i, n_i be the integer parts of a_i and b_i. We have

$$\begin{aligned} |G^{(i)}| &= \mathrm{vol}_{n-1}(B^{(i)}) + O(\sigma(B^{(i)})) \\ &= \frac{\mathrm{vol}_n(B)}{b_i - a_i}\left(1 + O\left(\frac{1}{\omega(B)}\right)\right), \end{aligned} \tag{7.9}$$

where $|G^{(i)}|$ is the cardinality of $G^{(i)}$. Define

$$\mathbf{P}_i(B) = \sum_{\mathbf{g} \in G^{(i)}} \mathbf{p}((m_i, \mathbf{g}), (n_i, \mathbf{g})) \tag{7.10}$$

with the convention that when $\mathbf{g}$ is on the boundary of $B^{(i)}$ the term for $\mathbf{g}$ is to be taken with an appropriate weight — $\frac{1}{2}$ when $\mathbf{g}$ is on an $(n-2)$-dimensional face, $\frac{1}{4}$ when it is on an $(n-3)$-dimensional face, and so on. The sum $\mathbf{P}_i(B)$ represents the sum of the displacements in address space due to traversing a number of paths in object space parallel to $\mathbf{e}_i$. The endpoints of these paths are points of the integer lattice $\mathbb{Z}^n$ close to the faces of B that are perpendicular to $\mathbf{e}_i$. It is readily seen that $\mathbf{P}_i$ is a local weight distribution on X. Part (a) of Definition 5.2 is a consequence of the Lipschitz property of ϕ and our estimate for $|G^{(i)}|$. Part (c) is satisfied without the error term (when the weights assigned to terms in the sum for $\mathbf{P}_i(B)$ arising from points on the boundary of $B^{(i)}$ are taken into account). The error in part (b) is due to the fact that, although $\mathbf{x}[(g, \mathbf{g})]$ lies within a radius R of $(g, \mathbf{g})$, which point of X it is may depend on nearby points of X outside B. In view of the Lipschitz condition, the total error is

$$O(|G^{(i)}|) = O(\mathrm{vol}_{n-1}(B^{(i)})) = O(\sigma(B)) \ .$$

Now Theorems 5.1 and 5.2 ensure the existence of a limiting value $L(\mathbf{e}_i)$ such that

$$\frac{\mathbf{P}_i(B)}{\mathrm{vol}_n(B)} = L(\mathbf{e}_i) + \epsilon(B) \tag{7.11}$$

with $\epsilon(B) \to 0$ as $\omega(B) \to \infty$ in both the linearly and densely repetitive cases. In the linearly repetitive case $\epsilon(B) = O(\omega(B)^{-\delta_1})$, where δ_1 is the δ of Theorem 5.1.

The second stage of the proof is to use this average value of $\mathbf{P}_i$ to estimate $\mathbf{p}(\mathbf{t}, U\mathbf{e}_i + \mathbf{t})$ for a sample path by applying it to the box B defined by

$$[a_k, b_k] = \begin{cases} [t_i, t_i + U] & \text{for } k = i, \\ [t_k - \frac{1}{2}\sqrt{U}, t_k + \frac{1}{2}\sqrt{U}] & \text{for } k \neq i, \end{cases}$$

where $t_1\mathbf{e}_1 + \cdots + t_n\mathbf{e}_n = \mathbf{t}$. By the Lipschitz property (1.15) every term on the right of (7.10) is

$$\mathbf{p}(\mathbf{t}, U\mathbf{e}_i + \mathbf{t}) + O(\sqrt{U}) \ .$$

So (7.10) and (7.9) give

$$\mathbf{P}_i(B) = U^{(n-1)/2}\mathbf{p}(\mathbf{t}, U\mathbf{e}_i + \mathbf{t}) + O(U^{n/2})$$

and by (7.11)

$$\frac{\mathbf{p}(\mathbf{t}, U\mathbf{e}_i + \mathbf{t})}{U} = L(\mathbf{e}_i) + \epsilon(B) + O(U^{-1/2}) \ ,$$

with $\epsilon(B) \to 0$ as $U \to \infty$ in both the linearly and densely repetitive cases and $\epsilon(B) = O(U^{-\delta_1/2})$ in the linearly repetitive case. This establishes the existence of the limit (7.6) in both cases and gives the estimate (7.8) in the linearly repetitive case with $\delta = \delta_1/2$. $\square$

The following example shows that for linearly repetitive sets X the bound

$$\|\phi(\mathbf{x}) - L(\mathbf{x})\| = O(\|\mathbf{x}\|^{1-\delta})$$

given in Theorem 7.1 cannot be made smaller than a power of $\|\mathbf{x}\|$.

Example 7.1. *There is a linearly repetitive Delone set X in $\mathbb{R}^n$ which has the property that there is a constant $\delta = \delta(X) < 1$ and a sequence $\{\mathbf{x}_i\} \subseteq X$ with $||\mathbf{x}_i|| \to \infty$, such that*

$$||\phi(\mathbf{x}_i) - L(\mathbf{x}_i)|| > ||\mathbf{x}_i||^{1-\delta} \ . \tag{7.12}$$

Proof. Such a set X is constructed as a set of control points of a self-similar tiling that has an expansion constant β which is not a Pisot or Salem number, see Kenyon [22]. The set of control points of any self-similar tiling is a linearly repetitive set, according to a result of Solomyak [67, Lemma 2.3]. The fact that the expansion constant β has an algebraic conjugate with absolute value > 1 produces property (7.12). It follows that such a set X cannot be a Meyer set, because (7.12) violates the Meyer set condition (1.16). $\square$

8. Linear Repetitivity, Diffraction and Model Sets

In this section we study linearly repetitive Delone sets, and consider them as models for "perfectly ordered quasicrystals."

We begin by stating a theorem proved in [27], which shows that linear repetitivity is the slowest possible growth rate of $M_X(T)$ for any translation-minimal set X that is not fully periodic.

Theorem 8.1 *Let X be a repetitive Delone set in $\mathbb{R}^n$. If its repetitivity function $M_X(T)$ satisfies*

$$M_X(T) < \frac{1}{3}T \tag{8.1}$$

for a single value of $T > 0$, then X is an ideal crystal.

This result implies that Conjectures 1.2a and 1.2b are equivalent, as follows. Any linearly repetitive set X in $\mathbb{R}^n$ has $N_X(T) = O(T^n)$ by Theorem 4.1. Conjecture 1.2a asserts that X is densely repetitive, which gives

$$M_X(T)^n < cN_X(T) \quad \text{for } T > T_0 \ , \tag{8.2}$$

with some positive c. Under the assumption that X is aperiodic Theorem 8.1 gives the linear lower bound $M_X(T) > T/3$, and (8.2) yields $N_X(T) > c'T^n$ for $T > T_0$, which is Conjecture 1.2b. In the other direction, if Conjecture 1.2b holds, then an aperiodic linearly repetitive set has $N_X(T) > c'T^n$ for $T > T_0$, and linear repetitivity then gives $M_X(T) < c_1T < c_2(N_X(T))^{1/n}$, so X is densely repetitive, which is Conjecture 1.2a.

We next establish some existence results for linearly repetitive sets. We construct a large class of such sets in the one-dimensional case, given in Theorem 8.2 below. It seems much harder to construct linearly repetitive sets in dimensions $n \geq 2$ that are irreducible in the sense of not being a product of lower-dimensional linearly repetitive sets. There do exist nontrivial examples of linearly repetitive sets in all dimensions $n \geq 2$, based on self-similar constructions.

The one-dimensional construction is based on Beatty sequences of numbers $\alpha \in [0, 1]$ whose continued fractions have bounded partial quotients. Given any symbol sequence $S = \{s_i\} \in \{0,1\}^{\mathbb{Z}}$ and any $\tau > 1$, we obtain a set $X_\tau(S)$ by assigning $x_0 = 0$ and then measuring distances $x_{i+1} - x_i$ to be 1 or τ according as s_i is 0 or 1. The set $X_\tau(S)$ is a Delone set of finite type and it is easy to see that whether or not $X_\tau(S)$ is linearly repetitive depends only on the symbol sequence S.

Lemma 8.1 *Let $S \in \{0,1\}^{\mathbb{Z}}$. Then for any $\tau > 1$, $X_\tau(S)$ is linearly repetitive if and only if S is minimal with recurrence function $\tilde{M}_S(n)$ satisfying $\tilde{M}_S(n) = O(n)$ as $n \to \infty$.*

Proof. This is immediate. □

The aperiodic 2-symbol sequences S with smallest word counts have exactly $n + 1$ words of length n for each $n \geq 1$, see Coven and Hedlund [6]. Of these, the ones which are minimal are exactly the Beatty sequences of irrational slope, as shown in Coven [5] and Paul [47]. It is easy to characterize those Beatty sequences that have a recurrence function with linear growth $O(n)$.

Theorem 8.2 *Let $B_\alpha \subseteq \{0,1\}^{\mathbb{Z}}$ be a Beatty sequence with $\alpha \in [0, 1]$. Then:*

(i) The recurrence function $\tilde{M}_{B_\alpha}(T)$ is bounded if and only if α is rational.

(ii) The recurrence function satisfies

$$\tilde{M}_{B_\alpha}(T) = O(T) \quad as \quad T \to \infty$$

if and only if α is badly approximable, i.e. α is a number whose regular continued fraction expansion has bounded partial quotients.

Recall that badly approximable numbers are characterized as the numbers α for which there exists a constant $c > 0$ such that $|\alpha - a/q| > c/q^2$ for every rational number a/q. For their various properties see Shallit [65].

Proof. When α is rational with denominator q then B_α has period q and $\tilde{M}_{B_\alpha}(T) = q$ for all $T \geq q$.

Now let a_k be the k-th partial quotient and p_k/q_k be the k-th convergent of the regular continued fraction expansion of α. By (4.4) we have

$$\tilde{M}_{B_\alpha}(\ell) = q_k + q_{k+1} \leq (a_{k+1} + 2)q_k \leq (a_{k+1} + 2)\ell$$

for $q_k \leq \ell < q_{k+1}$, and

$$\tilde{M}_{B_\alpha}(q_k) = q_k + q_{k+1} \geq (a_{k+1} + 1)q_k.$$

Also $q_k \to \infty$ as $k \to \infty$ when α is irrational, and in particular when $\{a_k\}$ is unbounded. Hence when α is irational $\tilde{M}_{B_\alpha}(T) = O(T)$ is equivalent to the sequence of partial quotients $\{a_k\}$ of α being bounded. $\square$

Theorem 8.2 implies that there are restrictions on the image in $\mathbb{R}^s$ of the linear form L that is associated by Theorem 7.1 to a linearly repetitive Delone set $X_\tau(B_\alpha)$: this image is necessarily a line in $\mathbb{R}^2$ whose slope is a badly approximable number.

The aperiodic Delone sets X in $\mathbb{R}^n$ that are linearly repetitive have several claims to be the "simplest" aperiodic sets. These claims include:

(1) Their patch-counting functions satisfy $N_X(T) = O(T^n)$. If Conjecture 2.1 is correct, this is the slowest possible growth rate of $N_X(T)$ for aperiodic sets.

(2) Among all aperiodic sets, they have minimal growth rate of the repetitivity function $M_X(T)$.

(3) They have long-range order in the sense of having strict uniform patch frequencies, and hence are diffractive.

(4) They include most, if not all, of the proposed examples of strongly ordered sets with quasicrystalline properties, including point sets associated to irreducible self-affine tilings (Solomyak [67, Lemma 2.3]). They also include various special cut-and-project sets such as Penrose tilings.

For these reasons we consider the class of linearly repetitive Delone sets to be a candidate for the concept of *perfectly ordered quasicrystals.* This class includes ideal crystals; we do not impose aperiodicity as a requirement for being quasicrystalline.

Many authors, however, consider that any definition of quasicrystalline order should also require a well-defined diffraction measure with a nontrivial pure point component ("Bragg peaks").

Definition 8.1 A Delone set X is said to have *strong long range order* or to be *strongly diffractive* if it has a unique diffraction measure $\hat{\gamma}_X$ which includes a pure point component that is relatively dense in $\mathbb{R}^n$.

Not all linearly repetitive Delone sets have this property. The construction of Example 7.1 can be used to produce a linearly repetitive Delone set X which is not a Meyer set and which has no pure point spectrum, using the criterion of Solomyak [66, Theorem 4.4], so is not strongly diffractive. In the opposite direction, cut-and-project sets with a nice window are always strongly diffractive, but need not be linearly repetitive.

We may further clarify the relations between linear repetitivity, strong diffraction, and being a Meyer set, by considering some properties of the address map and the associated topological dynamical system. The orbits of the topological dynamical system $(\mathcal{X}_X, \mathbb{R}^n)$ under the $\mathbb{R}^n$-action can be described using the address map $\phi : [X - X] \to \mathbb{Z}^s$ and a projection map $\pi : \mathbb{R}^s \to \mathbb{R}^n$ (not necessarily orthogonal). We have the following commutative diagram:

$$\begin{tikzcd} \mathbb{Z}^s \arrow[r, hook] & \mathbb{R}^s \arrow[d, "\pi"] \\ {[X-X]} \arrow[u, "\phi"] \arrow[r, hook] & \mathbb{R}^n \end{tikzcd} \tag{8.3}$$

Here the horizontal arrows are inclusions and π is the projection map, defined by $\pi(\mathbf{e}_j) = \mathbf{v}_j$ for $1 \le j \le s$ where $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ is the chosen basis of $X - X$ and $\mathbf{e}_j$ are the unit coordinate vectors. If $L : \mathbb{R}^n \to \mathbb{R}^s$ is the linear map given by Theorem 7.1 then $\pi \circ L$ is the identity. If $\mathbf{0} \in X$ then the addresses $Y = \phi(X)$ of X together with π permit X to be uniquely reconstructed using (8.3). If $(\mathcal{X}_X, \mathbb{R}^n)$ is minimal, then $[X' - X']$ is independent of the choice of $X' \in \mathcal{X}_X$, and a general $X' \in \mathcal{X}_X$ is described by data $(Y', \pi, \mathbf{x}')$, where we choose any point $\mathbf{x}' \in X'$ and $Y' = \phi(X' - \mathbf{x}')$ has $\mathbf{0} \in Y' \subseteq \phi[X' - X']$. This is unique up to choice of $\mathbf{x}'$, so the $\mathbb{R}^n$-orbit of X' is specified by the equivalence class $\{Y'\}$ of translates of Y' in $\mathbb{Z}^s$ that contain $\mathbf{0}$, i.e.

$$\{Y'\} := \{Y' - \mathbf{m} : \mathbf{m} \in Y\}.$$

We regard $\{Y'\}$ as describing a "symbolic dynamics" for this orbit.

The properties of minimality and linear repetitivity of $(\mathcal{X}_X, \mathbb{R}^n)$ are completely determined by the equivalence class $\{Y\}$ of $Y := \phi(X)$ (where we assume $\mathbf{0} \in X$). We can now obtain a new topological dynamical system by holding Y fixed and varying the projection π, to π' say. The pair (Y, π') uniquely determines the preimage X' by (8.3), and if X' is a Delone set then we obtain a new Delone dynamical system $(\mathcal{X}_{X'}, \mathbb{R}^n)$. If Y is linearly repetitive then the new topological dynamical system $(\mathcal{X}_{X'}, \mathbb{R}^n)$ is uniquely ergodic, hence gives rise to a unique metric dynamical system $(\mathcal{X}_{X'}, \mathbb{R}^n, d\mu')$, and all sets in $\mathcal{X}_{X'}$ are diffractive.

On the other hand, the existence of a pure point component of the diffraction measure $\hat{\gamma}_{X'}$ depends on the projection π'. Kolár et al. [23, Sect. 5.2] have one-dimensional examples in which $Y \subseteq \mathbb{Z}^2$ is linearly repetitive, and $X' = X_\pi$ seems to have some pure point spectrum for certain projections π but no pure point spectrum for other projections. (In this example the sets X' correspond to tilings of the line with tiles of two lengths, and varying π is equivalent to varying the ratios of the tile lengths while leaving the pattern of the tiles fixed.)

To summarize: *The property of being linearly repetitive depends only on the symbolic equivalence class $\{Y\}$, while the presence or absence of a pure point component in the diffraction spectrum depends on both $\{Y\}$ and the projection π.*

We now look at some one-dimensional examples, given by sets $X_\tau(S)$ constructed, as before, from a symbol sequence S and a real parameter τ. Lemma 8.1 showed that the linear repetitivity of $X_\tau(S)$ is determined by S alone, and is independent of τ. In contrast, the property of $X_\tau(S)$ being a Meyer set sometimes depends on whether τ is rational or irrational, see [25, Section 5]. This can occur because when τ is rational ϕ maps $[X]$ to $\mathbb{Z}$, but when τ is irrational it maps $[X]$ to $\mathbb{Z}^2$. For Beatty sequences B_α the sets $X_\tau(B_\alpha)$ are always Meyer sets by the criterion [25, Theorem 5.1].

We end with some open problems suggested by the observations made in this section.

Problem 8.1. Explicitly characterize all two-sided shifts Σ on a finite alphabet that are minimal and have recurrence function $\tilde{M}_\Sigma(\ell)$ bounded above by $C\ell$ for some constant C.

Problem 8.2. Characterize the possible images in $\mathbb{R}^s$ of linear forms $L : \mathbb{R}^n \to \mathbb{R}^s$ for which there exists a linearly repetitive Delone set X with $\mathbf{0} \in X$ that is associated to L in the manner of Theorem 7.1.

Problem 8.3 Explicitly characterize all cut-and-project sets that are linearly repetitive.

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Appendix A. Repetitivity function for cubical patches

We can define the *cubical repetitivity function* $\tilde{M}_X(T)$ to be the smallest value of M such that every cube of side $2M + T$ contains a translate of every cubical T-patch of X. Let $\tilde{N}_X(T)$ count the number of cubical patches of side T. Then a Delone set X is a *densely repetitive set for cubes* if

$$\tilde{M}_X(T) = O(\tilde{N}_X(T)^{1/n}) \ , \quad \text{as } T \to \infty.$$

This is equivalent to X being densely repetitive if $N_X(T)$ grows polynomially in T, or if the dimension $n = 1$; for $N_X(T)$ and $\tilde{N}_X(T)$ are then within a constant factor of each other. However, if $N_X(T)$ grows faster than polynomially the two concepts may differ, as far as we know. The proof of Theorem 6.1 also works in the context of densely repetitive Delone sets for cubes, and shows that they too have strict uniform patch frequencies.

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