Dynamics of a Family of Piecewise-Linear Area-Preserving Plane Maps
I. Invariant Circles

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ABSTRACT

This paper studies the behavior under iteration of the maps $T_{ab}(x, y) = (F_{ab}(x) - y, x)$ of the plane $\mathbb{R}^2$, in which $F_{ab}(x) = ax$ if $x \geq 0$ and bx if $x < 0$. This family of piecewise-linear maps has the parameter space $(a, b) \in \mathbb{R}^2$. These maps are area-preserving homeomorphisms of $\mathbb{R}^2$ that map rays from the origin into rays from the origin. The orbits of the map are solutions of the nonlinear difference operator of Schrödinger type $-x_{n+2} + 2x_{n+1} - x_n - V_\mu(x_{n+1})x_{n+1} = Ex_{n+1}$, with energy parameter $E = 2 - \frac{1}{2}(a + b)$ and with an antisymmetric step-function potential $V_\mu(x_n)$ specified by $\mu = \frac{1}{2}(a - b)$. We study the parameter set Ω_{bdd} of (a, b) where the map has at least one bounded orbit and the smaller parameter set where all orbits are bounded. We determine all cases where T_{ab} is purely periodic. We also determine special parameter values where T_{ab} has every orbit bounded and contained in an invariant circle with an irrational rotation number; The invariant circles are piecewise unions of arcs of conic sections. These parameter values yield quasiperiodic solutions to the nonlinear difference operator.

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1. Introduction

We study the behavior under iteration of the two parameter family of piecewise-linear homeomorphisms of $\mathbb{R}^2$ given by

$$T_{ab}(x, y) = \begin{cases} (ax - y, x) & \text{if} \quad x \geq 0, \\ (bx - y, x) & \text{if} \quad x < 0. \end{cases} \tag{1.1}$$

The parameter space is $(a,b) \in \mathbb{R}^2$. This map can be written

$$T_{ab}(x,y) = \begin{bmatrix} F_{ab}(x) & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{1.2}$$

in which

$$F_{ab}(x) = \begin{cases} a & \text{if} \quad x \geq 0, \\ b & \text{if} \quad x < 0; \end{cases} \tag{1.3}$$

we view elements of $(x,y) \in \mathbb{R}^2$ as column vectors. The formula (1.2) shows that $T_{ab}(x,y)$ is a homeomorphism, since

$$\begin{aligned} T_{ab}^{-1}(x,y) &= \begin{bmatrix} F_{ab}(y) & -1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & F_{ab}(y) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned} \tag{1.4}$$

It preserves the area form $d\omega = dx \wedge dy$, and it also maps rays from the origin into rays from the origin.

The maps T_{ab} have some claim to being the simplest family of nonlinear area-preserving maps of the plane, which gives one reason to study their dynamics under iteration. A second reason to consider the dynamics of T_{ab} (in terms of the parameters (a,b)), the one motivating this paper, is that iteration of a fixed map T_{ab} encodes the solutions of the second-order nonlinear recurrence

$$x_{n+2} = \mu |x_{n+1}| + \nu x_{n+1} - x_n \tag{1.5}$$

via

$$T_{ab}(x_{n+1}, x_n) = (x_{n+2}, x_{n+1}) \tag{1.6}$$

in which

$$\mu = \frac{1}{2}(a-b), \qquad \nu = \frac{1}{2}(a+b).$$

This recurrence can be interpreted as a solution to the one-dimensional nonlinear difference equation of Schrödinger type

$$-x_{n+2} + 2x_{n+1} - x_n + V_\mu(x_{n+1})x_{n+1} = Ex_{n+1}, \tag{1.7}$$

in which the "potential" is

$$V_\mu(x) := \begin{cases} -\mu & \text{if} \quad x \geq 0, \\ \mu & \text{if} \quad x < 0. \end{cases} \tag{1.8}$$

and the energy value

$$E := 2 - \nu\ . \tag{1.9}$$

Holding the potential [1] V_μ fixed and letting the parameter ν vary amounts to studying the set of solutions for all (real) energy values $E = 2 - \nu$.

The equation (1.7) may be compared with two much-studied classes of operators.

[1]Strictly speaking, the potential is $V_\mu(x)x$.

(i) The nonlinear Schrödinger operator with a general (real-valued) potential V takes the form

$$-\Delta u + V(|u|)u = i\frac{\partial}{\partial t}u,$$

where the function $u(t,x)$ is complex-valued. For stationary phase solutions $u(t,x) = e^{iEt}z(x)$ the phase component factors out and leads to

$$-\Delta z + V(|z|)z = -Ez.$$

The discretized version of this equation involves *discrete nonlinear Schrödinger operator*

$$\Phi_V(z)_n = -z_{n+2} + 2z_{n+1} - z_n - V(|z_{n+1}|)z_{n+1}$$

in the eigenvalue equation

$$-z_{n+2} + 2z_{n+1} - z_n - V(|z_{n+1}|)z_{n+1} = Ez_{n+1}, \tag{1.10}$$

for $z_n \in \mathbb{C}$, see [26]. The special case of solutions to this operator with constant phase $\mathbf{z}_n = e^{\imath\phi}x_n$ (with $0 \leq \phi < \pi$, x_n real), reduces to the general form (1.7) with a symmetric potential $V(x) = V(-x)$ (by cancelling out all phases). The recurrence (1.7) that we consider has a similar form to (1.10), with the difference that the potential $V(x)$ is antisymmetric, i.e. $V(x) = -V(-x)$.

(ii) Discrete linear Schrödinger operators on the line have been extensively studied, as simple models for conduction/insulation transitions of elections in metals, and more recently in quasicrystals, see [4], [10], [15], [16], [23], [24], [25]. The linear difference operator

$$-x_{n+2} + 2x_{n+1} - x_n + V(n+1)x_{n+1} = Ex_{n+1}, \tag{1.11}$$

with a potential $V(n)$ depending only on position is often called the "tight-binding" approximation to the Schrödinger operator on the line. Here one is interested in characterizing the values of E which have *extended states* [2]; these are bounded orbits, that is, real-valued orbits in $l_\infty(\mathbb{Z})$. We define the *l_∞-spectrum* for a fixed potential V to be

$$Spec_\infty[V] := \{E : \text{ The discrete Schrödinger equation (1.11) has a bounded orbit}\}.$$

In 1976 Hofstadter [15] computed numerically the l_∞-spectrum $\Sigma_{\lambda,\alpha}$ for the discrete Schrödinger operator with a quasiperiodic potential $V(n) = \lambda\cos(2\pi\alpha n)$, holding $\lambda = 2$ fixed, and letting α vary. (This equation models allowable conduction energies of an electron moving in a two-dimensional cubic crystal with a strong magnetic field applied perpendicularly.) He observed that the spectrum formed a two dimensional picture in the (α, E) plane resembling a fractal "butterfly". He gave a a conjectural explanation for the fractal structure using a kind of renormalization. For fixed irrational α the l_∞-spectrum appeared to be a Cantor set of measure zero. Hofstadter's model has been extensively studied, with one goal being to give a rigorous justification of this "butterfly" structure; see Bellissard [5] and Sjöstrand [22]. This spectrum for fixed parameters (λ, α) has a "band" structure when α is rational, and M. Kac conjectured that for irrational α and all $\lambda > 0$ it is a Cantor set; the status of this problem is given in

[2]Extended states are sometimes required not to be in l_2 but we do not impose this condition.

Helffer, Kerdelhué and Sjöstrand [14] and Helffer and Kerdelhué [13]. It has been proved for certain irrational α that the l_∞-spectrum is a Cantor set of zero measure when $\lambda = 1$, and of positive measure $2|1-\lambda|$, for $0 < \lambda < 1$. For general results on the spectra of discrete Schrödinger operators see Bougerol and Lacroix[7] and Pastur and Figotin [21].

In this paper and its sequel we study the orbits of T_{ab} for all parameter values, with the particular object of determining properties of the set Ω_{bdd} of parameter values having at least one bounded orbit (not identically zero), as well as the smaller set Ω_{tot} of parameter values having *all* orbits bounded. The set Ω_{bdd} is an analogue of the Hofstadter "butterfly", with $(\mu, 2-\nu) = (\frac{1}{2}(a-b), 2-\frac{1}{2}(a+b))$ playing the role of (α, E) in Hofstadter's model. For the nonlinear difference operator (1.7) we are able to obtain rigorous results by exploiting the piecewise-linear structure of the maps.

In part I we prove basic results about the dynamics of T_{ab} under iteration. This dynamics includes the induced dynamics on rays, which is described by a circle map S_{ab} (defined in §2), as well as a stretching/shrinking motion within rays. As is usual for circle maps, there is a dichotomy in the dynamics between the cases of rational rotation number and irrational rotation number. We classify the dynamics in the rational rotation number case, and characterize all parameter values where the motion is purely periodic. In the irrational rotation number case we explicitly determine cases where there is purely quasiperiodic motion.

Our interest in purely periodic maps T_{ab} was stimulated by the observation of Morton Brown [8] in 1983 that the recurrence $x_{n+1} = |x_n| - x_{n-1}$ has all solutions periodic of period 9; this is the case $\mu = 1$, $\nu = 0$ of (1.5). We show that this example belongs to a discrete infinite family $(\mu, \nu) = (2\cos\frac{\pi}{n}, 0)$, whose n-th member is periodic with period n^2 for $n \geq 3$, see Example 3.2. We show that the parameter values having purely periodic maps consist of a countable number of isolated points and one-parameter families. We explicitly exhibit some one-parameter families in which all members T_{ab} are purely periodic with a fixed period. (Example 3.3).

The most novel result of part I is the characterization of certain one-parameter families of T_{ab} whose generic members exhibit quasiperiodic motion. These families are those for which $(0,1)$ and $(0,-1)$ fall in the same orbit with a fixed symbolic dynamics between their occurrences, and which contain a parameter value having an irrational rotation number. We show that generic members of any such family have each orbit contained in an invariant circle which is a finite union of arcs of conic sections. We rigorously establish for specific examples that the irrational rotation number property holds (Examples 5.1 and 5.2).

We defer a detailed study of Ω_{bdd} to part II. There we show that Ω_{bdd} is a closed set which has certain properties like those ascribed to the Hofstadter "butterfly". The main result of part II shows that for all transcendental values of μ the analogue of the l_∞-spectrum associated to the potential V_μ of the nonlinear difference operator (1.7) is a Cantor set (perhaps of positive one-dimensional Lebesgue measure). Part II also presents numerical evidence suggesting that Ω_{bdd} has positive two-dimensional Lebesgue measure.

The results of part I have a bearing on the set Ω_{bdd}, because there appears to be a close relation between the narrow set of parameter values having piecewise conic invariant circles, which we show has two-dimensional Lebesgue measure zero, and the larger set Ω_{bdd}. Let Ω_{conic} denote the set of parameter values having a piecewise conic invariant circle as prescribed by Theorem 5.6, and let Ω_{per} denote the set of parameter values (a, b) for which T_{ab} is periodic. Then we have

$$\Omega_{per} \cup \Omega_{conic} \subseteq \Omega_{tot} \subseteq \Omega_{bdd}.$$

We formulate the following conjecture.

Conjecture A. *The set $\Omega_{per} \cup \Omega_{conic}$ of pairs (a, b) for which T_{ab} is either periodic or has a piecewise conic invariant circle is dense in Ω_{bdd}.*

In §6 we show that $\Omega_{per} \cup \Omega_{conic}$ has Hausdorff dimension 1, hence has two-dimensional Lebesgue measure zero. We also present nonrigorous numerical evidence in §5 of invariant circles for a variety of parameter values which appear to fall outside the set $\Omega_{per} \cup \Omega_{conic}$.

The contents of the paper are as follows. In §2 we summarize the main results, and then establish them in §3-§6 and the appendix. The appendix contains detailed proofs to rigorously justify Examples 5.1-5.3, exhibiting parameter values with irrational rotation number.

We conclude the introduction with two remarks. First, we note that another extensively studied two-parameter family of plane maps which are piecewise-affine with two pieces are the Lozi maps, introduced by Lozi [18]. These maps have the form

$$L(x, y) = (1 - a|x| + y, bx),$$

where a, b are real and $b < 0$. These maps are homeomorphisms of $\mathbb{R}^2$, but do not preserve area (except for $b = -1$), nor do they take rays through the origin to rays through the origin. Lozi maps are known to exhibit a wide range of chaotic behavior, including strange attractors, see Misiurewicz [19] and Collet and Levy [9]. The maps we consider in this paper are much simpler than the Lozi maps, but our results show they have nontrivial behavior under iteration.

Second, we note that there are analogies of (1.7) to conditions for ground states (locally minimal states) in Aubry-Mather theory, cf. Aubry and Le Daeron [2]. However the potentials involved in (1.7) do not seem very natural from the perspective of that theory.

We are indebted to T. Spencer for helpful comments relating (1.7) to nonlinear Schrödinger operators.

Notation. We write $\mathbf{v} = (\mathbf{v}_x, \mathbf{v}_y) \in \mathbb{R}^2$, to be viewed as a column vector. An interval $[\mathbf{v}_1, \mathbf{v}_2)$ of the unit circle, or corresponding sector $\mathbb{R}^+[\mathbf{v}_1, \mathbf{v}_2)$ of the plane $\mathbb{R}^2$, is the one specified by going counterclockwise from $\mathbf{v}_1$ to $\mathbf{v}_2$.

2. Summary of Results

The parameter space of the map can be taken to be either (a, b) or (μ, ν), as these are equivalent by

$$\mu = \frac{1}{2}(a - b), \qquad \nu = \frac{1}{2}(a + b). \tag{2.1}$$

Both coordinate systems have their advantages, and we write the map (1.1) as T_{ab}, $T_{\mu\nu}$ accordingly. It is convenient to represent the action of T_{ab}, acting on column vectors $\mathbf{v}_n = (x_{n+1}, x_n)$ as

$$T_n(\mathbf{v}_0) = \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = M_n(\mathbf{v}_0) \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}, \tag{2.2}$$

in which

$$M_n(\mathbf{v}_0) = \prod_{i=1}^{n} \begin{bmatrix} F_{ab}(x_i) & -1 \\ 1 & 0 \end{bmatrix} := \begin{bmatrix} F_{ab}(x_n) & -1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} F_{ab}(x_2) & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{ab}(x_1) & -1 \\ 1 & 0 \end{bmatrix} . \tag{2.3}$$

Conjugation by the involution $J_0 : (x, y) \to (-x, -y)$ gives

$$T_{ba}(x, y) = J_0^{-1} \circ T_{ab} \circ J_0 \ . \tag{2.4}$$

Thus, in studying dynamics, without loss of generality we can restrict to the closed half-space $\{(a, b) : a \geq b\}$ of the (a, b) parameter space. This corresponds to the region $\{(\mu, \nu) : \mu \geq 0\}$ of the $(\mu.\nu)$ parameter space, with $T_{\mu\nu}$ conjugate to $T_{-\mu,\nu}$.

Conjugation by the involution $R : (x, y) \to (y, x)$ takes T_{ab} to its inverse map T_{ab}^{-1}, i.e.

$$T_{ab}^{-1}(x, y) = R^{-1} \circ T_{ab} \circ R \ , \tag{2.5}$$

see Theorem 3.4. Thus, the dynamics of running the iteration backwards is essentially the same as running it forwards.

The map T_{ab} is homogeneous, so sends rays $[\mathbf{v}] := \{\lambda \mathbf{v} : \lambda \geq 0\}$ to rays $T_{ab}([\mathbf{v}])$. Therefore we obtain a well-defined circle map $S_{ab} : S^1 \to S^1$, for $0 \leq \theta \leq 2\pi$, given by

$$S_{ab}(e^{i\theta}) := \frac{T_{ab}(e^{i\theta})}{\|T_{ab}(e^{i\theta})\|} \ , \qquad 0 \leq \theta \leq 2\pi \ . \tag{2.6}$$

where $\|T_{ab}(e^{i\theta})\|$ is the Euclidean norm on $\mathbb{R}^2$, and we identity $e^{i\theta} = x + yi \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$. By abuse of language we shall also sometimes treat the circle map as $S_{ab}(\theta)$ having domain $\mathbb{R}/2\pi\mathbb{Z}$. Understanding the dynamics of the map T_{ab} subdivides into two problems: study of the dynamics of the circle map S_{ab}, and study of the motion of points inside the individual rays.

In §3 we study the circle map S_{ab}. It has a well-defined rotation number $r(S_{ab})$, and we determine the allowed range of the rotation number, as follows.

Theorem 2.1 *For fixed real a, and $-\infty < b < \infty$, the rotation number $r(S_{ab})$ is continuous and nonincreasing in b, and completely fills out the following intervals.*

(i) For $a < 0$,

$$r(S_{ab}) \in \left[0, \frac{1}{2}\right] \ . \tag{2.7}$$

(ii) Let $0 \leq a < 2$. Then for each integer $n \geq 2$, on the interval $2\cos\frac{\pi}{n} \leq a < 2\cos\frac{\pi}{n+1}$,

$$r(S_{ab}) \in \left[0, \frac{1}{n+1}\right] . \tag{2.8}$$

(iii) For $a \geq 2$ one has

$$r(S_{ab}) = 0 \ . \tag{2.9}$$

We next determine the range of the rotation number in (μ, ν)-space, for fixed μ.

Theorem 2.2 *For fixed real μ and $-\infty \le \nu < \infty$ the values of $r(S_{\mu\nu})$ are continuous and nonincreasing in ν and completely fill out the closed interval*

$$r(S_{\mu\nu}) \in \left[0, \frac{1}{2}\right] . \tag{2.10}$$

In §4 we study maps T_{ab} for which $r(S_{ab})$ is rational. The maps S_{ab} exhibit a "mode-locking" behavior so that there are open sets in the parameter space for which $r(S_{ab})$ takes a fixed rational value (for certain rationals), as in Theorem 2.1(iii) above. We first characterize those values of (a, b) where T_{ab} is periodic.

Theorem 2.3 *T_{ab} is of finite order if and only if the orbit of $(0, 1)$ is periodic.*

We use Theorem 2.3 to determine various curves in parameter space on which T_{ab} is purely periodic, see Example 3.3.

The main result of §4 characterizes all cases of rational rotation number, as follows.

Theorem 2.4 *If the rotation number $r(S_{ab})$ is rational, then S_{ab} has a periodic orbit, and one of the following three possibilities occurs.*

(i) S_{ab} has exactly one periodic orbit. Then T_{ab} has exactly one periodic orbit (up to scaling) and all other orbits diverge to $+\infty$ as $n \to \pm\infty$.

(ii) S_{ab} has exactly two periodic orbits. Then T_{ab} has no periodic orbits. All orbits of T_{ab} diverge in modulus to $+\infty$ as $n \to \pm\infty$, with the exception of orbits lying over the two periodic orbits of S_{ab}. These exceptional orbits have modulus diverging to $+\infty$ in one direction and to 0 in the other direction, with forward divergence for one, and backward divergence for the other.

(iii) S_{ab} has at least three periodic orbits. Then T_{ab} is of finite order, i.e. $T_{ab}^k = I$ for some $k \ge 1$, and all its orbits are periodic.

In §5 we study maps T_{ab} for which the rotation number $r(S_{ab})$ is irrational. We show the existence of an orbit with elements of modulus bounded away from 0 and ∞ implies that there exists an invariant circle. Then we show there exist parameter values having invariant circles with a striking structure.

Theorem 2.5 *Suppose that the rotation number $r(S_{ab})$ is irrational, and that the S_{ab} orbit of $(0, 1)$ contains $(0, -1)$. Then the following hold.*

(1) The T_{ab} orbit of $(0, 1)$ contains $(0, -1)$.

(2) The closure of every (nonzero) orbit of T_{ab} is an invariant circle, which is a piecewise union of arcs of conic sections. The conic sections occurring in such an invariant circle are all of the same type, either ellipses, hyperbolas or straight lines.

This result follows from Theorem 5.6, which gives more detailed information. We exhibit specific parameter values where the hypotheses of this theorem holds, for example $a = 2^{1/4}$, $b = -2^{1/4}$.

In §6 we show that the set Ω_{conic} has Hausdorff dimension 1 and discuss Conjecture A.

In the appendix we give detailed proofs of properties of Examples 5.1-5.3 in §5, giving explicit parameter values having piecewise conic invariant circles with irrational rotation number.

3. Associated Circle Map

We study the circle map $S_{\mu\nu}(\theta)$ in the (μ,ν)-coordinates.

Theorem 3.1 *Each $S_{\mu\nu}(\theta)$ is an orientation-preserving homeomorphism from S^1 to itself. The derivative $\frac{d}{d\theta}(S_{\mu\nu}(\theta))$ is continuous and of bounded variation.*

Proof. Coordinatizing S^1 as $(\cos(\theta),\sin(\theta))$, we have

$$S_{\mu\nu}(\theta) = \cot^{-1}\Big(\frac{\nu\cos(\theta)+\mu|\cos(\theta)|-\sin(\theta)}{\cos(\theta)}\Big) + \pi\,[\cos(\theta)<0]\,.$$

This is clearly continuous, mod 2π; the only cause for concern is when $\cos(\theta)=0$, where it can be easily verified that the pieces match up. Similarly, if we look at the derivative of $S_{\mu\nu}$, we get:

$$\frac{d}{d\theta}S_{\mu\nu}(\theta) = \frac{1}{(\nu\pm\mu)^2\cos^2(\theta)-2(\nu\pm\mu)\cos(\theta)\sin(\theta)+1},$$

with $+$ when $\cos(\theta)>0$ and $-$ when $\cos(\theta)<0$. This is piecewise continuous, and again, the pieces match up. Finally, the second derivative of $S_{\mu\nu}$ exists except when $\cos(\theta)=0$, and is bounded. We have

$$\frac{d^2}{d\theta^2}S_{\mu\nu}(\theta) = \frac{(\nu\pm\mu)^2\sin(2\theta)+2(\nu\pm\mu)\cos(2\theta)}{((\nu\pm\mu)^2\cos^2(\theta)-2(\nu\pm\mu)\cos(\theta)\sin(\theta)+1)^2},$$

and the numerator is bounded by

$$|(\nu\pm\mu)^2\sin(2\theta)+2(\nu\pm\mu)\cos(2\theta)| \le (|\nu|+|\mu|)\sqrt{(|\nu|+|\mu|)^2+4}$$

and the denominator by

$$|(\nu\pm\mu)^2\cos^2(\theta)-2(\nu\pm\mu)\cos(\theta)\sin(\theta)+1|^{-1} \le 1+\frac{1}{2}((|\nu|+|\mu|)^2+(|\nu|+|\mu|)\sqrt{(|\nu|+|\mu|)^2+4}).$$

Thus, we can conclude that $\frac{d}{d\theta}S_{\mu\nu}(\theta)$ is continuous and of bounded variation. Finally, $S_{\mu\nu}$ is orientation-preserving, since $\frac{d}{d\theta}S_{\mu\nu}(\theta)$ is always positive, and is a homeomorphism, since $S_{\mu\nu}^{-1}$ obtained by conjugating $S_{\mu\nu}$ by reflection through the line $x=y$. ∎

The dynamics of iteration of maps on the circle is well understood. A basic result ([20, p.33]) is that every orientation-preserving homeomorphism $S : S^1 \to S^1$ has a well-defined *rotation number*, defined by

$$r(S) := \lim_{n\to\infty}\frac{\tilde{S}^{(n)}(x)-x}{n}, \tag{3.1}$$

where $\tilde{S}$ is any lift of S to $\mathbf{R}$ (2π lifts to 1); here $r(S)$ is independent of the initial value x (mod 1) and of the choice of lift. In particular, if S has a periodic point, then $r(S)$ is rational (and conversely, as well).

This section gives general results on the rotation number $r(S_{ab})$; the following two sections cover the case $r(S_{ab})$ rational and $r(S_{ab})$ irrational, respectively.

Theorem 3.2 (i) *For fixed b, the rotation number $r(S_{ab})$ is nonincreasing in a, and for fixed a it is nonincreasing in b.*

(ii) *For fixed μ, the rotation number $r(S_{\mu\nu})$ is nonincreasing in ν.*

Proof. (i). It suffices to show that $r(S_{ab})$ is nonincreasing in a, the result for b follows from the relation $r(S_{ab}) = r(S_{ba})$.

Now, consider the behavior of $S_{ab}(\theta)$ as a increases. If we can show that it moves clockwise as a increases, for every θ, then we are done, by inspection of equation (3.1). To show that $S_{ab}(\theta)$ moves clockwise with a, we need only show that

$$L = T_{ab}(\mathbf{v})_x \frac{\partial}{\partial a} T_{ab}(\mathbf{v})_y - T_{ab}(\mathbf{v})_y \frac{\partial}{\partial a} T_{ab}(\mathbf{v})_x \leq 0,$$

for all $\mathbf{v} \in \mathbb{R}^2$. For $\mathbf{v}_x < 0$, this is clearly 0, since $T_{ab}(\mathbf{v})$ is independent of a in that case. For $\mathbf{v}_x \geq 0$, $T_{ab}(\mathbf{v}) = (a\mathbf{v}_x - \mathbf{v}_y, \mathbf{v}_y)$; a simple calculation gives

$$L = -\mathbf{v}_x^2 \leq 0,$$

so (i) is proved.

(ii). As ν increases, both a and b increase, so the result essentially follows from (i). More precisely, for fixed θ the point $S_{\mu\nu}(\theta)$ moves clockwise as ν increases. ∎

The fact that the rotation number is monotonic in b allows us to give fairly strong bounds on the rotation number, for a given a.

Theorem 3.3 *For each fixed a and $-\infty \leq b \leq \infty$ the values of $r(S_{ab})$ completely fill out the following intervals, including their endpoints.*

(i) For $a < 0$, the rotation number satisfies

$$r(S_{ab}) \in \left[0, \frac{1}{2}\right] . \tag{3.2}$$

(ii) Let $0 \leq a \leq 2$. For each $n \geq 2$ on the interval $2\cos\frac{\pi}{n} \leq a < 2\cos\frac{\pi}{n+1}$, the rotation number satisfies

$$r(S_{ab}) \in \left[0, \frac{1}{n+1}\right] . \tag{3.3}$$

(iii) For $a \geq 2$, the rotation number satisfies

$$r(S_{ab}) = 0 . \tag{3.4}$$

Proof. We first consider the lower bounds in (i)–(iii). Now $r(S_{ab})$ is nonincreasing in b, and S_{ab} has a fixed point whenever $b \geq 2$; the existence of a fixed point implies $r(S_{ab}) = 0$. This establishes the lower bound and shows it is attained for all real a.

For the upper bound in case (iii), if $a \geq 2$ then $r(S_{ab}) = 0$ since $r(S_{ab}) = r(S_{ba})$ by the conjugacy (2.4), and $r(S_{ba}) = 0$.

For the upper bound in the other cases, these will follow if we show they are attained for all sufficiently large negative b. Suppose first that $-2 \leq a \leq 2$, and define θ so that $a = 2\cos\theta$, $0 \leq \theta \leq \pi$. Let $x_i(\mathbf{v}) = (T_{ab}^{(i)}\mathbf{v})_x$. Then we claim that, for b sufficiently small, there is a periodic point $\mathbf{v}$ of S_{ab} such that $\mathbf{v}_x < 0$, $x_i(\mathbf{v}) > 0$, $1 \leq i \leq n$, and $S_{ab}^{(n+1)}(\mathbf{v}) = (\mathbf{v})$, where n is as in the hypothesis ($n = 1$ for $a < 0$). Note that $n = \lfloor \frac{\pi}{\theta} \rfloor$.

Now, note that a periodic point of S_{ab} satisfying the constraints must be a real eigenvector of the matrix $M = \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} b & -1 \\ 1 & 0 \end{bmatrix}$. Since this matrix has determinant 1, a necessary condition is that its trace be at least 2 (if the trace were negative, the sign condition would necessarily be violated). This gives a polynomial inequality in a and b; in terms of θ, this is

$$\sin((n+1)\theta)b \geq 2\sin(n\theta) + 2\sin(\theta).$$

Note that the choice of n forces $\sin((n+1)\theta) \leq 0$, so we have:

$$b \leq 2\frac{\sin(n\theta) + \sin(\theta)}{\sin((n+1)\theta)}.$$

Now, define $c \geq 0$ by $c\sin(\theta) = b\sin((n+1)\theta) - 2\sin(n\theta) - 2\sin(\theta)$. Then the eigenvector of M of eigenvalue ≥ 1 is:

$$\mathbf{v} = (\frac{\sin((n+1)\theta)}{\sin(\theta)}, \frac{1}{2}(c - \sqrt{c(c+4)}) + 1 + \frac{\sin(n\theta)}{\sin(\theta)}).$$

It remains only to verify that $\mathbf{v}$ satisfies the sign conditions. Now, $v_x \leq 0$ by inspection, so we need consider only x_i, $1 \leq i \leq n$. By explicitly performing the appropriate matrix multiplication, we get:

$$x_m = \frac{\sin(m\theta)}{\sin(\theta)}k + \frac{\sin((n-m+1)\theta)}{\sin(\theta)},$$

for $1 \leq m \leq n$, where $k = 1 + \frac{1}{2}(c + \sqrt{c(c+4)})$. This is clearly positive for all $1 \leq m \leq n$, so the result is shown.

The remaining upper bound case for $a < -2$ is handled by rewriting the proof above explicitly in terms of a for the $n = 1$ case; we omit the details. ■

Proof of Theorem 2.1. By Theorem 3.2(i) the rotation number $r(S_{ab})$ is nonincreasing in b, and by Theorem 3.2 it has rotation number attaining the endpoint values of the given intervals, in cases (i)–(iii). It remains only to show that for each a, $r(S_{ab})$ is continuous in b, for it must then fill out the whole interval.

We first show that for varying b, $r(S_{ab})$ assumes every rational value in the given interval. The proof of Theorem 3.2 shows that $S_{ab}(0)$ moves clockwise continuously in a as a increases. It follows that $S_{ab}^{(n)}(0)$ has the same behavior for each n. If $r = \frac{p}{q}$ is interior to the interval, then examining $S_{ab}^{(q)}(0)$ as b increases, its continuous variation in b requires it to pass through a fixed point $S_{ab}^{(q)}(0) = 0$ such that $\theta = 0$ is a periodic point of rotation number $\frac{p}{q}$, because if it does not, the rotation number never increases beyond $\frac{p}{q}$, a contradiction.

Any nondecreasing function $f : \mathbb{R} \to [a, b]$ which takes every rational value in $[a, b]$ is continuous. The nondecreasing property guarantees that the left limit

$$\lim_{x \searrow x_0^+} f(x) = f^+(x_0)$$

and right limit

$$\lim_{x \nearrow x_0^-} f(x) = f^-(x_0)$$

both exist, with $f^-(x_0) \leq f^+(x_0)$. If $f^-(x_0) < f^+(x_0)$, then since $f(x_0)$ can take only one rational value in the interval $(f^-(x_0), f^+(x_0))$, some rational value is omitted from the range of $f(x)$, a contradiction. Thus $f^-(x_0) = f^+(x_0)$ and $f(x)$ is continuous at x_0. ∎

Proof of Theorem 2.2. Let μ be fixed. Theorem 3.2(i) shows that $r(S_{\mu\nu})$ is nondecreasing in ν, and $r(S_{\mu\nu}) \in [0, \frac{1}{2}]$ by Theorem 3.4. We have $r(S_{\mu\nu}) = 0$ when $\nu \geq 2 - \mu$ by Theorem 3.4(iii) since $a = \mu + \nu \geq 2$. For ν negative and large the matrix $M = \begin{bmatrix} b & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix}$ has real eigenvalues with eigenvector of sign $(+,-)$ hence $S_{\mu\nu}$ has periodic point of period 2 and $r(S_{\mu\nu}) = 2$.

To establish continuity of $r(S_{\mu\nu})$ in ν, it suffices to show that $S_{\mu\nu}$ has periodic points with rotation number $\frac{p}{q}$ for all rational values $0 < \frac{p}{q} < \frac{1}{2}$. This follows similarly to the proof of Theorem 2.1. (We will later prove a stronger result in Lemma 4.1.) ∎

For later use we characterize behavior of $T_{\mu\nu}$ in the special cases where $S_{\mu\nu}^{(n)}(0,1) = (0, \pm 1)$ or $S_{\mu\nu}^{(n)}(0,-1) = (0, \pm 1)$.

Theorem 3.4 *Let $n \geq 1$ and suppose that $S_{ab}^{(n)}(0,1) = (0, \pm 1)$, or $S_{ab}^{(n)}(0,-1) = (0, \pm 1)$, Then one of the following two relations holds:*

$$T_{ab}^{(n)}(0,1) = (0, \lambda), \tag{3.5}$$

$$T_{ab}^{(n)}(0,-1) = (0, -\lambda^{-1}). \tag{3.6}$$

where λ is a nonzero real number.

(i) If $\lambda > 0$, then both relations above hold. In addition,

$$T_{ab}^{(n)}(-1,0) = (-\lambda, 0) \quad \text{and} \quad T_{ab}^{(n)}(1,0) = (\lambda^{-1}, 0). \tag{3.7}$$

The rotation number $r(S_{ab})$ is rational.

(ii) If $\lambda < 0$, then necessarily $\lambda = -1$. In the first case

$$T_{ab}^{(n)}(0,1) = (0,-1) \quad \text{and} \quad T_{ab}^{(n)}(-1,0) = (1,0), \tag{3.8}$$

while in the second case,

$$T_{ab}^{(n)}(0,-1) = (0,1) \quad \text{and} \quad T_{ab}^{(n)}(1,0) = (-1,0). \tag{3.9}$$

The rotation number $r(S_{\mu\nu})$ can be irrational or rational.

Proof. We check that the involution $R(x,y) = (y,x)$ has

$$T_{ab}^{-1}(x,y) = R^{-1} \circ T_{ab} \circ R\ , \tag{3.10}$$

We have $T_{ab}(x,y) = (ax - y, x)$ if $x \geq 0$, and $(bx - y, x)$ if $x < 0$. This gives

$$T_{ab}(x,y) = (z,w) \quad \Rightarrow \quad T_{ab}(w,z) = (y,x),$$

where in fact $w = x$, which implies (3.10). By induction on $n \geq 1$ this yields the implication

$$T_{ab}^{(n)}(x,y) = (z,w) \quad \Rightarrow \quad T_{ab}^{(n)}(w,z) = (y,x). \tag{3.11}$$

Now suppose the first relation (3.5) holds. This yields

$$T_{ab}^{(n)}(-1,0) = T_{ab}^{(n+1)}(0,1) = T_{ab}(0,\lambda) = (-\lambda,0).$$

Using (3.11) applied to the first relation and to this relation yields

$$\begin{aligned} T_{ab}^{(n)}(\lambda,0) &= (1,0), & (3.12)\\ T_{ab}^{(n)}(0,-\lambda) &= (0,-1). & (3.13) \end{aligned}$$

If $\lambda > 0$ then rescaling these by a factor λ^{-1} gives (i), e.g.

$$T_{ab}^{(n)}(0,-1) = \lambda^{-1} T_{ab}^{(n)}(0,-\lambda) = (0,-\lambda^{-1}).$$

If $\lambda < 0$, then we obtain:

$$(-\lambda,0) = T_{ab}^{(n)}(-1,0) = |\lambda|^{-1} T_{ab}^{(n)}(\lambda,0) = (-\lambda^{-1},0)$$

and thus $\lambda = \lambda^{-1} = -1$. By hypothesis $T_{ab}^{(n)}(0,1) = (0,-1)$ and applying T_{ab} to both sides gives the other relation.

Now suppose the second relation (3.6) holds. This yields

$$T_{ab}^{(n)}(1,0) = T_{ab}^{(n+1)}(0,-1) = T_{ab}(0,-\lambda^{-1}) = (\lambda^{-1},0).$$

Using (3.11) applied to these relations yields

$$\begin{aligned} T_{ab}^{(n)}(-\lambda^{-1},0) &= (0,-1) & (3.14)\\ T_{ab}^{(n)}(0,\lambda^{-1}) &= (0,1). & (3.15) \end{aligned}$$

If $\lambda > 0$ rescaling these equalities by λ gives (i). The case $\lambda < 0$ giving the second case in (ii). is done similarly to the first case.

If $\lambda > 0$ in the first or second relation then $S_{ab}^{(n)}(0,1) = (0,1)$ (resp. $S_{ab}^{(n)}(0,-1) = (0,-1)$) so the rotation number $r(S_{ab})$ is rational.

If $\lambda < 0$ in these relations there is no constraint on the rationality of the rotation number $r(S_{ab})$. Example 5.1 below gives a case (ii) with irrational rotation number, and a case (ii) example with rational rotation number is given in remark (2) after Theorem 5.6. ∎

There are several special cases in which we can give explicit formulae for $r(S_{ab})$. These are of interest in part because we have no formulae for generic a and b; they also provide several examples of parameter values for which T_{ab} is periodic.

Example 3.1 *If $a = b = 2\cos(\theta)$, with $0 < \theta < \pi$, then $r(S_{ab}) = \frac{\theta}{2\pi}$.*

Proof. Note that T_{ab} is a linear map, since $a = b$. If we conjugate T_{ab} by the matrix $M = \begin{bmatrix} 1 & 0 \\ \cos(\theta) & \sin(\theta) \end{bmatrix}$, then we get the matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, which is rotation by θ. Since M is orientation-preserving for $0 < \theta < \pi$, conjugation by M preserves the rotation number of S_{ab}; rotation by θ has rotation number $\frac{\theta}{2\pi}$, thus the formula given. Note that when $\frac{\theta}{2\pi}$ is rational, $T_{0\nu}$ is periodic, and the period is computed from the rotation number. ∎

Example 3.2 *If $a = 2\cos(\frac{\pi}{n})$, for $n \geq 2$, and $b = 2\cos(\theta)$ with $0 < \theta < \pi$, then $r(S_{ab}) = \frac{\theta}{\pi + n\theta}$. When $\frac{\theta}{2\pi} = \frac{p}{q}$ is rational, then T_{ab} is periodic with period $2np + q$, for q odd, and half this if q is even.*

Proof. Here, we have a slightly more complicated situation, in that our map is no longer linear. Again, we conjugate by $M = \begin{bmatrix} 1 & 0 \\ \cos(\theta) & \sin(\theta) \end{bmatrix}$, to get

$$T'(\mathbf{v}) = \begin{cases} \mathbf{v}\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, & \text{if } (\cos(\theta), \sin(\theta)) \cdot \mathbf{v} \leq 0, \\ \mathbf{v}M\begin{bmatrix} 2\cos(\frac{\pi}{n}) & -1 \\ 1 & 0 \end{bmatrix}M^{-1}, & \text{if } (\cos(\theta), \sin(\theta)) \cdot \mathbf{v} \leq 0. \end{cases}$$

The important thing to note is that the second case will always occur exactly n times in a row, and that the nth power of that matrix is -1. If, in formula (3.1), we consider only the subsequence of iterates not in the middle of such a block, we get the following:

$$r(S_{ab}) = \lim_{i \to \infty} \frac{i\frac{\theta}{2\pi} + \frac{1}{2}\lfloor i\frac{\theta}{\pi} \rfloor}{i + \lfloor i\frac{\theta}{\pi} \rfloor n}.$$

In the limit, we can ignore the floors, to get $\frac{\theta}{\pi + n\theta}$, as claimed. Here, again, when $\frac{\theta}{2\pi}$ is rational, T_{ab} is periodic. ∎

As a special case of Example 3.2, in (μ, ν) parameters, taking $\nu = 0$, $\mu = 2\cos\frac{\pi}{n}$, for $n \geq 3$, yields a $T_{\mu\nu}$ which is periodic with period n^2. This corresponds to the parameters $a = 2\cos\frac{\pi}{n}$, $b = 2\cos\frac{(n-1)\pi}{n}$ in Example 3.2.

Example 3.3 *If both $a, b \leq 0$, and $ab = 4\cos^2(\frac{\pi}{2n})$, for $n \geq 2$, then $r(S_{ab}) = \frac{2n-1}{4n}$, and T_{ab} is periodic.*

Proof. For this case, the simplest approach is to explicitly calculate $T_{ab}^{(m)}(\mathbf{v})$, for $1 \leq m \leq 4n$, and plug that information into equation (3.1). It is most convenient to take $\mathbf{v}_x, \mathbf{v}_y < 0$. We can now calculate:

$$\begin{aligned} x_{2i} &= \frac{\sin(\frac{(i+1)\pi}{n}) + \sin(\frac{i\pi}{n})}{\sin(\frac{\pi}{n})}\mathbf{v}_x - \frac{\sin(\frac{i\pi}{n})}{\sin(\frac{\pi}{n})}a\mathbf{v}_y \leq 0, \\ x_{2i+1} &= \frac{\sin(\frac{(i+2)\pi}{n}) + 2\sin(\frac{(i+1)\pi}{n}) + \sin(\frac{i\pi}{n})}{\sin(\frac{\pi}{n})}\frac{\mathbf{v}_x}{a} - \frac{\sin(\frac{(i+1)\pi}{n}) + \sin(\frac{i\pi}{n})}{\sin(\frac{\pi}{n})}\mathbf{v}_y \geq 0, \end{aligned}$$

for $0 \leq i \leq (n-1)$. One can now calculate that $T_{ab}^{(2n)}(\mathbf{v}) = -\mathbf{v}$; since, by symmetry, we could as easily have started with $\mathbf{v}_x, \mathbf{v}_y > 0$, we can conclude that $T_{ab}^{(4n)} = 1$. By inspecting the sequence of quadrants we pass through in a period, we conclude that in $4n$ steps, we wrap around the origin $2n-1$ times; $r(S_{ab})$ is thus $\frac{2n-1}{4n}$. ∎

Example 3.3 gives an uncountable number of parameter values for which T_{ab} is periodic; the question of whether other any other continuous families of periodic parameter values exist is still open. However these examples by no means exhaust the supply of periodic parameter values; other examples of isolated periodic parameter values appear in §5.

4. Rational Rotation Number

We first characterize when T_{ab} is periodic.

Proof of Theorem 2.3. If T_{ab} is periodic, then $(0,1)$ is a periodic point.

Suppose, conversely, that $(0,1)$ is a periodic point of T_{ab} of minimal period p. Then so is $(1,0)$, with the same period p, because T_{ab}^{-1} is obtained from T_{ab} by interchanging x and y coordinates, cf. Theorem 3.4. Next $T_{ab}((0,-1)) = (1,0)$ hence $(0,-1)$ is a periodic point. We use the piecewise linear nature of T_{ab} to prove that T_{ab} is periodic. An iterate of $T_{ab}^{(n)}$ is linear on a sector Σ as long as the x-axis never intersects the interior of $T_{ab}^{(j)}(\Sigma)$ for $0 \le j < n$. Because $(0,1)$ and $(0,-1)$ are both periodic, the image of the x-axis under all iterates $\{T^{(n)} : n > 0\}$ splits the plane into a finite number of sectors, and the image of each of these sectors under T is another sector. It follows that $T_{ab}^{(p)}$ permutes these sectors and is linear on each one of them. But each sector contains two linearly independent fixed points of $T_{ab}^{(p)}$, from which it follows that $T_{ab}^{(p)}$ is the identity map on each sector. Since these sectors partition the entire plane, T_{ab} is periodic with period p. ∎

Remark. The parameter values for periodic points can be subdivided into two types.

Type (i). $(0,1)$ and $(0,-1)$ are in the same periodic orbit of T_{ab}, necessarily of even period.

Type (ii). $(0,1)$ and $(0,-1)$ are in separate periodic orbits of T_{ab}, having the same (even or odd) period.

Both types of periodic orbit occur.

To prove Theorem 2.4 we use the (μ,ν)-parameter space. The following auxiliary lemma is a main tool in proving Theorem 2.4.

Lemma 4.1 *For every rational $r = \frac{p}{q}$ with $0 < r < \frac{1}{2}$ and each fixed $\mu \in \mathbb{R}$, the following holds. For each $\mathbf{v} \in e^{i\theta} \in S^1$ there is a unique parameter value $\nu \in \mathbb{R}$ such that $\mathbf{v}$ is a periodic point of $S_{\mu\nu}$ of rotation number r.*

Proof. Let μ be fixed. We first show that for fixed θ, the second iterate $S_{\mu\nu}^{(2)}(\theta)$ moves strictly clockwise as ν increases. The proof of Theorem 3.2(ii) shows that $\frac{d}{d\nu}S_{\mu\nu}^{(2)}(\theta) \le 0$. If $x_0 = \mathbf{v}_x$ and $x_1 = (S_{\mu\nu}(\mathbf{v}))_x$ then for $\frac{d}{d\nu}S_{\mu\nu}^{(2)}(\mathbf{v}) = 0$ it is necessary that $x_0 = x_1 = 0$. But if $x_0 = 0$, then $x_1 = -\mathbf{v}_y$ which is impossible for $\mathbf{v} \in S^1$. By the same argument, for each $n \ge 2$, and fixed θ, $S_{\mu\nu}^{(n)}(\theta)$ moves strictly clockwise as ν increases.

The families of Example 3.3 give us some ν_0 such that $\mathbf{v}$ is a periodic point of $S_{\mu\nu_0}$ of rotation number $\frac{2q-1}{4q}$; consider, now, the behavior of $S_{\mu\nu}^{(4q)}(\mathbf{v})$ as ν changes from ν_0. If ν decreases from

ν_0, $r(S_{\mu\nu}) > \frac{2q-1}{4q} > r$, so we need consider only $\nu > \nu_0$. As we increase ν, $\mathbf{w}_\nu := S^{(4q)}_{\mu\nu}(\mathbf{v})$ moves smoothly (and strictly) clockwise. If we view this as a function of ν to the point θ_ν with $\mathbf{w}_\nu = e^{2\pi i\theta_\nu}$, we can take the lift $\tilde{S}(\nu) = \theta_\nu$ of this function to $\mathbf{R}$, with the initial condition $\tilde{S}(\nu_0) = 2q-1$; this is continuous, strictly decreasing, and for large enough ν attains a value below 4. ($\tilde{S}(\nu) \geq 4$ implies $r(S_{\mu\nu}) \geq \frac{1}{q}$.) Since $2q-1 > 4p \geq 4$, there is a unique ν such that $\tilde{S}(\nu) = 4p$. It is then easy to see that $\mathbf{v}$ is a periodic point of $S_{\mu\nu}$ of rotation number $r = \frac{p}{q}$. ∎

Proof of Theorem 2.4. We use the (μ,ν)-parameter space. We fix μ, and allow ν to vary; we can use the monotonicity in ν to deduce the behavior for every ν (for that rotation number) from the behavior from one special ν. In particular, let ν_0 be the unique ν such that $(0,1)$ is a periodic point of $S_{\mu\nu_0}$ of rotation number $r(S_{\mu\nu}) = \frac{p}{q}$. Now, if $(0,1)$ is a periodic point of $T_{\mu\nu_0}$, then $T_{\mu\nu_0}$ is periodic, by Theorem 2.3; since rational rotation number implies existence of a periodic point, the uniqueness part of Lemma 4.1 then tells us that $\nu = \nu_0$ is the only value of ν for which $r(S_{\mu\nu}) = \frac{p}{q}$.

Thus, assume $(0,1)$ is not a periodic point of $T_{\mu\nu_0}$. Then there must be some positive real $\lambda \neq 1$ such that $T^{(q)}_{\mu\nu_0}(0,1) = (0,\lambda)$. We can, without loss of generality, assume that $\lambda > 1$; the proof for $\lambda < 1$ is analogous. For the orbit of $(0,-1)$, we also have by Theorem 3.4(i) that $T^{(q)}_{\mu\nu_0}(0,-1) = (0,-\lambda^{-1})$.

Note first that if we plot the orbits of $(0,1)$ and $(0,-1)$ under $S_{\mu\nu_0}$, then the two orbits are alternating as we move around the circle. Also, the sector bounded by a point of one orbit and a neighboring point from the other orbit is linear under the action of $T^{(q)}_{\mu\nu_0}$. Now $T^{(q)}_{\mu\nu_0}$ can have no other eigenvector in the sector, because it already has two. Thus, for $\nu = \nu_0$, there are no other periodic orbits.

Now, for any $0 \leq n < q$, let $\mathbf{v}_0(\nu) \in S^1$ be $S^{(-n)}_{\mu\nu}(0,1)$, and let $\mathbf{v}_1(\nu)$ and $\mathbf{v}_{-1}(\nu)$ be the points of the form $S^{(-i)}_{\mu\nu}(0,-1)$, $0 \leq i < q$ immediately counterclockwise and clockwise of $\mathbf{v}_0(\nu)$, respectively. For every point in the interval from $\mathbf{v}_{-1}(\nu)$ to $\mathbf{v}_1(\nu)$, label the point as clockwise if $S^{(q)}_{\mu\nu}$ moves the point clockwise, and similarly label it as counterclockwise if $S^{(q)}_{\mu\nu}$ moves the point counterclockwise (fixed points will receive no label). Now, at $\nu = \nu_0$, the points between $\mathbf{v}_{-1}$ and $\mathbf{v}_0$ will be labelled clockwise, and the points between $\mathbf{v}_0$ and $\mathbf{v}_1$ will be labelled counterclockwise, since $\mathbf{v}_0$ is an attracting fixed point of $S^{(q)}_{\mu\nu}$, and $\mathbf{v}_{\pm 1}$ are repulsing fixed points of $S^{(q)}_{\mu\nu}$. As ν decreases, every point labelled counterclockwise will remain labelled counterclockwise (by the proof of Theorem 2.2); further, $\mathbf{v}_0$ and $\mathbf{v}_{\pm 1}$ will be labelled counterclockwise. Now, $T^{(q)}_{\mu\nu}$ is linear on the interval from $\mathbf{v}_0$ to $\mathbf{v}_1$, so $S^{(q)}_{\mu\nu}$ can have at most two fixed points in the interval. As long as there are points labelled clockwise in the interval, $S^{(q)}_{\mu\nu}$ must have exactly two fixed points; these must move continuously toward each other; when the two fixed points coincide, we are left with one fixed point; after that point, there can be no fixed points in the interval. Thus, we can conclude that the interval from $\mathbf{v}_{-1}$ to $\mathbf{v}_1$ always has either two, one, or zero fixed points of $S^{(q)}_{\mu\nu}$, in the range of $\nu \leq \nu_0$ such that $r(S_{\mu\nu}) = \frac{p}{q}$; the proof for $\nu \geq \nu_0$ is analogous. Now, the proof applies for any $\mathbf{v}_0$; there are exactly q possible choices of $\mathbf{v}_0$, and the intervals from $\mathbf{v}_{-1}$ to $\mathbf{v}_1$ around these choices of $\mathbf{v}_0$ exhaust S^1; therefore, we can conclude that the number of distinct periodic orbits of $S_{\mu\nu}$ must equal the number of distinct fixed points of $S^{(q)}_{\mu\nu}$ between $\mathbf{v}_{-1}$ and $\mathbf{v}_1$, which is at most two. There are then three cases:

Case 1: $S_{\mu\nu}$ *has no periodic orbit.*

Then $r(S_{\mu\nu})$ is irrational (a contradiction); this case cannot occur.

Case 2: $S_{\mu\nu}$ *has exactly two periodic orbits.*

Let $\mathbf{w}$ be a fixed point corresponding to one of the orbits, and $\mathbf{w}'$ a fixed point corresponding to the other orbit. Then there must be some λ such that $T^{(q)}_{\mu\nu}(\mathbf{w}) = \lambda\mathbf{w}$, and $T^{(q)}_{\mu\nu}(\mathbf{w}') = \lambda^{-1}\mathbf{w}'$. For every point $\mathbf{v}$, $S^{(i)}_{\mu\nu}(\mathbf{v})$ comes arbitrarily close to $\mathbf{w}$ (for $i > 0$); there is an interval around $\mathbf{w}$ which behaves linearly under $T^{(q)}_{\mu\nu}$, so every point eventually is transformed by the same map. $T^{(q)}_{\mu\nu}$ has an eigenvector of eigenvalue > 1; it is easy to see that every point must diverge exponentially except for the eigenvector of eigenvalue < 1, which converges exponentially to 0. Similarly, in the other direction, every point diverges exponentially except for the eigenvector of eigenvalue > 1, which converges exponentially to 0.

Case 3: $S_{\mu\nu}$ *has exactly one periodic orbit.*

If we reflect a periodic orbit of $S_{\mu\nu}$ around the line $x = y$, we must get a periodic orbit; thus, in this case, the periodic orbit must be symmetrical around $x = y$. It follows, then, that the orbit lifts to a periodic orbit of $T_{\mu\nu}$ ($T^{(q)}_{\mu\nu}$ and $T^{(-q)}_{\mu\nu}$ must multiply the vectors in the orbit by the same amount). In a sufficiently small neighborhood of a point of the orbit, $T^{(q)}_{\mu\nu}$ is linear; it has exactly one eigenvector of eigenvalue one, but has eigenvalue one with multiplicity two; it follows that every point in that neighborhood diverges linearly.

Every point on the circle has a value of ν associated to it by the above reasoning such that it is a periodic point of $S_{\mu\nu}$. By Lemma 4.1, there can be no other ν such that $S_{\mu\nu}$ has a periodic point of rotation number $\frac{p}{q}$. The result then follows. ∎

5. Irrational Rotation Number

If the rotation number is irrational, then the situation becomes significantly more complicated. The existence of smoothness conditions on S_{ab} help somewhat.

Theorem 5.1 *If the rotation number* $r(S_{ab}) = r$ *is irrational, then there exists a homeomorphism* $h : S^1 \to S^1$ *such that* $h \circ S_{\mu\nu} \circ h^{-1} = \Theta_r$*, where* $\Theta_r : S^1 \to S^1$ *is rotation by* r *(*$\theta \mapsto \theta + 2\pi r$*).*

Proof. By Theorem 3.1 $\frac{d}{d\theta}S_{ab}(\theta)$ is continuous and of bounded variation. The conjugacy result then follows from Denjoy's Theorem, see Nitecki [20, p. 41 and Theorem, p. 45]. ∎

This effectively characterizes the possible behavior of S_{ab}, but it still leaves several possibilities open for T_{ab}. In particular, it does not eliminate the possibility of a localized orbit (one that converges to 0) or a divergent orbit. One major question is that of when T_{ab} has an invariant circle not containing 0 which, by scale invariance, gives us an invariant circle containing every point.

Theorem 5.2 *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be any homeomorphism which is scale-invariant; that is* $T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$ *for all* $\lambda \geq 0$*, all* $\mathbf{v} \in \mathbb{R}^2$*. Let* $S(\mathbf{v}) = \frac{T(\mathbf{v})}{\|T(\mathbf{v})\|}$ *be its associated circle map, and suppose that* S *is topologically conjugate to an irrational rotation. If* T *has a forward orbit*

$\mathcal{O}^+(\mathbf{v}_0) = \{\mathbf{v}_n : n \geq 0\}$ *such that*

$$\frac{1}{C} \leq \|\mathbf{v}_n\| \leq C, \qquad all \quad n \geq 0 \ ,$$

for some positive constant $C > 1$, then T has an invariant circle, and all orbits are bounded.

Proof. For each $n \geq 0$, let L_n be the line segment from 0 to $T^{(n)}(\mathbf{v})$, and let

$$I_m = \bigcup_{n \geq m} L_n, \quad \text{and} \quad I' = \bigcap_{m \geq 0} \overline{I_m}.$$

Now, $T(I_m) = I_{m+1}$, so $T(\overline{I_m}) = \overline{I_{m+1}}$, and $\overline{I_m}$ is a decreasing sequence of sets. Therefore, we can conclude that I' is an invariant set of T. Furthermore, $\overline{I_m}$ contains a ball of radius $1/C$ around 0 for each m, since $S^{(n)}(\mathbf{v})$ is dense in S^1, so I' contains a ball of radius $1/C$.

Now, consider the set $B := \partial I'$, the boundary of I'. B is also an invariant set of T which does not contain 0, and it contains at least one point in every direction from 0. If we can show that B is homeomorphic to S^1, we will be done. It suffices to show that B is connected, and contains at most one point in every direction from 0; projection onto the unit circle then provides a homeomorphism to S^1. Connectedness of B follows from the fact that I' is closed, bounded, and star-convex, so we have left only to show that every ray from the origin hits B in exactly one point.

Suppose that $\mathbf{v}_1, \mathbf{v}_2 \in B$, with $\mathbf{v}_2 = k\mathbf{v}_1$, $k > 1$. Then, for all $n \geq 0$,

$$\frac{k}{C} \leq |T^{(n)}(\mathbf{v}_2)| \leq C \,,$$

and $|T^{(n)}(\mathbf{v}_2)| = k|T^{(n)}(\mathbf{v}_1)|$. Now, B contains the line segment from $\mathbf{v}_1$ to $\mathbf{v}_2$, and thus contains the line segment from $T^{(n)}(\mathbf{v}_1)$ to $T^{(n)}(\mathbf{v}_2)$ for all $n \geq 0$. These line segments have length bounded from below, all lie within the closed ball of radius C around 0, and lie on a dense set of rays. It thus follows that $\bar{B}$ has non-empty interior. But, B is a boundary, so this is impossible. Therefore, every ray must hit B in at most one point, so B is an invariant circle. ∎

Theorem 5.2 applies to T_{ab}, and characterizes when it has invariant circles. We next establish a symmetry of invariant circles (when they exist); this symmetry is manifest in the plots of invariant circles given later on.

Theorem 5.3 *If the map T_{ab} has associated irrational rotation number $r(S_{ab})$, they any invariant circle of T_{ab} is symmetric under the involution $R(x, y) = (y, x)$.*

Proof. Let $\mathcal{C}$ denote an invariant circle. The conjugacy (2.5) by R shows that the set $R(\mathcal{C}) = R^{-1}(\mathcal{C})$ is invariant under $T_{ab}^{-1} = R^{-1} \circ T_{ab} \circ R$ and hence under T_{ab}, so that it too is an invariant circle of T_{ab}. The invariant circle $\mathcal{C}$ intersects the line $x = y$, and points on this line are left fixed by R, so that $\mathcal{C}$ and $R(\mathcal{C})$ have at least one common point. For irrational rotation number, any two invariant circles having a common point must be identical, because they have a dense orbit in common; thus $\mathcal{C} = R(\mathcal{C})$. ∎

We now study the irrational rotation case in more detail by considering first-return maps to suitable sectors of the plane.

Theorem 5.4 *Suppose that the rotation number $r(S_{ab})$ is irrational.*

(1) For any half-open sector $J = \mathbb{R}^+[\mathbf{v}, \mathbf{v}')$ the first return map $T_J^{(1)} : J \to J$ of T_{ab} to J is piecewise linear with at most five pieces.

(2) Let m_+, m_-, n_+, n_- be nonnegative integers, and set $\mathcal{O} := \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$, where:

$$\begin{aligned}
\mathcal{O}_1 &= \{T_{ab}^{(1+i)}(0,1) : 0 \le i < m_+\} \\
\mathcal{O}_2 &= \{T_{ab}^{(-i)}(0,1) : 0 \le i < m_-\} \\
\mathcal{O}_3 &= \{T_{ab}^{(1+i)}(0,-1) : 0 \le i < n_+\} \\
\mathcal{O}_4 &= \{T_{ab}^{(-i)}(0,-1) : 0 \le i < n_-\}\,.
\end{aligned}$$

Let $J = \mathbb{R}^+[\mathbf{v}, \mathbf{v}')$ be any half-open sector determined by two elements $\mathbf{v}, \mathbf{v}' \in \mathcal{O}$ such that J contains no points of $\mathcal{O}$ in its interior. Then the first return map $T_J : J \to J$ is piecewise linear with at most three pieces.

Proof. (1) Note first that for any half-open sector J, the first return map for T_{ab} is well-defined, since the assumption that S_{ab} has irrational rotation number implies that the orbit of any point under S_{ab} is dense.

Let $\mathbf{v}_1$ be the first preimage $T_{ab}^{(-i)}(0,1)$, $i \ge 0$, of $(0,1)$ that falls in J; let $\mathbf{v}_2$ be the first preimage of $(0,-1)$ with the same property. Let $\mathbf{v}_3$ be the first point of the form $T_{ab}^{(-i)}(\mathbf{v})$, $i > 0$ that falls in J, and let $\mathbf{v}_4$ be the first point of the form $T_{ab}^{(-i)}(\mathbf{v}')$, $i \ge 0$ that falls in J. The $\mathbf{v}_i$ split J into at most five intervals. We claim that the first return map for T_{ab} on the corresponding sector is linear on each corresponding subsector.

Consider first the (at most) three half-open subsectors into which J is subdivided by $\mathbf{v}_3$ and $\mathbf{v}_4$. The number of steps until the first return will be constant on each subsector, since between any two points that take a different number of steps to return, there must be a point that hits the boundary of J at or before the first return. If $\mathbf{v}_5$ were such a point other than $\mathbf{v}_3$ or $\mathbf{v}_4$, then the iterates of $\mathbf{v}_5$ would hit the ray determined by $\mathbf{v}_3$ or $\mathbf{v}_4$ before hitting the ray determined by $\mathbf{v}$ or $\mathbf{v}'$, which gives a contradiction, since hitting $\mathbf{v}_3$ or $\mathbf{v}_4$ constitutes a return to J.

Now, we restrict our attention to one of these three subsectors, on which the number of iterations until return to J is constant. For any two points in the subsector whose iterates until first return to J are always on the same side of the y-axis, the first return map on J is linear on the sector they induce. Thus the only points where linearity can fail are points which hit the y-axis on iteration before their first return, namely $\mathbf{v}_1$ and $\mathbf{v}_2$. (Any other such point on J would hit the ray determined by $\mathbf{v}_1$ or $\mathbf{v}_2$ before it hit the y-axis, a contradiction.) Thus, the subsector between each two neighboring such $\mathbf{v}_i$ is a subsector on which the first return map on J is linear, and (1) follows.

(2) First note that $T_{ab}(0,1) = (-1,0)$; similarly $T_{ab}(0,-1) = (1,0)$. Suppose now that $\mathbf{v} \in \mathcal{O}_1$. by definition $\mathbf{v}_3$ is the first strict preimage of $\mathbf{v}$ that falls in J. This preimage cannot belong to $\mathcal{O}_1$ because by hypothesis the only point of $\mathcal{O}$ in the half-open interval J is the endpoint $\mathbf{v}$. Thus this preimage is a strict preimage of $(-1,0)$, hence is $(0,1)$ or a preimage, so we conclude that $\mathbf{v}_3 = \mathbf{v}_1$.

We deduce the following facts by similar reasoning: (1) If $\mathbf{v} \in \mathcal{O}_2$ then $\mathbf{v}_1 = \mathbf{v}$; (2) If $\mathbf{v} \in \mathcal{O}_3$ then $\mathbf{v}_3 = \mathbf{v}_2$; (3) If $\mathbf{v} \in \mathcal{O}_4$ then $\mathbf{v}_2 = \mathbf{v}$.

The preimage $\mathbf{v}_4$ of $\mathbf{v}'$ must also be equal to one of these other values. We distinguish three cases: (a) $\mathbf{v}$ and $\mathbf{v}'$ are in the same orbit, with $\mathbf{v}$ coming first, (b) $\mathbf{v}$ and $\mathbf{v}'$ are in the same orbit, with $\mathbf{v}'$ coming first, (c) $\mathbf{v}$ and $\mathbf{v}'$ are in different orbits. In case (a), $\mathbf{v}_4 = \mathbf{v}$, while in case (b), $\mathbf{v}_4 = \mathbf{v}_3$. In case (c) one deduces that if $\mathbf{v}' \in \mathcal{O}_1 \cup \mathcal{O}_2$ then $\mathbf{v}_4 = \mathbf{v}_1$, while if $\mathbf{v}' \in \mathcal{O}_3 \cup \mathcal{O}_4$ then $\mathbf{v}_4 = \mathbf{v}_2$. In all cases the five points $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ take at most three different values, and therefore T_J has at most three linear pieces on $\mathbb{R}^+[J]$. ∎

If it so happens that the T_{ab}-orbit of $(0,1)$ contains $(0,-1)$, then one can obtain sectors on which the first return map is piecewise linear with at most two pieces, and a surprising thing happens:

Theorem 5.5 *Suppose that S_{ab} has irrational rotation number, and that $(0,1)$ and $(0,-1)$ are in the same T_{ab}-orbit. Let $\mathcal{O} = \{T_{ab}^{(j)}(\mathbf{v}_0) : 1 \le j \le m\}$ be any finite segment of that orbit which includes all of the points strictly between $(0,1)$ and $(0,-1)$, as well as whichever of $(0,1)$ and $(0,-1)$ comes last in the orbit. If $J = \mathbb{R}^+[\mathbf{v}, \mathbf{v}')$ is a half-open sector determined by two members of $\mathcal{O}$ containing no elements of $\mathcal{O}$ in its interior, then the first return map T_J on J is piecewise linear with exactly two pieces, and the linear transformations corresponding to the two pieces commute.*

Proof. Let $J = \mathbb{R}^+[\mathbf{v}, \mathbf{v}')$ be a sector determined by two members of $\mathcal{O}$, such that no element of $\mathcal{O}$ lies in its interior. We show that the first return map T_J of T_{ab} on the sector J is piecewise linear with exactly two pieces. Let $T_{ab}^{(n)}(0,1) = (0,-1)$. We first treat the case $n > 0$, when $(0,1)$ comes earlier in the orbit than $(0,-1)$. In the notation of Theorem 5.4, we take $\mathcal{O}_i$ as large as possible inside $\mathcal{O}$, and obtain the following relations: $\mathcal{O}_2 \subset \mathcal{O}_4$, $\mathcal{O}_3 \subset \mathcal{O}_1$, and $\mathcal{O}_1 \cup \mathcal{O}_2 = \mathcal{O}_3 \cup \mathcal{O}_4$. Theorem 5.4(ii) states that the first return map T_J is piecewise linear with at most three pieces. We show that the relations between the sets $\mathcal{O}_i$ cause two of the pieces to become identified. Indeed, from the proof of Theorem 5.4(ii), we have the following three possibilities:

(a) $\mathbf{v} \in \mathcal{O}_1 \cap \mathcal{O}_4$: $\mathbf{v}_3 = \mathbf{v}_1$, $\mathbf{v}_2 = \mathbf{v}$.
(b) $\mathbf{v} \in \mathcal{O}_2 \cap \mathcal{O}_4$: $\mathbf{v}_2 = \mathbf{v}_1 = \mathbf{v}$.
(c) $\mathbf{v} \in \mathcal{O}_3 \cap \mathcal{O}_1$: $\mathbf{v}_3 = \mathbf{v}_2 = \mathbf{v}_1$.

In case (a), $\mathbf{v}_4 = \mathbf{v}$, while in case (b), $\mathbf{v}_4 = \mathbf{v}_3$; case (c) is excluded, since $\mathbf{v}$ and $\mathbf{v}'$ are in the same orbit. Thus the five points $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ take on only two values ($\mathbf{v}$ and $\mathbf{v}_3$), which divides the sector J into two pieces. We cannot have a further collapse to one piece because the equality $\mathbf{v} = \mathbf{v}_3$ would make $\mathbf{v}$ a periodic point of $S_{\mu\nu}$, whence $S_{\mu\nu}$ has a rational rotation number, contradicting the irrational rotation hypothesis. We conclude that T_J is piecewise linear on the sector J with exactly two pieces.

In the other case $n < 0$ we have $\mathcal{O}_4 \subset \mathcal{O}_2$, $\mathcal{O}_1 \subset \mathcal{O}_3$, and $\mathcal{O}_1 \cup \mathcal{O}_2 = \mathcal{O}_3 \cup \mathcal{O}_4$. This yields the three possibilities:

(a) $\mathbf{v} \in \mathcal{O}_2 \cap \mathcal{O}_3$: $\mathbf{v}_3 = \mathbf{v}_2$, $\mathbf{v}_1 = \mathbf{v}$.
(b) $\mathbf{v} \in \mathcal{O}_2 \cap \mathcal{O}_4$: $\mathbf{v}_2 = \mathbf{v}_1 = \mathbf{v}$.
(c) $\mathbf{v} \in \mathcal{O}_3 \cap \mathcal{O}_1$: $\mathbf{v}_3 = \mathbf{v}_2 = \mathbf{v}_1$.

By similar arguments to the case $n > 0$, we conclude that T_J is piecewise linear with exactly two pieces.

Now let $J = \mathbb{R}^+[\mathbf{v}, \mathbf{v}')$ be a sector of the type above. The first return map T_J is piecewise linear on the sector with exactly two pieces, and we let $\mathbf{v}^*$ denote the breakpoint in the interior of J (so $\mathbf{v}^* = \mathbf{v}_3$.) Let M_1 be the linear transformation corresponding to $J_1 = \mathbb{R}^+[\mathbf{v}, \mathbf{v}^*)$, and let M_2 be the linear transformation corresponding to $J_2 = \mathbb{R}^+[\mathbf{v}^*, \mathbf{v}')$, so that

$$T_J(\mathbf{w}) = \begin{cases} M_1\mathbf{w} & \text{if } \mathbf{w} \in \mathbb{R}^+[\mathbf{v}, \mathbf{v}^*) \\ M_2\mathbf{w} & \text{if } \mathbf{w} \in \mathbb{R}^+[\mathbf{v}^*, \mathbf{v}') \end{cases}.$$

Note that $M_2\mathbf{v}^* = \mathbf{v}$, and, by continuity, $M_1\mathbf{v}^* = \mathbf{v}'$. Now, consider the action of $T_J^{(2)}$. In some neighborhood of $\mathbf{v}^*$, $T_J^{(2)}$ acts as M_2M_1 on points clockwise of $\mathbf{v}^*$ and as M_1M_2 on points counterclockwise of $\mathbf{v}^*$. If we can show that $T_J^{(2)}$ is linear on this neighborhood, then $M_1M_2 = M_2M_1$ and we are done.

To compute $T_J^{(2)}$, one multiplies a sequence of matrices of the form $\left[\begin{smallmatrix} c & -1 \\ 1 & 0 \end{smallmatrix}\right]$, with $c = a$ or $c = b$ at step i if the ith x-coordinate is positive or negative, respectively. Now, as we pass through $\mathbf{v}^*$, the sequence will change only at the two steps $j = j_1, j_2$ where $T_{ab}^{(j)}(\mathbf{v}^*) = (0, \pm 1)$. Furthermore, away from $\mathbf{v}^*$ in the neighborhood, if $c = a$ at one of those two steps, we must have $c = b$ at the other. Thus we find that (say)

$$M_1M_2 = M_A \begin{bmatrix} b & -1 \\ 1 & 0 \end{bmatrix} M \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix} M_B$$

and

$$M_2M_1 = M_A \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix} M \begin{bmatrix} b & -1 \\ 1 & 0 \end{bmatrix} M_B,$$

for certain matrices M_A, M_B and M, in which $\tilde{M} := M \left[\begin{smallmatrix} * & -1 \\ 1 & 0 \end{smallmatrix}\right]$ is a product of matrices corresponding to the sequence of iterations between the two places where $T_{ab}^{(j)}(\mathbf{v}^*) = (0, \pm 1)$. Note that the value of $*$ (a or b) has no effect on the first step of the iteration of T_{ab}. The matrix equation $M_1M_2 = M_2M_1$ boils down to showing that

$$\begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix} M \begin{bmatrix} b & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & -1 \\ 1 & 0 \end{bmatrix} M \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix}.$$

We now recall from Theorem 3.4(ii), the key fact that if $S_{ab}^{(n)}(0,1) = (0,-1)$ for some (positive or negative) integer n, then

$$T_{ab}^{(n)}(0,1) = (0,-1).$$

If $n > 0$, this gives $\tilde{M}(0,1) = M(-1,0) = (0,-1)$; if $n < 0$ we obtain instead $\tilde{M}(0,-1) = M(1,0) = (0,1)$. In either case, we can conclude that

$$M = \begin{bmatrix} 0 & -1 \\ 1 & d \end{bmatrix},$$

for some value d, and it follows that

$$\begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix} M \begin{bmatrix} b & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -a-b-d & 1 \\ -1 & 0 \end{bmatrix},$$

This is symmetric in a and b, hence $M_1M_2 = M_2M_1$, and the theorem is proved. ∎

In preparation for the next result, we recall the following well-known fact, see Arnold and Avez [1, Appendix 27]. .

Lemma 5.1 *Let* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R})$*, and suppose that* $M \neq \pm I$*. Then the map* $(x,y) \to M(x,y)$ *leaves invariant the quadratic form*

$$Q(x,y) = cx^2 + (d-a)xy - by^2,$$

and the only quadratic forms it leaves invariant are scalar multiples of $Q(x,y)$*. The level sets* $cx^2 + (d-a)xy - by^2 = \lambda$ *for real* λ *are invariant sets, and fill the plane. Furthermore:*

(1) If $|Tr(M)| < 2$*, then the nonempty level sets are ellipses, or else a single point.*

(2) If $|Tr(M)| > 2$*, then the level sets are hyperbolas, except for* $\lambda = 0$*, where they are their asymptotes, consisting of two straight lines through the origin.*

(3) If $|Tr(M)| = 2$*, then the nonempty level sets are either two parallel lines whose vector sum is a fixed parallel line through the origin; or, for* $\lambda = 0$*, this fixed line.*

Any matrix M' *commuting with* M *also preserves the quadratic form* Q*.*

Proof. The condition for invariance of the quadratic form $Q(x,y) = Ax^2 + 2Bxy + Cy^2$ is that $M^T Q M = Q$, where $Q = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$; an equivalent matrix condition is that $M^T Q = Q M^{-1}$. Letting $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$, the latter condition yields the linear equations $(a-d)A = -2cB$, $cC = -bA$, and $(a-d)C = 2bB$. At least one of $b, c, a-d$ is nonzero, otherwise $M = \pm I$ which we have excluded. Then this linear system has rank at least two, so has at most a one-parameter family of solutions. For $\det(M) = 1$ (and $M \neq \pm I$) the system has rank two and has the one-parameter family of solutions $(A, 2B, C) = \alpha(c, d-a, -b)$ for $\alpha \in \mathbb{R}$. Choosing $\alpha = 1$, its discriminant is

$$\mathrm{Disc}(Q) := (d-a)^2 + 4bc = Tr(M)^2 - 4.$$

Cases (1)-(3) correspond to the discriminant being negative, positive or zero.

Diagonalizing M over the complexes gives either $\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$ or $\pm \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$; in either case, we find that any matrix M' commuting with M is of the form $M' = eM + fI$. Substituting this in, we find that M' also preserves Q. ∎

Theorem 5.6 *Suppose that* S_{ab} *has irrational rotation number and that the* S_{ab} *orbit of* $(0,1)$ *contains* $(0,-1)$*. Then the* T_{ab} *orbit of* $(0,1)$ *contains* $(0,-1)$*, and* T_{ab} *has a piecewise conic invariant circle. If* $S_{ab}^{(n)}(0,1) = (0,-1)$ *for* $n \in \mathbb{Z}$*, then the number of conic pieces of the invariant circle is at most* $|n|$*, and all conic pieces are of the same type, either arcs of ellipses, arcs of hyperbolas or line segments, respectively.*

Proof. The fact that the T_{ab} orbit of $(0,1)$ contains $(0,-1)$ follows from Theorem 3.4, since irrational rotation number can only occur in case (ii).

Let $\mathcal{C}$ be the closure of the orbit containing $\mathbf{v}_0 := (0,1)$; we prove below that $\mathcal{C}$ is an invariant circle with the required properties. If so, it follows that all T_{ab} orbits are scaled copies of this invariant circle.

By hypothesis $S_{ab}^{(n)}(0,1) = (0,-1)$ with $n \in \mathbb{Z}$; the value of n is unique since S_{ab} has irrational rotation number. Set $\mathcal{O} = \{S_{ab}^{(i)}(0,1) : 0 < i \leq n\}$ if $n > 0$ and set $\mathcal{O} = \{S_{ab}^{(i)}(0,1) : n < i \leq 0\}$ if $n < 0$; this is the minimal set $\mathcal{O}$ to which Theorem 5.5 applies. It follows that the rays determined by the points of $\mathcal{O}$ partition of the plane into $|n|$ sectors, and on each sector the first return map of $T_{\mu\nu}$ is piecewise linear, with two commuting pieces M_1, M_2.

We consider the sector $J = \mathbb{R}^+[\mathbf{v}, \mathbf{v}')$ in which $\mathbf{v}$ is whichever of $(0,1)$ or $(0,-1)$ occurs first in $\mathcal{O}$. We have each $M_j \neq \pm I$ because, if not, then the rotation map S_{ab} would have a periodic point, and hence a rational rotation number, a contradiction. Now Lemma 5.1 applies, and shows that M_1 and M_2 each leave invariant a one-parameter family of quadratic forms, multiples of Q_1 and Q_2, respectively; that is, $M_1^T Q_1 M_1 = Q_1$ and $M_2^T Q_2 M_2 = Q_2$. Moreover, since M_1 and M_2 commute, we can take $Q_1 = Q_2$.

The claim shows that all the return visits of $\mathbf{v}$ to J lie on a single level set of the quadratic form Q_1, with the level set parameter λ specified by requiring that $\mathbf{v}$ lie on this conic. Since S_{ab} has irrational rotation number the return visits are dense on that part of this conic section in the sector J, and the closure is the full arc of the conic section, call it $\mathcal{C}_1$, that intersects the sector. In the hyperbola case $|Tr(M_1)| > 2$, the sector J cannot include an eigenvector of M_1 (an asymptote), because if it did, then the map S_{ab} would have a periodic point, and hence a rational rotation number, a contradiction; thus the sector J contains an arc of one connected component of a hyperbola.

The arc $\mathcal{C}_1$ is either an arc of an ellipse or a hyperbola, or a line segment. By similar reasoning, on each other sector J_i, the visits of the orbit of $\mathbf{v}_0$ fill out the arc of a unique conic on that sector. The image of the arc $\mathcal{C}_1$, when iterated under T_{ab} will visit each sector; this visit is given by a linear map, so the conic $\mathcal{C}_i$ is the same type of conic (linear, ellipse, or hyperbola) as $\mathcal{C}_1$.

It remains to show that the union $\mathcal{C} = \cup_{j=1}^n \mathcal{C}_j$ of these pieces is a connected set. To see this, consider the preimage $\mathbf{v}_{-1} = T_{ab}^{-1}(\mathbf{v})$. This point is not in $\mathcal{O}$, so it necessarily lies in the interior of some sector J_i, hence is contained in the conic piece $\mathcal{C}_i$. Now, the forward iteration $T_{ab}(\mathcal{C}_i)$ will cover the point $\mathbf{v}_0$ in its interior. This image is a conncected set, so the conic pieces $\mathcal{C}$ and $\mathcal{C}_j$ in the sector J_j adjacent to J, moving clockwise, must touch at the point $\mathbf{v}_0$. Next, forward iteration of T_{ab} of a connected neighborhood of $\mathbf{v}$ over the full set $\mathcal{O}$ shows that all segments $\mathcal{C}_j$ glue together into a connected set. Thus $\mathcal{C}$ is an invariant circle. ∎

Remark. The conclusion of Theorem 5.6 need not hold without the requirement of irrational rotation number, see Example 5.3 below.

We now give three examples of one-parameter families where $(0,1)$ and $(0,-1)$ are in the same orbit, which illustrate these theorems; Example 3.2 above is another such example, with piecewise elliptic invariant circle when not periodic.

Example 5.1 *If $1 < a < \sqrt{2}$, and $b = 2\frac{a^2-1}{a(a^2-2)}$, then T_{ab} is either periodic or has a piecewise elliptical invariant circle. (These parameter values come from assuming that $T_{ab}^{(8)}(0,-1) = (0,1)$, with appropriate sign conditions.)*

A detailed analysis of this example is given in the appendix, where the following facts are proved. One chooses $\mathcal{O} = \{T_{ab}^{(j)} : 1 \leq j \leq 8\}$. When the rotation number is rational, $T_{\mu\nu}$ is periodic; when it is irrational Theorem 5.6 applies to show that it has a piecewise conic invariant circle with at most eight pieces. The conic sections are always ellipses.

There is an explicit formula for the rotation number. Set $a = 2\cos\theta$, with $\pi/4 < \theta < \pi/3$.

The rotation number $r(S_{ab})$ is explicitly given by

$$r(S_{ab}) = \frac{3\pi - 7\theta}{14\pi - 32\theta}.$$

It follows that $r(S_{ab})$ is rational if and only if θ is a rational multiple of π.

The special case $\nu = 0$, $\mu = \sqrt[4]{2}$ (corresponding to $a = -b = \sqrt[4]{2}$) has a rotation number which is provably irrational. Indeed, one checks that $z = e^{i\theta}$ is a root of $z^8 + 4z^6 + 4z^4 + 4z^2 + 1 = 0$; this equation has no roots of unity as roots, so θ is not a rational multiple of π. An invariant circle of $T_{\mu\nu}$ is pictured in Figure 5.1; it is a piecewise union of eight arcs of ellipses. (In this case several of the arcs are parts of a single ellipse.)

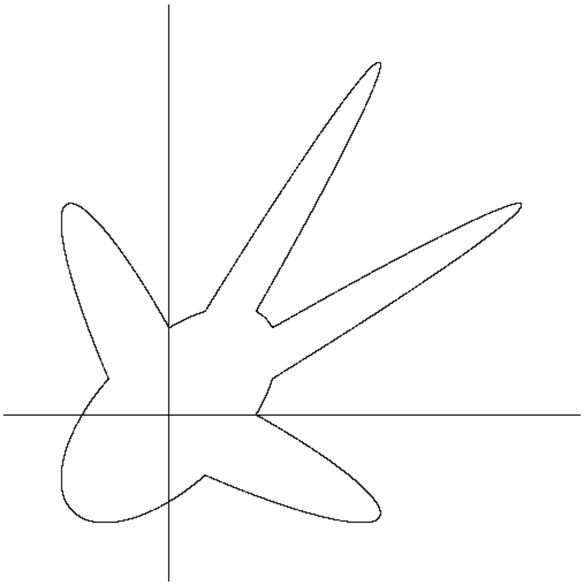

Figure 5.1: Invariant circle for $a = \sqrt[4]{2}$, $b = -\sqrt[4]{2}$ ("Bugs Bunny").

Example 5.2 *If* $0 < a < 1$, *and* $b = \frac{1}{2}\frac{3a^2-3+\sqrt{a^4-2a^2+9}}{a(a^2-2)}$, *then* T_{ab} *is either periodic or has a piecewise conic invariant circle. (These parameter values come from assuming that* $T_{ab}^{(10)}(0,1) = (0,-1)$, *with appropriate sign conditions.)*

A detailed analysis of this example is given in the appendix, where the following facts are proved. We take $\mathcal{O} = \{T_{ab}^{(j)}(0,1) : 1 \leq j \leq 10\}$. In the case of rational rotation number the map T_{ab} is periodic. For irrational rotation number Theorem 5.6 applies, and there are three cases, depending on the trace of M_1. One has $0 < Tr(M_1) < 2$ for $1 < a < \alpha_0$ where $\alpha_0 \approx 0.3802775690976\ldots$ is the real root of $x^4 + 3x^3 + 3x^2 + x - 1$ in the unit interval; invariant circles are then piecewise arcs of ellipses. One has $Tr(M_1) = 2$ for $a = \alpha_0$; the invariant circle in this case is piecewise linear. One has $Tr(M_1) > 2$ for $\alpha_0 < a < 1$; invariant circles are then piecewise arcs of hyperbolas.

The following three figures picture invariant circles for cases of Example 5.2. Figure 5.2 pictures the case $a = \sqrt{\frac{\sqrt{5}-1}{2}}) \approx 0.78615$, which is the case $b = -a$. The rotation number is irrational (in fact, transcendental) and the invariant circle consists of ten arcs of hyperbolas.

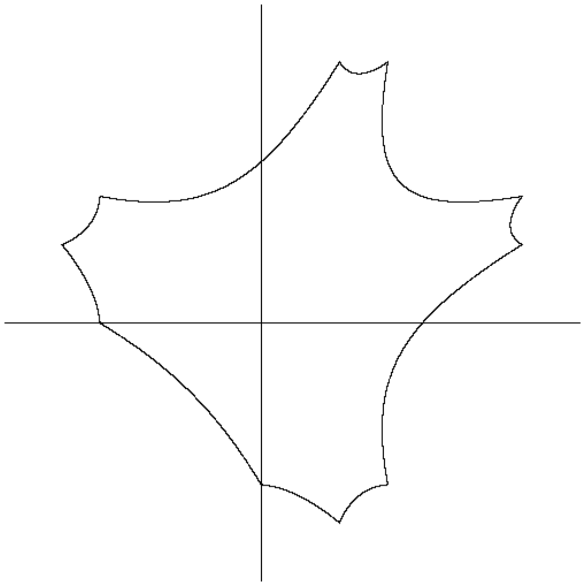

Figure 5.2: Invariant circle for $a = \left(\frac{\sqrt{5}-1}{2}\right)^{1/2}$, $b = -\left(\frac{\sqrt{5}-1}{2}\right)^{1/2}$ ("Elmer Fudd").

Figure 5.3 pictures the case of the critical value $a = \alpha_0 \approx 0.3802$. The rotation number

$$r(T_{\mu\nu}) = \frac{2\alpha_0^2 + 1}{9\alpha_0^2 + 4}$$

is an irrational algebraic number. Theorem 5.6 applies to show it has an invariant circle, which is piecewise linear with ten pieces.

Figure 5.4 pictures the case $\alpha = 1/10$. In this case the rotation number is irrational and the invariant circle consists of ten arcs of ellipses.

Example 5.3 *If $1 < a < \sqrt{2}$, and $b = \frac{(a-1)(2a^3-4a-1)}{a(a^2-2)(a^2-a-1)}$, then $T_{ab}^{(13)}(0,-1) = (0,1)$, but T_{ab} has no invariant circles, and all orbits diverge.*

The range of b is $-\infty < b < 0$. Theorem 5.5 and 5.6 do not apply, because all S_{ab} in this family have constant rational rotation number $r = \frac{1}{5}$. The following facts are proved in the appendix. If we take $\mathcal{O} = \{T^{(j)}(0,-1) : 1 \le j \le 13\}$ then part of the conclusion of Theorem 5.5 still applies: there exists a sector $J = \mathbb{R}^+[\mathbf{v}, \mathbf{v}')$ given by two elements of $\mathcal{O}$ which has a well defined first return map, which consists of exactly two pieces, and the resulting matrices M_1 and M_2 commute. (However not all sectors have a well-defined first return map.) For the sector J the matrices M_j have $|Tr(M_j)| > 2$, so their invariant sets restricted to J are hyperbolic arcs; however the asymptotes of the hyperbolas also lie in J. These maps fall under case (ii) of Theorem 2.4; all orbits are unbounded. Each T_{ab} in this family satisfy the following weak form of the invariant circle property: Each orbit of T_{ab} either falls on a finite number of rays, or else lies on a finite union of (unbounded) hyperbolic arcs.

A specific example is the parameter values $a = -b \approx 1.235877977$ where a is a root of $x^6 - x^5 - x^4 - 2x^2 + 3x + 1 = 0$.

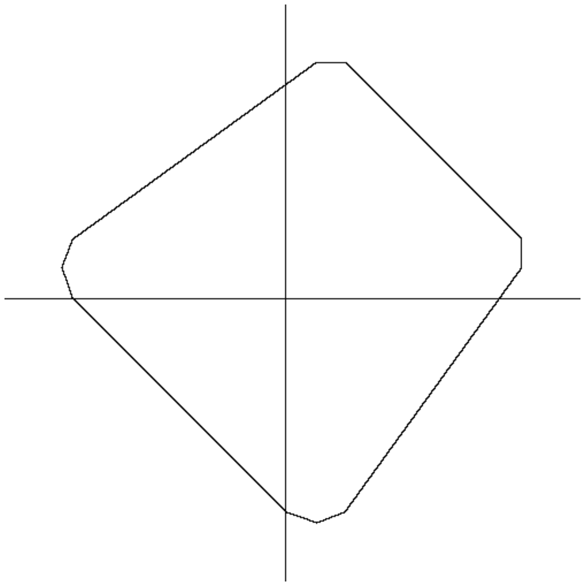

Figure 5.3: Piecewise linear invariant circle for critical value $a = \alpha_0 = 0.380277$, b as in example 5.2.

Theorem 5.6 gives conditions under which invariant circles exist. It appears plausible that there are many more parameter values having invariant circles. We present several non-rigorous examples showing numerical plots of orbits for other parameter values; these are intended to illustrate the structure of (possible) general invariant circles. The parameter values exhibited here were found by iterative trial-and-error search, in which the first 10000 iterates appeared to form an invariant circle. These include the values (a, b) used in making the Figures 5.5–5.9 below. One should view these figures with a grain of salt; a sufficiently slowly divergent periodic orbit could produce a picture that looked like an invariant circle; qualitatively, the longer the period of the periodic orbit, the closer the corresponding eigenvalue seems to tend to be to 1, so an orbit with very high period will tend to diverge very slowly, and will, furthermore, look like a curve. However Conjecture A together with various theorems in §2 would imply that the set of parameter values having an irrational rotation number has positive two-dimensional Lebesgue measure. If so, this makes it plausible that the computer plots actually approximate invariant circles. That is, even if the parameter values below don't correspond to invariant circles (as seems likely), then very small parameter changes should give some T_{ab} having an invariant circle exhibiting the same qualitative appearance.

Example 5.4 $a = 0.2$, $b = -0.7$. (Figure 5.5) Note that the invariant circle appears to be smooth; it is not clear if this is actually the case.

Example 5.5 $a = 1.4$, $b = -1.4$. (Figure 5.6) This highlights a typical behavior of the generic invariant circle; there are oscillations on several frequency scales (the thickness of the curve results from oscillations of wavelength smaller than the pixel size). The "hair" is usually not as pronounced as this.

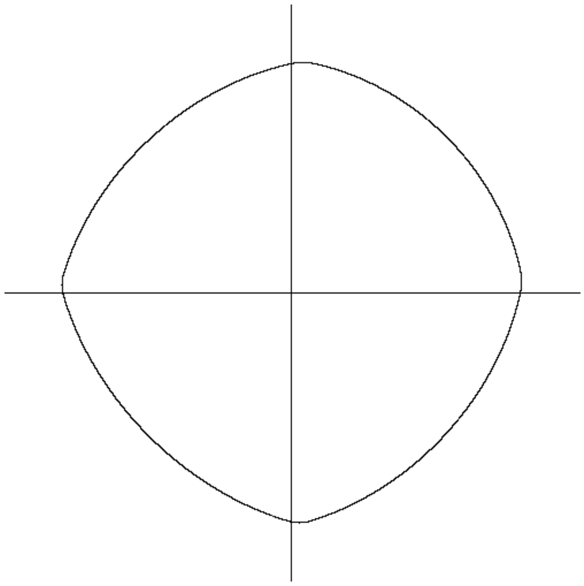

Figure 5.4: Invariant circle for $a = 0.1$, b as in example 5.2.

Example 5.6 $a = -.9$, $b = -4$. (Figure 5.7) The long spikes in the second and fourth quadrants are typical of invariant circles when $a, b < 0$, although, again, not usually as pronounced, for larger values of b.

Example 5.7 $a = 1.5$, $b = 1.1$. (Figure 5.8) For $a, b > 0$, on the other hand, the first and third quadrants features tend to be pronounced.

Example 5.8 $a = 1.9$, $b = -.2$. (Figure 5.9) For $a > 0 > b$, the first quadrant alone tends to be pronounced if $a >> -b$ (otherwise, as in figure 5.5), no quadrant seems to dominate).

6. The Set Ω_{conic}

In the case of irrational rotation number, aside from the invariant circle case, we have not determined any other information for the structure of bounded orbits. There remain several other possible types of bounded orbits: one is that the orbit approaches arbitrarily close to 0 as $n \to \infty$, but does not converge to 0; alternatively the orbit can converge to 0 as $n \to \infty$; these can be combined in any order with similar conditions as $n \to -\infty$. We do not know if any of these possibilities occurs. We also do not know whether or not all orbits must be bounded in the irrational rotation case. In part II we obtain some further information about the structure of invariant sets the case where all orbits are bounded, but there is no invariant circle.

A heuristic (from KAM theory) is that if the rotation number has very good rational approximations, then the orbit should tend to diverge, since it is very close to a number of divergent parameter values. On the other hand, if the rotation number r is irrational, then the $S_{\mu\nu}$ orbit of $(0, 1)$ will approach arbitrarily close to $(0, -1)$ infinitely often; this ought to place some

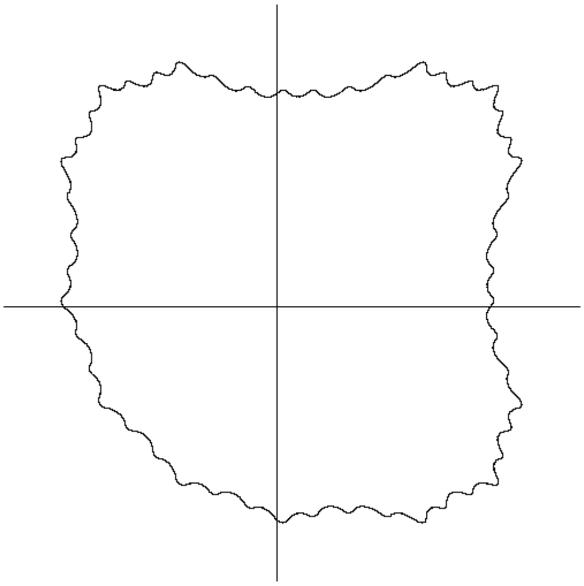

Figure 5.5: Apparent invariant circle for $a = 0.2$, $b = -0.7$.

constraints on the extent to which the size of the iterates of $(0, 1)$ can vary, and perhaps under some Diophantine condition on r this might lead to an invariant circle. These heuristics suggest that both these possibilities occur on uncountable sets of rotation number values. Note that the existence of an invariant circle might depend not only on the irrational rotation number value r but also on the particular parameter choice μ.

Recall that we defined Ω_{conic} to be the set of parameter values (a, b) for which T_{ab} has a piecewise conic invariant circle of the type given in Theorem 5.5, i.e. parameter values such that the S_{ab}-orbit of $(0, 1)$ contains $(0, -1)$, and S_{ab} has irrational rotation number. The fact that that the orbit of $(0, 1)$ is restricted makes this a rather small set.

Theorem 6.1 *The set $\Omega_{per} \cup \Omega_{conic}$ has Hausdorff dimension* 1.

Proof. We have exhibited parametric curves in this set in the examples, so it remains to show that Ω_{per} and Ω_{conic} each have Hausdorff dimension at most one.

We treat Ω_{conic}, the argument for Ω_{per} being similar. Theorem 5.5 gives the necessary condition that either $S_{ab}^{(k)}(0, 1) = (0, -1)$ or $S_{ab}^{(k)}(0, -1) = S_{ab}(0, 1)$, for some $k \in \mathbb{Z}_{>0}$. These conditions (namely $T_{ab}^{(k)}(0, 1)_x = 0$, subject to the constraint $T_{ab}^{(k)}(0, 1)_y < 0$, for each $k \in \mathbb{Z}\backslash\{0\}$) cut out a countable number of pieces of real algebraic curves. The resulting set contains Ω_{conic}. It follows that Ω_{conic} has Hausdorff dimension at most 1. ∎

In the introduction we proposed the following conjecture.

Conjecture A. *The set $\Omega_{per} \cup \Omega_{conic}$ is a dense subset of Ω_{bdd}.*

We give some justification for this conjecture. For any positive ϵ, if $r(S_{\mu\nu})$ is irrational, there is a perturbation of ν smaller than ϵ such that $T_{\mu\nu'}$ maps $(0, 1)$ to $(0, -1)$; more precisely,

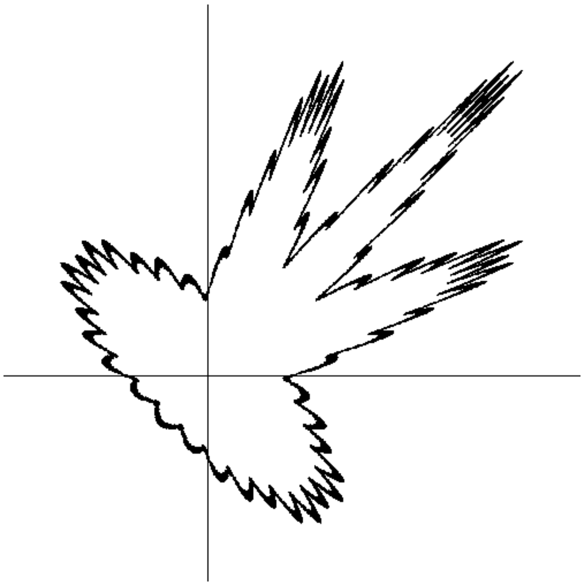

Figure 5.6: Apparent invariant circle for $a = 1.4$, $b = -1.4$.

for any rational, there is such a point having rotation number between that rational and $r(S_{\mu\nu})$; experimentation suggests that this point will be in $\Omega_{per} \cup \Omega_{conic}$.

As an example, we consider the problem of finding points of $\Omega_{per} \cup \Omega_{conic}$ close to the parameter values $(a, b) = (1.4, -1.4)$ which has the (apparent) invariant circle pictured in Figure 5.6. We found numerically that

$$a = -b = 1.3999999999968698551388950626942617779494$$

has a piecewise elliptic invariant circle (or is periodic), with $T^{(810096)}(0, -1) = (0, 1)$, complex eigenvalues. We also found numerically that

$$a = -b = 1.4000000000000003666426603872542419579986$$

has a piecewise hyperbolic invariant circle (or is periodic), with $T^{(35676178)}(0, 1) = (0, -1)$, no eigenvectors in interval.

Both these parameter sets produce plots basically indistinguishable from Figure 5.6. Since the simpler case has 810096 conic pieces, the piecewise conic structure is invisible in the computer plot.

In fact, $\Omega_{per} \cup \Omega_{conic}$ is contained in a countable union of real algebraic curves; the argument above would say that any 2-dimensional ball would contain a segment of such a curve. In nearly all cases, as exhibited in Examples 5.1 and 5.2, it appears that curve does not have constant rotation number; if so, then it contains a dense set of points with irrational rotation number, and one can guarantee a point of Ω_{conic} in any neighborhood, so that Conjecture A would follow.

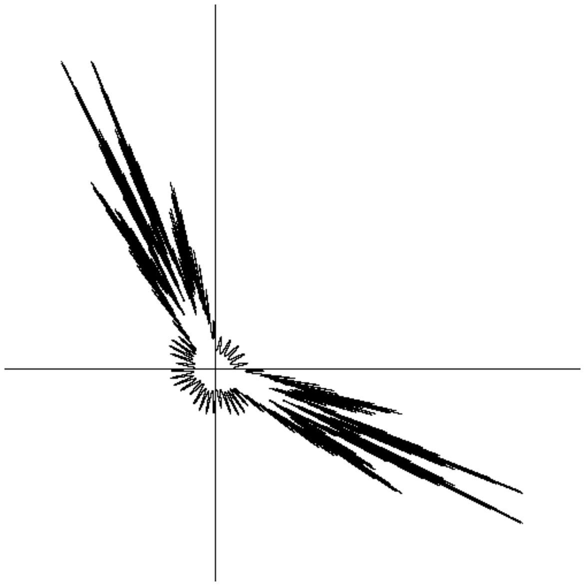

Figure 5.7: Apparent invariant circle for $a = -0.9$, $b = -4$.

7. Appendix: Irrational Rotation Numbers

In this appendix we give proofs for the assertions made in §5 about Examples 5.1- 5.3. The calculations for Example 5.1 below can be verified by hand, but those for Example 5.2 and 5.3 require a computer. The packages MAPLE and MAGMA were used for some of the calculations below.

Example 5.1. *T_{ab} with $1 < a < \sqrt{2}$, with $b = 2\frac{a^2-1}{a^3-2a}$. Here $b \in (-\infty, 0)$, and $T_{ab}^{(8)}(0,-1) = (0,1)$.*

Proof. We first verify that one has $T_{ab}^{(8)}(0,-1) = (0,1)$. Set $\mathbf{v}_0 = (0,-1)$ and $\mathbf{v}_j = T_{ab}^{(j)}(\mathbf{v}_0)$; the sequence of signs $S_j = \mathrm{Sign}(\mathbf{v}_j)_x)$ determine the sequence of iterates. The resulting $\mathbf{v}_j$ and their sign values are given in Table 7.1.

vector	$\mathrm{Sign}(\mathbf{v}_j)_x)$
$\mathbf{v}_1 = (1,0)$	$+$
$\mathbf{v}_2 = (a,1)$	$+$
$\mathbf{v}_3 = (a^2-1,a)$	$+$
$\mathbf{v}_4 = (a^3-2a,a^2-1)$	$-$
$\mathbf{v}_5 = (a^2-1,a^3-2a)$	$+$
$\mathbf{v}_6 = (a,a^2-1)$	$+$
$\mathbf{v}_7 = (1,a)$	$+$
$\mathbf{v}_8 = (0,1)$	$\pm$

Table 7.1: Orbit $\mathcal{O}$ for example 5.1

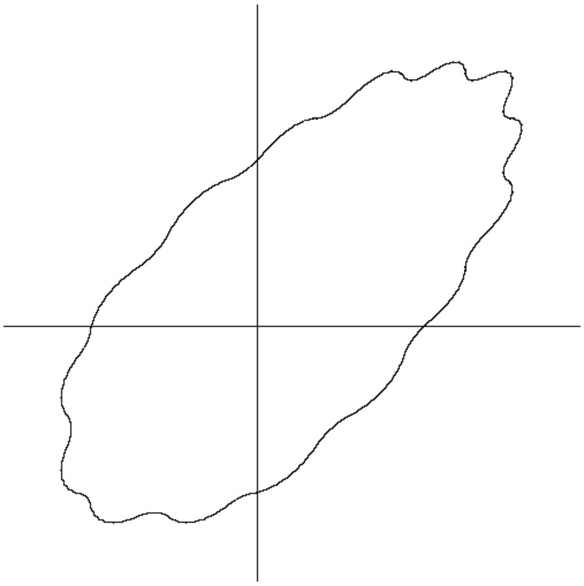

Figure 5.8: Apparent invariant circle for $a = 1.5$, $b = 1.1$.

To verify the entries in Table 7.1, note that the condition $1 < a < \sqrt{2}$ determines $\{\mathbf{v}_j : 1 \leq j \leq 4\}$ and their signs, and in particular

$$\mathbf{v}_5 = ((a^3 - 2a)b + 1 - a^2, a^3 - 2a.)$$

We impose the condition that $\mathbf{v}_5 = R(\mathbf{v}_4)$, and this requires that

$$(a^3 - 2a)b + 2 - 2a^2 = 0, \tag{7.1}$$

which specifies b uniquely. Now the relation $T_{ab}^{-1} = R \circ T_{ab} \circ R^{-1}$ yields $\mathbf{v}_j = R(\mathbf{v}_{9-j})$ for $5 \leq j \leq 8$, and also determines their symbol sequence. This completes Table 7.1, and shows $T_{ab}^8(0, -1) = (0, 1)$.

We choose $\mathcal{O} = \{\mathbf{v}_j : 1 \leq j \leq 8\}$, and the hypotheses of Theorem 5.5 and 5.6 apply whenever the rotation number $r(S_{ab})$ is irrational. The elements of $\mathcal{O}$ appear in counterclockwise angular order as

$$\mathbf{v}_1, \mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_7, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_4, \mathbf{v}_5.$$

We choose the sector $J := \mathbb{R}^+[\mathbf{v}, \mathbf{v}') = \mathbb{R}^+[\mathbf{v}_4, \mathbf{v}_5)$. This sector includes the entire third quadrant with signs $(-, -)$; here $\mathbf{v}_4$ and $\mathbf{v}_5$ have signs $(-, +)$ and $(+, -)$, respectively. The breakpoint for J is the ray determined by

$$\mathbf{v}^* := (0, -1). \tag{7.2}$$

The first return maps on the subsectors $J_1 = \mathbb{R}^+[\mathbf{v}, \mathbf{v}^*)$ and $J_2 = \mathbb{R}^+[\mathbf{v}^*, \mathbf{v}')$ are well-defined regardless of whether the rotation number is rational or irrational.

The first return map on J_1 is $T_{ab}^{(5)}$ and has symbol seqence $(- + + + +)$; its associated matrix[3] M_1 is:

$$M_1 = \begin{bmatrix} a & -a^2 + 1 \\ a^2 - 1 & -a^3 + 2a \end{bmatrix}. \tag{7.3}$$

[3] To express the entries of M_1 in terms of the variable a alone, the variable b is eliminated using (7.1).

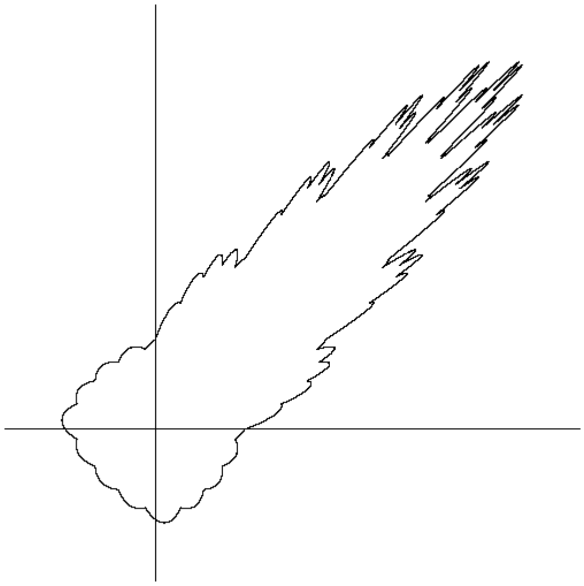

Figure 5.9: Apparent invariant circle for $a = 1.9$, $b = -0.2$.

The first return map on J_2 is $T_{ab}^{(4)}$ with symbol sequence $(++++)$ and its associated matrix M_2 is:

$$M_2 = \begin{bmatrix} a^4 - 3a^2 + 1 & -a^3 + 2a \\ a^3 - 2a & -a^2 + 1 \end{bmatrix}. \tag{7.4}$$

We have $Tr(M_1) = -a^3 + 3a$ so

$$0 < Tr(M_1) < 2 \qquad \text{for} \quad 1 < a < \sqrt{2}. \tag{7.5}$$

We parametrize the eigenvalues and eigenvectors of M_1 and M_2 using the parameter θ, with

$$a = 2\cos\theta \quad \text{for} \quad \frac{\pi}{4} < \theta < \frac{\pi}{3}.$$

The common eigenvectors $\mathbf{w}^+$ and $\mathbf{w}^-$ of M_1 and M_2 are given by

$$\mathbf{w}^\pm = (e^{\pm i\theta}, 1).$$

A calculation (using $a = e^{i\theta} + e^{-i\theta}$) gives

$$M_1\mathbf{w}_+ = -e^{3i\theta}\mathbf{w}^+, \quad M_1\mathbf{w}^- = -e^{3i\theta}\mathbf{w}^-.$$

$$M_2\mathbf{w}^+ = e^{4i\theta}\mathbf{w}^+, \quad M_2\mathbf{w}^- = e^{-4i\theta}\mathbf{w}^-.$$

The matrices M_1 and M_2 leave invariant a common family of ellipses centered at the origin. Such a family of ellipses centered at the origin is specified by a single parameter ϕ with $0 \le \phi < 2\pi$, and the family elements parametrized by $r > 0$, with

$$E = E(r, \phi) := \{\mathbf{v}(\omega) = (r\cos(\phi + \omega), r\cos(\omega)) : 0 \le \omega < 2\pi\}.$$

The action of M_1 on that portion of the ellipse in J is to take $\mathbf{v}(\omega)$ to $\mathbf{v}(\omega+\pi-3\theta)$, while the action of M_2 there takes $\mathbf{v}(\omega)$ to $\mathbf{v}(\omega-2\pi+4\theta)$; the multiples of π are specified to keep the image angle in $[0, 2\pi]$.

We now show that the rotation number $r(S_{ab})$ is given by

$$r(S_{ab}) = \frac{3\pi - 7\theta}{14\pi - 32\theta}. \tag{7.6}$$

To establish this, we consider iteration $T_J^{(p)}$ of the first return map T_J on the sector J. Pick $\mathbf{w} \in J$ lying on $E(r, \phi)$ and set

$$T_J^{(p)}(\mathbf{w}) = M_1^{m_1} M_2^{m_2} \mathbf{w},$$

with $m_1 = m_1(p), m_2 = m_2(p)$, both going to $+\infty$ as $p \to \infty$. Since each appearance of M_1 or M_2 corresponds to one counterclockwise revolution of the circle, while M_1 and M_2 correspond to 5 and 4 iterations of T_{ab}, respectively, we have

$$r(S_{ab}) = \lim_{p\to\infty} \frac{m_1(p) + m_2(p)}{5m_1(p) + 4m_2(p)}. \tag{7.7}$$

However the condition that the angle ω remains inside the sector J of $E(r, \phi)$ at each iteration of T_J yields

$$0 < \phi + m_1(p)(\pi - 3\theta) + m_2(p)(-2\pi + 4\theta) < 2\pi.$$

Letting $p \to \infty$, this constraint implies that

$$\lim_{p\to\infty} \frac{m_2(p)}{m_1(p)} = \frac{\pi - 3\theta}{2\pi - 4\theta}. \tag{7.8}$$

Rewriting (7.7) as

$$r(S_{ab}) = \lim_{p\to\infty} \frac{1 + \frac{m_2(p)}{m_1(p)}}{5 + 4\frac{m_2(p)}{m_1(p)}}$$

and substituting (7.8) yields the desired rotation number formula (7.6).

Suppose that $r(S_{ab})$ is rational. By (7.6) this occurs if and only if θ is a rational multiple of π. We show that T_{ab} is periodic. Indeed, for rational rotation number S_{ab} has a periodic orbit, and this orbit necessarily contains some point $\mathbf{w} \in J$. Since $S_{ab}^{(q)}(\mathbf{w}) = \mathbf{w}$ where q is the period, we have $T_{ab}^{(q)}(\mathbf{w}) = \lambda \mathbf{w}$ for some real λ. Since this value is an iterate of a first return map to J, there is some p such that

$$T_{ab}^{(q)}(\mathbf{w}) = T_J^{(p)}(\mathbf{w}) = M_1^{m_1} M_2^{m_2} \mathbf{w}.$$

Now $M_1^{m_1} M_2^{m_2}$ has $\mathbf{w}$ as an eigenvector with a real eigenvalue, but it also has $\mathbf{w}^+$ and $\mathbf{w}^-$ as eigenvectors with complex conjugate eigenvalues of norm 1. It follows that $M_1^{m_1} M_2^{m_2} = \pm I$. so that

$$T_{ab}^{(2q)}(\mathbf{w}) = T_J^{(2r)}(\mathbf{w}) = M_1^{2m_1} M_2^{2m_2} \mathbf{w}.$$

It follows that the first return map $T_J^{(2p)}$ is the identity on a subsector of J of positive width, so S_{ab} must have infinitely many periodic points of period $2q$, and Theorem 2.4(iii) shows that T_{ab} is periodic, with period dividing $2q$.

Suppose that $r(S_{ab})$ is irrational. Then Theorem 5.6 applies to show that T_{ab} has a piecewise conic invariant circle with at most 8 pieces, and the pieces are arcs of ellipses by (7.5).

We consider the special parameters $a = -b = \sqrt[4]{2}$. Defining θ by $a = 2\cos\theta$ shows that $x = e^{i\theta}$ satisfies $x + 1/x = \sqrt[4]{2}$, hence it also satisfies

$$x^8 + 4x^6 + 4x^4 + 4x^2 + 1 = 0. \tag{7.9}$$

We show the rotation number $r(S_{ab})$ is irrational by showing that $\frac{\theta}{\pi}$ is irrational, i.e. that $e^{i\theta}$ is not a root of unity. It suffices to show that (7.9) has no root that is a root of unity. One checks using MAPLE that this polynomial is irreducible over $\mathbb{Q}$, and that it is not a cyclotomic polynomial since it has a root off the unit circle. Thus it has no roots of unity as roots, and $r(S_{ab})$ is irrational. Thus T_{ab} has a piecewise elliptical invariant circle, pictured in Figure 5.1. ∎

Example 5.2. *T_{ab} with $0 < a < 1$, with $b = 2\frac{a^2-1}{a^3-2a}$. Here $T_{ab}^{(10)}(0,1) = (0,-1)$.*

Proof. We first verify that one has $T_{ab}^{(10)}(0,1) = (0,-1)$. Set $\mathbf{v}_0 = (0,-1)$ and $\mathbf{v}_j = T_{ab}^{(j)}(\mathbf{v}_0) = \mathbf{v}_j$,; the signs $S_j = \text{Sign}(\mathbf{v}_j)_x)$ determine the sequence of iterates. The resulting $\mathbf{v}_j$ and their sign values are given in Table 7.2.

vector	$\text{Sign}(\mathbf{v}_j)_x$
$\mathbf{v}_1 = (-1, 0)$	$-$
$\mathbf{v}_2 = (-b, -1)$	$+$
$\mathbf{v}_3 = (-ab + 1, -b)$	$+$
$\mathbf{v}_4 = (-a^2b + a + b, 1 - ab)$	$+$
$\mathbf{v}_5 = (a^3b + a^2 + 2ab - 1, -a^2b + a + b)$	$-$
$\mathbf{v}_6 = (-a^2b + a + b, a^3b + a^2 + 2ab - 1)$	$+$
$\mathbf{v}_7 = (-ab + 1, -a^2b + a + b)$	$+$
$\mathbf{v}_8 = (-b, -ab + 1)$	$+$
$\mathbf{v}_9 = (-1, -b)$	$-$
$\mathbf{v}_{10} = (0, -1)$	$\pm$

Table 7.2: Orbit $\mathcal{O}$ for example 5.2

To verify the entries in Table 7.2, note that the condition $0 < a < 1$ and the given sign sequence determine $\{\mathbf{v}_j : 1 \le j \le 5\}$, and in particular

$$\mathbf{v}_6 = (a^3b^2 + 2a^2b + 2ab^2 - a - 2b, a^2 + 2ab + a^3b - 1)$$

We impose the condition that $\mathbf{v}_6 = R(\mathbf{v}_5)$, and this requires that

$$(a^3 - 2a)b^2 - 3(a^2 - 1)b + 2a = 0. \tag{7.10}$$

The correct choice of root of this quadratic equation determines b as given; one has $-\sqrt{2} < b < 0$ and it produces the sign sequence as given above for $1 \le j \le 5$. As in Example 3.1 the relation $\mathbf{v}_6 = R(\mathbf{v}_5)$ forces $\mathbf{v}_j = \mathbf{v}_{11-j}$ for $6 \le j \le 10$, and determines the remaining sign sequence, and gives $T_{ab}^{(10)}(0,-1) = (0,1)$.

We take $\mathcal{O} = \{\mathbf{v}_j : 1 \le j \le 10\}$, and the hypotheses of Theorem 5.5 and 5.6 apply whenever the rotation number $r(S_{ab})$ is irrational. The elements of $\mathcal{O}$ appear in counterclockwise angular order as

$$\mathbf{v}_1, \mathbf{v}_{10}, \mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_7, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_5.$$

We choose the sector $J = \mathbb{R}^+[\mathbf{v}, \mathbf{v}'] := \mathbb{R}^+[\mathbf{v}_1, \mathbf{v}_{10})$, which is exactly the third quadrant, with sign $(-,-)$. The breakpoint is

$$\mathbf{v}^* = (a^2 - 1, (a^2 - 1)b - a). \tag{7.11}$$

The first return maps on the subsectors $J_1 = \mathbb{R}^+[\mathbf{v}, \mathbf{v}^*)$ and $J_2 = \mathbb{R}^+[\mathbf{v}^*, \mathbf{v}')$ are well-defined regardless of whether the rotation number is rational or irrational.

The first return map on J_1 is $T_{ab}^{(4)}$ with symbol sequence $(-++-)$ and its associated matrix M_1 is:

$$M_1 = \begin{bmatrix} (a^2-1)b^2 - 2ab + 1 & (1-a^2)b + a \\ (a^2-1)b - a & 1 - a^2 \end{bmatrix}.$$

The first return map on J_2 is $T_{ab}^{(13)}$ with symbol sequence $(-+++-+++-+++-)$ and its associated matrix M_2 is:

$$M_2 = \begin{bmatrix} (1-a^2)b + a & a^2 - 1 \\ 1 - a^2 & \frac{a^2(a^2-2)}{(a^2-1)b-a} \end{bmatrix}.$$

We have $Tr(M_1) = (a^2-1)(b^2-1) - 2ab + 1$ and using the relation (7.10) and the fact that $b < 0$, we obtain [4]

$$Tr(M_1) = 2 \quad \text{for} \quad a = \alpha_0 \approx 0.3802,$$

where α_0 is the unique root in the unit interval of

$$x^4 + 3x^3 + 3x^2 + x - 1 = 0. \tag{7.12}$$

One can check that $0 < Tr(M_1) < 2$ for $0 < a < \alpha_0$ and $Tr(M_1) > 2$ for $\alpha_0 < a < 1$.

Subcase 1. *Piecewise hyperbolic case ($\alpha_0 < a < 1$)*

The matrix M_1 is hyperbolic and has real eigenvectors and eigenvalues; therefore M_2 does also, since it commutes with M_1 and has the same eigenvectors. One checks $Tr(M_2) > 2$ as well. The first return map T_J for J leaves invariant hyperbolic arcs in J, even in the rational rotation number case. These are bounded arcs, for one can verify numerically that the asymptotes of the hyperbola fall strictly outside J. Thus the orbit of a given point $\mathbf{w} \in J$, lies on some bounded hyperbolic arc $\mathcal{C}_1$ inside J.

Let $\lambda_1 = \lambda_1(a) > 1$ be the large eigenvalue of M_1, the other eigenvalue being $1/\lambda_1$; let $\lambda_2 > 1$ be the large eigenvalue of M_2, the other being $1/\lambda_2$. We choose the eigenvectors so that $M_1\mathbf{w}_+ = \lambda_1\mathbf{w}_+$ and $M_1\mathbf{w}_- = 1/\lambda_1\mathbf{w}_-$. We now claim that

$$M_2\mathbf{w}_+ = \frac{1}{\lambda_2}\mathbf{w}_+. \tag{7.13}$$

[4] Take the resultant of (7.10) and the polynomial $Tr(M_1) - 2$. It is a polynomial of degree 8 in a which has two degree 4 factors over $\mathbb{Q}$. The other degree 4 factor produces extraneous roots.

To establish this, choose a point $\mathbf{w}$ in J, whose first return iterates $T_J^{(p)}$ necessarily all lie on a fixed hyperbolic arc $\mathcal{C}_1$ in J. Write

$$T_J^{(p)}(\mathbf{w}) = M_1^{m_1} M_2^{m_2} \mathbf{w},$$

with $m_1 = m_1(p), m_2 = m_2(p)$, both going to $+\infty$ as $p \to \infty$. Express $\mathbf{w}$ in terms of the eigenvectors as

$$\mathbf{w} = c_1 \mathbf{w}_+ + c_2 \mathbf{w}_-,$$

with $c_1 c_2 \neq 0$ since $\mathbf{w}$ cannot be an eigenvector. We now argue by contradiction, and suppose $M_2 \mathbf{w}_+ = \lambda_2 \mathbf{w}_+$. Then

$$T_J^{(p)}(\mathbf{w}) = c_1 \lambda_1^{m_1} \lambda_2^{m_2} \mathbf{w}_+ + c_2 \lambda_1^{-m_1} \lambda_2^{-m_2} \mathbf{w}_-,$$

The first term on the right gets large as $p \to \infty$, while the second gets small, hence $||T_J^{(p)}(\mathbf{w})|| \to \infty$ as $p \to \infty$. This contradicts $T_J^{(p)}(\mathbf{w}) \in \mathcal{C}_1$ for all p; the claim follows.

We show that the rotation number $r(S_{ab})$ is given by

$$r(S_{ab}) = \frac{\log \lambda_2 + 3 \log \lambda_1}{4 \log \lambda_2 + 13 \log \lambda_1}. \tag{7.14}$$

We pick a point $\mathbf{w} \in J$ and study the first return map iterates, with $T_J^{(p)}(\mathbf{w}) = M_1^{m_1} M_2^{m_2} \mathbf{w}$ and $m_1 = m_1(p), m_2 = m_2(p) \to \infty$ as $p \to \infty$. The return map for J_1 goes around the circle once, while that for M_2 goes around three times, and using the number of steps of T_{ab} each takes we obtain

$$r(S_{ab}) = \lim_{p \to \infty} \frac{m_1(p) + 3m_2(p)}{4m_1(p) + 13m_2(p)}. \tag{7.15}$$

Expressing $\mathbf{w} = c_1 \mathbf{w}_+ + c_2 \mathbf{w}_-$ in terms of eigenvectors and using (7.13) yields

$$T_J^{(p)}(\mathbf{w}) = c_1 \lambda_1^{m_1} \lambda_2^{-m_2} \mathbf{w}_+ + c_2 \lambda_1^{-m_1} \lambda_2^{m_2} \mathbf{w}_-.$$

In order for these vectors to remain on $\mathcal{C}_1$ their norms must be bounded away from 0 and ∞, so there is a positive constant C such that $1/C < \lambda_1^{m_1} \lambda_2^{-m_2} < C$. This yields

$$\lim_{p \to \infty} \frac{m_1(p)}{m_2(p)} = \frac{\log \lambda_2}{\log \lambda_1}.$$

Substituting this in (7.15) yields the rotation number formula (7.14).

When $r(S_{ab})$ is rational, then T_{ab} is periodic. To show this, we use the fact that the eigenvectors of M_1 and M_2 fall on rays lying outside the sector J. If $r(S_{ab})$ is rational, then S_{ab} has a periodic orbit, which necessarily visits the sector J. Choosing an element $\mathbf{w} \in J$ on this orbit, one has some first return value $T_J^{(p)}(\mathbf{w}) = \lambda \mathbf{w}$, for some real value λ. Now $T_J^{(p)}(\mathbf{w}) = M_1^{m_1} M_2^{m_2} \mathbf{w}$, and the matrix $M = M_1^{m_1} M_2^{m_2}$ now has three distinct eigenvectors, $\mathbf{w}_+$, $\mathbf{w}_-$ and $\mathbf{w}$, so $M = \pm I$. Now we find that $T_J^{(2p)}(\mathbf{w}') = M_1^{2m_1} M_2^{2m_2} \mathbf{w}' = \mathbf{w}'$ for all $\mathbf{w}'$ on a subsector of positive angular width. It follows that S_{ab} has infinitely many periodic orbits, so T_{ab} is periodic by Theorem 2.4(iii).

When $r(S_{ab})$ is irrational, Theorem 5.6 applies to show that T_{ab} has an invariant circle which is a piecewise union of at most 10 arcs of hyperbolas.

We consider the special parameter values $a = -b = \sqrt{\frac{\sqrt{5}-1}{2}}$. Here $Tr(M_1) = a^4 + 2 > 2$. The larger eigenvalue λ_1 of M_1 satisfies $x^2 - (a^4 + 2)x + 1 = 0$, and hence one finds it is an algebraic unit that satisfies

$$x^4 - 7x^3 + 13x^2 - 7x + 1 = 0.$$

A similar calculation shows that λ_2 is an algebraic unit that satisfies

$$x^8 + 23x^6 - 77x^4 + 23x^2 + 1 = 0.$$

The rotation number formula (7.14) gives

$$r(S_{ab}) = \frac{\log(\lambda_1^3 \lambda_2)}{\log(\lambda_1^{13} \lambda_2^4)}, \tag{7.16}$$

and the condition for $r(S_{ab})$ to be irrational is that

$$\lambda_1^{n_1} \lambda_2^{n_2} \neq 1 \quad \text{for} \quad (n_1, n_2) \in \mathbb{Z}^2 \backslash (0,0).$$

That is, λ_1 and λ_2 must be multiplicatively independent algebraic units in the algbraic number field over $\mathbb{Q}$ that they generate. This can be verified by checking that their logarithms and the logarithms of the absolute values of their algebraic conjugates span a two dimensional lattice. Given that $r(S_{ab})$ is irrational, Theorem 5.6 applies to show T_{ab} has an invariant circle, which consists of at most 10 arcs of hyperbolas, as pictured in Figure 5.2.

Given that $r(S_{ab})$ is irrational, we can show it is transcendental. Clearing denominators in (7.16) and exponentiating gives

$$(\lambda_1^{13} \lambda_2^4)^r = \lambda_1^3 \lambda_2.$$

The right side of this equation is an algebraic number, and the left side is an algebraic number raised to the r-th power. If $r = r(S_{ab})$ were an irrational algebraic number, then the left side would be transcendental by the criterion [5] of Theorem 2.4 in Baker [3]. This is a contradiction, so we conclude that $r(S_{ab})$ is transcendental.

Subcase 2. *Piecewise linear case ($a = \alpha_0$)*

Here $a = \alpha_0, b = \alpha_0^3 + 2\alpha_0^2 + \alpha_0 - 1$, where α_0 satisfies (7.12). The sector $J = \mathbb{R}^+[(-1,0)),(0,-1))$ and in this case one finds that

$$M_1 = I + \begin{bmatrix} \alpha_0^2 & \alpha_0^2 \\ -\alpha_0^2 & -\alpha_0^2 \end{bmatrix},$$

$$M_2 = I + \begin{bmatrix} \alpha_0^2 - 1 & \alpha_0^2 - 1 \\ 1 - \alpha_0^2 & 1 - \alpha_0^2 \end{bmatrix}.$$

Consequently

$$M_1^{m_1} M_2^{m_2} = I + \begin{bmatrix} x & x \\ -x & -x \end{bmatrix}, \tag{7.17}$$

[5] In [3, Theorem 2.4] take $\alpha_1 = \lambda_1^{13} \lambda_2^4$ and $\beta = r$, and $[1, \beta]$ are linearly independent over $\mathbb{Q}$ since r is irrational.

with $x = (m_1 + m_2)\alpha_0^2 - m_2$. By Lemma 5.1 these commuting maps leave invariant a family of parallel straight lines, and in the sector J the first return map iterates of any given point $\mathbf{w}$ lie on a certain finite line segment $\mathcal{C}_1$.

We show that the rotation number

$$r(S_{ab}) = \frac{2\alpha_0^2 + 1}{9\alpha_0^2 + 4}. \tag{7.18}$$

For this, picking a point $\mathbf{w} \in J$ and using the fact that the orbit $T_J^{(p)}(\mathbf{w}) = M_1^{m_1} M_2^{m_2}\mathbf{w}$ Just as in subcase 1) we have

$$r(S_{ab}) = \lim_{p\to\infty} \frac{m_1(p) + 3m_2(p)}{4m_1(p) + 13m_2(p)}. \tag{7.19}$$

Next, using the fact that the orbit $T_J^{(p)}(\mathbf{w})$ lies on the line segment $\mathcal{C}_1$, so has norm bounded away from 0 and ∞, using (7.17) we have $|m_1(p)\alpha_0^2 + m_2(p)(\alpha_0^2 - 1)| < C$ for some positive C. Letting $p \to \infty$ yields

$$\lim_{p\to\infty} \frac{m_1(p)}{m_2(p)} = \frac{1 - \alpha_0^2}{\alpha_0^2},$$

and substituting this in (7.19) yields (7.18).

The rotation number $r(S_{ab})$ is irrational, as is verified by computing the irreducible polynomial it satisfies over $\mathbb{Q}$, which has degree 4. Now Theorem 5.6 applies to show that T_{ab} has a piecewise linear invariant circle with at most 10 pieces. This justifies Figure 5.3.

Subcase 3. *Piecewise ellipse case* $(0 < a < \alpha_0)$

This case is analyzed similarly to the case of Example 5.1 and we omit the details. The invariant circle is piecewise conic with arcs of ellipses when the rotation number is irrational, and T_{ab} is periodic when the rotation number is rational.

Consider the special parameter values $a = \frac{1}{10}$, where $b = 5(\frac{297-\sqrt{89801}}{199})$. One finds that the (complex conjugate) eigenvalues $\lambda_1, \bar{\lambda}_1$ of M_1 and $\lambda_2, \bar{\lambda}_2$ of M_2 are roots of degree 4 polynomials over $\mathbb{Z}[x]$ with large coefficients. Verifying irrationality of the rotation number $r(S_{ab})$ reduces to showing that $< \lambda_1, \lambda_2 >$ generate multiplicatively a free abelian group of rank 2. These numbers are not algebraic integers, and it suffices to give a set of prime ideals such that the valuations of λ_1 and λ_2 at those primes span a lattice of rank 2. In particular, at the four prime ideals above 2, λ_1 has valuations $(2, -2, 2, -2)$, while λ_2 has valuations $(7, -3, 3, -7)$. Thus $\lambda_1^i \lambda_2^j$ is an algebraic integer only when $i = j = 0$. The irrationality of $r(S_{ab})$ shows that this T_{ab} has a piecewise elliptic invariant circle, as pictured in Figure 5.4. ■

Example 5.3. *T_{ab} with $1 < a < \sqrt{2}$, with $b = \frac{(a-1)(2a^3-4a-1)}{a(a^2-2)(a^2-a-1)}$. Here $T_{ab}^{(13)}(0,-1) = (0,1)$.*

Proof. One verifies that one has $T_{ab}^{(13)}(0,-1) = (0,1)$ in a manner similar to the earlier examples. Set $\mathbf{v}_0 = (0,-1)$ and $\mathbf{v}_j = T_{ab}^{(j)}(\mathbf{v}_0)$, and define the sequence of symbols $S_j = \mathrm{Sign}(\mathbf{v}_j)_x$ which determine the sequence of iterates. The resulting $\mathbf{v}_j$ and their symbol values S_j are given in Table 7.3. The value $b = \frac{(a-1)(2a^3-4a-1)}{a(a^2-2)(a^2-a-1)}$ is determined by the requirement that $R(\mathbf{v}_7) = \mathbf{v}_7$, plus the given sign seqence $\{S_i : 1 \le i \le 6\}$. This value of b has been substituted in the earlier iterates; one verifies retrospectively that the symbol sequence $S_j: 1 \le \mathbf{v}_j \le 6\}$ is as indicated.

vector	$\mathrm{Sign}(\mathbf{v}_j)_x$
$\mathbf{v}_1 = (1,0)$	$+$
$\mathbf{v}_2 = (a,1)$	$+$
$\mathbf{v}_3 = (a^2-1,a)$	$+$
$\mathbf{v}_4 = (a(a^2-2),a^2-1)$	$-$
$\mathbf{v}_5 = (a(a-1)(a^2-2)/(a^2-a-1),a(a^2-2)$	$+$
$\mathbf{v}_6 = (a(a^2-2)/(a^2-a-1),a(a-1)(a^2-2)/(a^2-a-1))$	$+$
$\mathbf{v}_7 = (a(a^2-2)/(a^2-a-1),a(a^2-2)/(a^2-a-1))$	$+$
$\mathbf{v}_8 = (a(a-1)(a^2-2)/(a^2-a-1),a(a^2-2)/(a^2-a-1))$	$+$
$\mathbf{v}_9 = (a(a^2-2),a(a-1)(a^2-2)/(a^2-a-1))$	$-$
$\mathbf{v}_{10} = (a^2-1),a(a^2-2))$	$+$
$\mathbf{v}_{11} = (a,a^2-1)$	$+$
$\mathbf{v}_{12} = (1,a))$	$+$
$\mathbf{v}_{13} = (0,1)$	$\pm$

Table 7.3: Orbit $\mathcal{O}$ for example 5.3

The points appear around the circle in counterclockwise order

$$\mathbf{v}_1,\mathbf{v}_6,\mathbf{v}_{11},\mathbf{v}_2,\mathbf{v}_7,\mathbf{v}_{12},\mathbf{v}_3,\mathbf{v}_8,\mathbf{v}_{13},\mathbf{v}_4,\mathbf{v}_9,\mathbf{v}_5,\mathbf{v}_{10}.$$

Of the thirteen arcs these determine, five have a well-defined first return map and eight do not. The ones that do are divided into two intervals by a breakpoint and the conclusion of Theorem 5.5 holds for them; they are the intervals with left endpoint $\mathbf{v}_9,\mathbf{v}_{10},\mathbf{v}_{11},\mathbf{v}_{12},\mathbf{v}_{13}$, respectively. The other eight intervals are $[v_i,v_{i+5})$ for $1 \le i \le 8$; they are the images under $T_{ab}^{(i)}$ of the interval $[(0,-1),v_5)$, $1 \le i \le 8$. These intervals are entirely transient; that is, any orbit contains at most one point from each of the eight intervals.

We consider the interval $J = \mathbb{R}^+[\mathbf{v},\mathbf{v}') := \mathbb{R}^+[\mathbf{v}_9,\mathbf{v}_5)$. The break point $\mathbf{v}^* = (0,-1)$. The first return map on $J_1 = [\mathbf{v},\mathbf{v}^*)$ is $T_{ab}^{(5)}$ and has symbol sequence $(-+++-)$; its associated matrix M_1 is

$$M_1 = \begin{bmatrix} (a^3-2a)b^2+(-2a^2+2)b+a & (-a^3+2a)b+a^2-1 \\ (a^3-2a)b-a^2+1 & -a^3+2a \end{bmatrix}.$$

The first return map on $J_2 = [\mathbf{v}^*,\mathbf{v}')$ is $T_{ab}^{(9)}$ and has symbol sequence $(++++-++++)$; its associated matrix M_2 is

$$M_2 = \frac{a^2-a-1}{a-1}M_1 + \frac{a^2(a-2)(a^2-2)^2}{(a-1)(a^2-a-1)}I.$$

The argument of Theorem 5.5 applies to show these two matrices commute. One checks that $|Tr(M_1)| > 2$ for $1 < a < \sqrt{2}$, hence the associated matrices are hyperbolic.

We show that the rotation number

$$r(T_{ab}) = \frac{1}{5}.$$

This follows because the subsector $J' = \mathbb{R}^+[(-1,0),(-1,-1))$ is mapped into itself by T_J and therefore contains an attracting periodic point; since it is in J_1 this point has periodic symbol sequence $(-+++-)$ and therefore has rotation number $1/5$.

Lemma 5.1 applies to show that on the sector J the iterates of a given point lie on arcs of a hyperbola, or on asymptotes. One can show that the interval J_1 contains two asymptotes of a hyperbola. Each of these defines a periodic point of order 5, one attracting and one repelling, defining the two periodic orbits of period 5 of S_{ab}. It follows that the first return map orbits of all points diverge in norm to $+\infty$ under forward iterations, except for those on the repelling fixed ray, which go to $+\infty$ under backwards iteration.

The points in J not on an asymptote under iteration of the first return map will approach an asymptote, which is an attracting fixed point of the first return map. For the map T_{ab} it is an attracting periodic orbit, and there are five copies of the asymptotes. The orbit of any point not on a periodic orbit of S_{ab} is contained in a finite union of arcs of hyperbolas, even though there are no invariant circles. Theorem 5.6 does not apply to bound the number of pieces, because the first return map on J has three pieces; however the number of pieces appears to be at most 23. ∎

References

[1] V. I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics,* W. A. Benjamin: New York 1968.

[2] S. Aubry and P. Y. Le Daeron, The discrete Frenkel-Kontorova model and its extensions I. Exact results for the ground state, Physica D **8** (1983), 381–422.

[3] A. Baker, *Transcendental Number Theory*, Cambridge Univ. Press, Cambridge 1975.

[4] J. Bellissard, Gap labelling theorems for Schrödinger operators, in: *From Number Theory to Physics*, (M. Waldschmidt, P. Moussa, J.-M. Luck, C. Itzykson, Eds.), Springer-Verlag: New York 1992, pp. 538–630.

[5] J. Bellissard, Le papillon de Hofstadter (d'apres B. Helffer et J. Sjöstrand), Seminar Bourbaki, Vol. 1991/92, Astérisque No. 206 (1992), Exp. No. 745.pp. 7–39.

[6] J. Bellissard, A. Bouvier, J.-M. Gher, Spectral properties of a tight-binding Hamiltonian with period-doubling potential, Comm. Math. Phys. **135** (1991), 379–399.

[7] P. Bougerol and J. Lacroix, *Products of Random Matrices with Applications to Schrödinger Operators,* Birkhäuser: Boston, 1985. (see p. 187ff)

[8] M. Brown, Problem 6349, Amer. Math. Monthly **90** (1983), 569. [Solution, ibid **92** (1985), 218–219.]

[9] P. Collet and Y. Levy, Ergodic properties of the Lozi mappings, Comm. Math. Phys. **93** (1984), 461–481.

[10] F. Delyon and D. Petritis, Absences of localization in a class of Schrödinger operators with quasiperiodic potential, Comm. Math. Phys. **103** (1986), 441–444.

[11] K. Falconer, *The Geometry of Fractal Sets*, Cambridge Tracts in Math. 85, Cambridge University Press: Cambridge 1985.

[12] J. Fröhlich, T. Spencer, P. Wittwer, Localization for a class of one-dimensional Schrödinger operators, Comm. Math. Phys. **132** (1990), 5–25.

[13] B. Helffer, P. Kerdelhué, On the total bandwidth for the rational Harper's equation, Comm. Math. Phys. **173** (1995), 335–356.

[14] B. Helffer, P. Kerdelhué, and J. Sjöstrand, Le papillon de Hofstadter revisité, Mem. Soc. Math. France (N. S.) No. 43, 1990, 87pp.

[15] D. Hofstadter, Energy levels and wave functions of Bloch electrons in a rational or irrational magnetic field, Phys. Rev. B **14** (1976), 2239–2249.

[16] S. Kotani, Jacobi matrices with random potential taking finitely many values, Reviews in Math. Phys. **1** (1990), 129–133.

[17] J. C. Lagarias and E. Rains, Dynamics of a family of piecewise-linear area-preserving plane maps II. Cantor set spectra of associated difference operators, preprint.

[18] R. Lozi, Strange attractors: a class of mappings of $\mathbb{R}^2$ which leaves some Cantor sets invariant, in: *Intrinsic stochasticity in plasmas (Internat. Workshop, Inst. Etudes Sci. Cargèse, Cargèse 1979*, Ecole Polytech., Palaiseau, 1979, pp. 373–381.

[19] M. Misiurewicz, Strange attractors for the Lozi mappings, in: *Nonlinear dynamics, New York 1979*, Ann. New York Acad. Sci. **357**, New York Acad. Sci. 1980, pp. 348–358.

[20] Z. Nitecki, *Differentiable Dynamics: An Introduction to the Orbit Structure of Diffeomorphisms,* MIT Press: Cambridge, MA 1971.

[21] L. Pastur and A. Figotin, *Spectra of random and almost-periodic operators*, Grund. Math. Wiss. 297, Springer-Verlag: Berlin 1992.

[22] J. Sjóstrand, Microlocal analysis for the periodic magnetic Schrödinger equation and related questions, pp. 237–332 in: *Microlocal analysis and applications (Motecatini Terme, 1989),* Lecture Notes in Math. 1495, Springer-Verlag, Berlin 1991.

[23] T. Spencer, Random and Quasiperiodic Schrödinger Operators, Proc. Int. Congress Math., Vol II (Berkeley 1986), pp. 1312–1318.

[24] B. Sutherland and M. Kohmoto, Resistance of a one-dimensional quasicrystal: Power-law growth, Phys. Rev. B **36** (1987), 5877–5886.

[25] A. Sütö, Schrödinger difference equation with deterministic ergodic potentials, in: *Beyond Quasicrystals (Les Houches 1994)*, Springer-Verlag: New York 1995, pp. 481–549.

[26] Yi Wan and C. M. Soukoulis, One-dimensional nonlinear Schrödinger equation: a nonlinear dynamics approach, Phys. Rev. A **41** (1990), 800–809.