The Lap Counting Function and Zeta Function of the Tent Map

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ABSTRACT

The tent map $t_\beta : [0,1] \to [0,1]$ with parameter $1 < \beta \le 2$ is defined by

$$t_\beta(x) = \begin{cases} \beta x & 0 \le x \le \frac{1}{2} \ , \\ \beta(1-x) & \frac{1}{2} \le x \le 1 \ . \end{cases}$$

This paper derives formulae for its dynamical zeta function and lap counting function, which exhibits the renormalization structure of such maps. It relates these functions to the centrally symmetric linear mod one transformation

$$f_\beta(x) = \beta x + 1 - \frac{\beta}{2} \pmod{1} \ .$$

The singularities of these functions on the circle $|z| = \frac{1}{\beta}$ are explicitly determined.

1. Introduction

The *symmetric tent map* $t_\beta : [0,1] \to [0,1]$ with parameter $1 < \beta \le 2$ is defined by

$$t_\beta(x) := \begin{cases} \beta x & 0 \le x \le \frac{1}{2} \ , \\ \beta(1-x) & \frac{1}{2} \le x \le 1 \ . \end{cases} \tag{1.1}$$

The dynamics under iteration of the tent map have been extensively studied as one of the simplest examples of hyperbolic discrete dynamical systems. Its name comes from the tent-like appearance of its graph, see Figure 1.

In Lorenz's 1963 paper on deterministic aperiodic flow, which introduced the "Lorenz attractor", Lorenz computed a statistic of orbits on the "Lorenz attractor" that appeared to be described by iterating a tent-like map, see [23]. Later Parry [28] exhibited a symmetric tent map as a factor map in the dynamics of a simplified model of the Lorenz attractor. He observed that iteration of $t_\beta(x)$ exhibited a "renormalization" behavior depending on the value of k such that

$$2^{2^{-k-1}} \le \beta \le 2^{2^{-k}} \ . \tag{1.2}$$

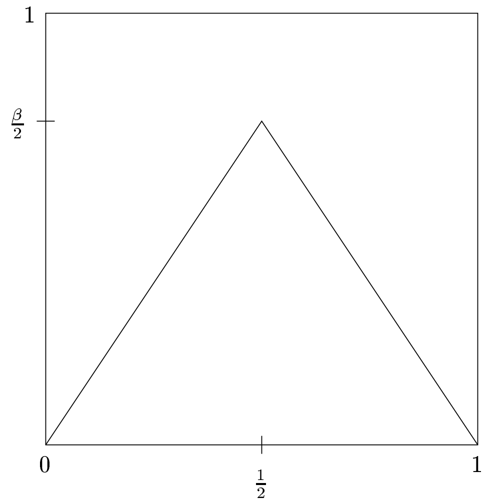

Figure 1: Tent Map

The tent map also arises in study of the *quadratic map* (or *logistic map*)

$$q_a(x) := ax(1-x) \ . \tag{1.3}$$

For the range of parameters $0 \le a \le 4$, the interval $[0,1]$ is an invariant set. Milnor and Thurston [24, Theorem 7.4] showed that the map $q_a(x)$ is *topologically semiconjugate* to some tent map $t_{h(a)}(x)$. That is, there is a continuous nondecreasing function $\varphi_a : [0,1] \to [0,1]$ with $f_a(0) = 0$, $f_a(1) = 1$ such that

$$\varphi_a \circ q_a(x) = t_{h(a)} \circ \varphi_a(x) \ . \tag{1.4}$$

Here φ_a is generally not invertible: it may have "flat spots." The map $t_{h(a)}$ essentially captures the dynamics of $q_a(x)$ on the subset Σ_a of $[0,1]$ where it is "expanding," this set Σ_a is the intersection of the Julia set $J(q_a)$ in $\mathbb{C}$ with the interval $[0,1]$, and is generally a Cantor-like set, sometimes of positive Lebesgue measure. The value $\log h(a)$ is the topological entropy of the map q_a on $[0,1]$. In the extreme case $a = 4$, we have $\Sigma_a = [0,1]$ and the map q_a is topologically conjugate and even smoothly conjugate to $t_2(x)$. The function $h(a) = 0$ for $0 \le a \le c_0$, with $c_0 = 3.5699...$ and only for $c_0 < a \le 4$ does the Julia set $J(q_a)$ intersect $[0,1]$, see [24, pp. 471 and 546]. The exact behavior of the function $h(a)$ is apparently quite complicated, and it is not completely understood, see [7], [24, Section 13].

Two of the simplest dynamical invariants associated to $t_\beta(x)$ are its lap counting function and zeta function. The *lap counting function* of a continuous function $f : [0,1] \to [0,1]$ whose graph consists of a finite number of monotone segments ("laps") is the formal power series

$$L_f(z) := \sum_{k=0}^{\infty} L_k(f) z^k \ , \tag{1.5}$$

in which $L_0(f) = 1$ and $L_k(f)$ counts the number of laps of the iterate f^k. This function was introduced by Milnor and Thurston [24]. The (dynamical) *zeta function* of f is the formal power series

$$\zeta_f(z) := \exp\left(\sum_{k=1}^{\infty} \frac{P_k(f)}{k} z^k\right) \ , \tag{1.6}$$

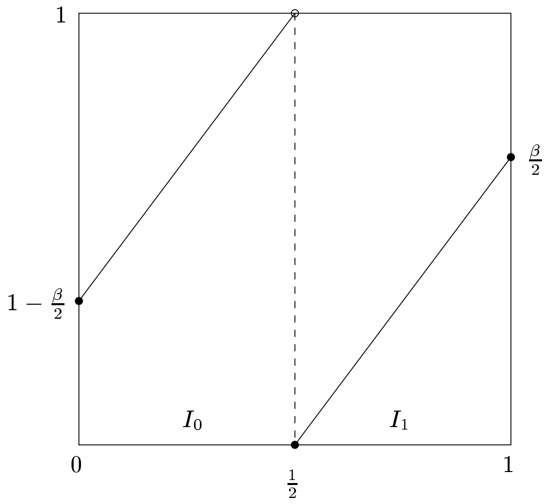

Figure 2: Symmetric Linear Mod One Transformation

in which $P_k(f)$ counts the number of fixed points of f^k. That is, $P_k(f)$ counts the number of periodic points of f with a period dividing k. This function was introduced by Artin and Mazur [1], with variations and extensions in Smale [30], Williams [32] and Ruelle [29]. The sequences $L_k(f)$ and $P_k(f)$ both have at most exponential growth in k under suitable smoothness conditions on f, in which case these expressions will converge to analytic functions in some open neighborhood of $z = 0$.

This paper derives explicit formulas for the lap-counting function and zeta function of tent maps t_β. For these purpose we introduce an auxiliary function, the (centrally) *symmetric linear mod one transformation*

$$f_\beta(x) = \beta x + 1 - \frac{\beta}{2} \pmod{\text{one}} . \tag{1.7}$$

This map is piecewise linear but discontinuous, and is pictured in Figure 2. Its name is justified by the appearance of its graph, which is symmetric around $x = 1/2$, in the sense that

$$f_\beta(1-x) = 1 - f_\beta(x) . \tag{1.8}$$

except at the discontinuity points of f_β. We now set

$$\Phi_\beta(z) := 1 + \sum_{n=1}^{\infty} [f_\beta^n(1^-) - f_\beta^n(0)] z^n , \tag{1.9}$$

in which

$$f_\beta^n(1^-) = \lim_{x \nearrow 1} f_\beta^n(x).$$

This function is clearly analytic for $|z| < 1$ since its power series coefficients are bounded, and $\Phi_\beta(0) = 1$. The relation (1.8) implies[1] that

$$f_\beta^n(1^-) = 1 - f_\beta^n(0) , \tag{1.10}$$

[1]In general $f_\beta^n(1^-) = f_\beta^n(1)$ unless some iterate $f_\beta^k(1) = \frac{1}{2}$. In this case let k_0 be the smallest such k. Then $f_\beta^n(1^-) = f_\beta^n(1)$ for $n < k_0$, while $f_\beta^n(1^-) = f_\beta^{n-k_0}(0) = 1 - f_\beta^n(1)$ for all $n \geq k_0$.

hence

$$\Phi_\beta(z) = \frac{1}{1-z} - 2\sum_{n=0}^{\infty} f_\beta^n(0) z^n \ . \tag{1.11}$$

It is relatively easy to prove that $\Phi_\beta(z)$ is nonzero in the open disk $|z| < \frac{1}{\beta}$, see [12, Theorem 3.3]. Its singularities on the circle $|z| = \frac{1}{\beta}$ are closely related to the dynamics of $f_\beta(x)$.

Theorem 1.1. *The lap counting function L_{t_β} of the tent map t_β for $1 < \beta \le 2$ is meromorphic in the open unit disk $|z| < 1$. If $2^{2^{-k-1}} < \beta \le 2^{2^{-k}}$ and $\beta^* = \beta^{2^k}$, then*

$$L_{t_\beta}(z) = \frac{1}{1-(\beta z)^{2^k}} \frac{z\prod_{j=0}^{k-1}(1+z^{2^j})}{(1-z)\prod_{j=0}^{k}(1-z^{2^j})\Phi_{\beta^*}(z^{2^k})} + \frac{1}{1-z} \ , \tag{1.12}$$

for $|z| < 1$. The function $\Phi_{\beta^*}(z^{2^k})$ is nonzero in the closed disk $|z| \le \frac{1}{\beta}$.

The formula explicitly exhibits a "renormalization" phenomenon well-known for the tent map, which was observed as a self-similar structure in its symbolic dyamics in Derrida et al [9] in 1978. It also shows that for $2^{2^{-k-1}} < \beta \le 2^{2^{-k}}$ the closest singularities to the origin of $L_{t_\beta}(z)$ lie at radius $\frac{1}{\beta}$ and consist of simple poles at the 2^k points

$$\left\{ \frac{1}{\beta} \exp\left(\frac{2\pi i m}{2^k}\right) : 1 \le m \le 2^k \right\} \ . \tag{1.13}$$

The fact that $\Phi_{\beta^*}(z^{2^k})$ is nonzero in the closed disk $|z| \le \frac{1}{\beta}$ is a consequence of the work in [14, Theorem 1.1] for the lap counting function of linear mod one transformations.

Theorem 1.2. *The zeta function $\zeta_{t_\beta}(z)$ of the tent map t_β for $1 < \beta \le 2$ is meromorphic in the open unit disk $|z| < 1$. Let p be the smallest positive integer such that $t_\beta^p\left(\frac{1}{2}\right) = \frac{1}{2}$ and set $p = \infty$ if no such integer exists. If $2^{2^{-k-1}} < \beta \le 2^{2^{-k}}$ and $\beta^* = \beta^{2^k}$, then*

$$\zeta_{t_\beta}(z) = \frac{1}{1-(\beta z)^{2^k}} \frac{\prod_{j=0}^{k-1}(1+z^{2^j})}{(1-z^p)\prod_{j=0}^{k}(1-z^{2^j})\Phi_{\beta^*}(z^{2^k})} \ , \tag{1.14}$$

for $|z| < 1$, using the convention that $z^p \equiv 0$ if $p = \infty$.

This formula shows that in the open unit disk $|z| < 1$ the function $\zeta_{t_\beta}(z)$ has its poles at exactly the same locations as $L_{t_\beta}(z)$, a fact which already follows from results of Milnor and Thurston detailed in §2.

The lap-counting function and zeta function of the tent map are either both rational functions, or else both have the unit circle as a natural boundary to analytic continuation. This follows from a similar dichotomy for linear mod one transformations given in [12, Theorem 3.1], which carries over through the function $\Phi_{\beta^*}(z)$. The result for linear mod one transformations also implies that these functions are rational functions if and only if the iterates $\{f_\beta^n(0)\}$ of the symmetric linear mod one transformation are eventually periodic, using (1.11). In §2 we observe that this is equivalent to the condition that the iterates $\{t_\beta^n(\frac{1}{2}) : n = 1, 2, ...\}$ of the turning point of the tent map are eventually periodic.

There is a relation between the singularities of the lap counting function (or, equivalently, the zeta function) of t_β on the circle $|z| = \frac{1}{\beta}$ and the dynamics of iterating t_β. It is known that t_β has a unique absolutely continuous invariant measure $d\mu_\beta$ on $[0,1]$, see Li and Yorke [22].

The location of the poles of $\zeta_{t_\beta}(z)$ on the circle $|z| = \frac{1}{\beta}$ correspond exactly to the complete set of eigenvalues of the *Koopman operator* U on $L^2([0,1], d\mu_\beta)$ given by

$$U\phi(x) = \phi(t_\beta(x)), \quad \phi \in L^2([0,1], d\mu_\beta) \ . \tag{1.15}$$

This follows from similar facts about f_β, see [12, p. 456 bottom]. This operator is the formal adjoint of the *Frobenius-Perron operator* or *transfer operator*

$$\mathcal{F}\phi(x) = \sum_{t_\beta(y)=x} \frac{\phi(y)}{|t'_\beta(y)|} \ , \tag{1.16}$$

acting on a space of smooth functions contained in $L^2([0,1], d\mu)$. The zeta function can be related to the Fredholm determinant $\det(I - z\mathcal{F})$ of this operator on a suitable function space, see Hofbauer and Keller [17]. Recently Hasegawa and Saphir [16] suggested a relation of other poles of such operators to "irreversibility" of the dynamics of such maps. For some general information on eigenvalues of Frobenius-Perron operators and Koopman operators, see Ding [10]. Eigenvalues of the Frobenius-Perron operator for tent maps were studied in Dörfle [11].

We prove Theorem 1.1 by relating both functions f_β and t_β to an auxiliary function g_β, given by

$$g_\beta(x) = \begin{cases} \beta x + 2 - \beta, & \text{for} \quad 0 \le x \le 1 - \frac{1}{\beta} \ , \\ -\beta x + \beta, & \text{for} \quad -\frac{1}{\beta} \le x \le 1 \ , \end{cases} \tag{1.17}$$

We call this function a *peak function* with parameter $1 < \beta \le 2$. Basic relations between the tent map t_β, peak function g_β and symmetric linear mod one transformation f_β are described in §3. The function g_β is a factor of f_β, while in turn g_β captures the non-wandering dynamics of t_β. These relations were first observed by Parry [28] in 1979. The relations between t_β, f_β, and g_β imply relations between their respective lap counting functions. We derive in Theorem 4.1 a functional equation relating the lap-counting function of the symmetric linear mod one transformation f_β to that of the peak function g_β. In Theorem 5.1 we give a functional equation relating the lap counting function of the tent map t_β to that of the peak function g_β. We are then able to apply the results of our previous analysis of the lap-counting functions of linear mod one transformations in Flatto and Lagarias [12]–[14]. These results include a renormalization formula

$$L_{f_\beta}(z) = \frac{1+z}{1-z} L_{f_{\beta^2}}(z^2), \quad \text{when} \quad 1 < \beta \le \sqrt{2} \tag{1.18}$$

which is a special case of [12, Theorem 4.2]. The analysis in [14] shows that $\Phi_\beta(z)$ has no zeros in $|z| \le \frac{1}{\beta}$ for $\sqrt{2} < \beta \le 2$.

There are a number of different approaches available to compute the zeta function $\zeta_{t_\beta}(z)$. There is a method of Milnor and Thurston [24] which expresses both the zeta function and lap counting functions in terms of a *kneading determinant* $D_\beta(z)$, which for unimodal maps of a kind including the tent map family is a power series whose coefficients are given in terms of the iterates of the turning point $x = \frac{1}{2}$ under iteration by t_β. This implies a functional relation between the lap counting function and the zeta function of the tent map, and we use it to show that Theorem 1.1 implies Theorem 1.2. In §6 we show that the kneading determinant is explicitly given by

$$D_\beta(z) = (1 - \beta z)\Phi_\beta(z) \ , \tag{1.19}$$

for $1 < \beta \le 2$. Another approach to computing $\zeta_{t_\beta}(z)$ can be based on the use of a transfer operator, see Hofbauer and Keller [17], Mori [26], and Baladi and Ruelle [2].

The formulas for $L_{t_\beta}(z)$ and $\zeta_{t_\beta}(z)$ in Theorem 1.1 and Theorem 1.2 make manifest the smallest singularities of $L_{t_\beta}(z)$ and $\zeta_{t_\beta}(z)$ on the circle $|z| = \frac{1}{\beta}$. These singularities do not seem to be readily apparent from the kneading determinant. In this respect, the formula (1.19) represents a main result of the paper, because we conclude using it that $\Phi_\beta(z) \neq 0$ for $z \leq \frac{1}{\beta}$ whenever $\sqrt{2} < \beta \leq 2$, by employing the Markov chain machinery for f_β developed in [12] - [14]. This approach to determining the singularities of the lap counting function on the circle $|z| = \frac{1}{\beta}$ seems quite roundabout, and one may ask whether it is possible to obtain this result directly from t_β, without considering f_β.

Many other properties of tent maps have been studied. The self-similar nature of the symbolic dynamics was analyzed in Derrida et al. [9]. The unique absolutely continuous invariant measure $d\mu_\beta$ of the tent map was explicitly determined by Ito et al [18, Theorem 1.1], along with properties of their periodic points. There has also been work showing that tent maps have the shadowing property for $\sqrt{2} < \beta \leq 2$,, see Coven et al [8, Theorem 4.2]. Various information has been obtained about semi-conjugacies of smooth maps to tent maps, see [31, Sect. 2.5] and [5]. The behavior of iterates of the turning point for "generic" tent maps has recently been studied in Brucks and Misiurewicz [4] and Bruin [6].

2. Kneading Determinants and Zeta Functions

This section is based on the results of Milnor and Thurston [24]. Let $h : [0,1] \to [0,1]$ be a continuous map of the interval such that $h(x)$ has exactly two monotone segments ("laps") with one turning point at $0 < c < 1$, and assume further that either $h(0) = h(1) = 0$, or $h(0) = h(1) = 1$. We call such a map a *pinned unimodal map*, to signify that its endpoints are pinned to the boundary, see Figure 3. Let $I_0(c)$ and $I_1(c)$ denote the open intervals $(0, c)$ and

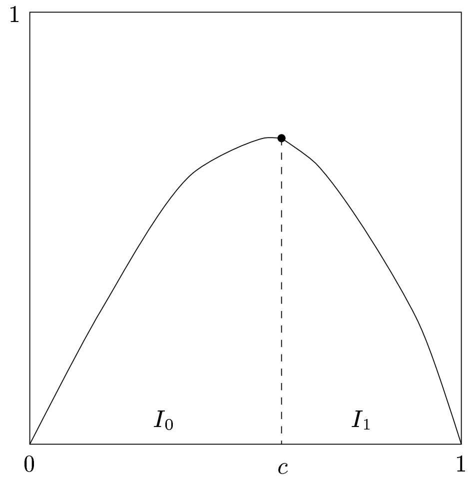

Figure 3: Pinned Unimodal Map h

$(c, 1)$, respectively, and set

$$\epsilon(x) := \begin{cases} 1 & \text{if} \quad x \in I_0 \\ -1 & \text{if} \quad x \in I_1 \ . \end{cases} \tag{2.1}$$

For $n \geq 1$ we define the *kneading sequences* of h by

$$\epsilon_n := \begin{cases} \epsilon(h^n(c)) & \text{if} \quad h^n(c) \neq c \ , \\ \epsilon_1 \epsilon_2 \cdots \epsilon_{n-1} & \text{if} \quad h^n(c) = c \ . \end{cases} \tag{2.2}$$

The graph of $h(x)$ with $h(0) = h(1) = 0$ is pictured schematically in Figure 3. We define

$$D_n := \epsilon_1 \epsilon_2 \cdots \epsilon_n \quad \text{for} \quad n \geq 1, \tag{2.3}$$

and set $D_0 = 1$. Then we define the *kneading determinant* by

$$D_h(z) := \sum_{n=0}^{\infty} D_n z^n = 1 + \sum_{n=1}^{\infty} \epsilon_1 \cdots \epsilon_n z^n \ . \tag{2.4}$$

Milnor and Thurston [24] give a different and more general definition of kneading determinant for continuous maps of the interval with finitely many turning points, but it agrees with the one above for pinned unimodal maps by [24, Lemma 4.5].

Proposition 2.1. (Milnor-Thurston) *For a pinned unimodal map h of the interval, the lap counting function $L_h(z)$ is given by*

$$L_h(z) = \frac{z}{(1-z)^2 D_h(z)} + \frac{1}{1-z} \ , \tag{2.5}$$

in which $D_h(z)$ is the kneading determinant of h.

Proof. This follows from Corollary 5.8 and 5.9 of [24]. ∎

The tent map t_β for $1 < \beta \leq 2$ is a pinned unimodal map. On comparing Proposition 2.1 with $h = t_\beta$ to Theorem 1.1, we see that for $\sqrt{2} < \beta \leq 2$ it gives

$$D_{t_\beta}(z) = (1 - \beta z)\Phi_\beta(z) \ .$$

In §6 we show that this relation holds for the full range $1 < \beta \leq 2$.

Milnor and Thurston [24, Section 9] introduced a *reduced zeta function* $\hat{\zeta}_\beta(z)$, which is closely related to the zeta function $\zeta_\beta(z)$. The reduced zeta function $\hat{\zeta}_h(z)$ differs from $\zeta_h(z)$ in only counting monotone equivalence classes of fixed points of h^n, e.g., it identifies all fixed points that have the same symbolic dynamics given by lap intervals.

Proposition 2.2. (Milnor-Thurston) *The reduced zeta function $\hat{\zeta}_h(z)$ of a pinned unimodal map is*

$$\hat{\zeta}_h(z) = \frac{1}{(1-z)D_h(z)} \tag{2.6}$$

if the sequence of coefficients of the power series $D_h(z)$ is not periodic. It is

$$\hat{\zeta}_h(z) = \frac{1}{(1-z^p)(1-z)D_h(z)} \ , \tag{2.7}$$

if this sequence is periodic with minimal period p.

Proof. This is Corollary 10.7 of [24]. ∎

We remark that the sequence of coefficients $\{D_n : n \geq 1\}$ is periodic if and only if some iterate $h^n(c) = c$, and if so then the minimal period p of $\{D_n : n \geq 1\}$ is the minimal period $n \geq 1$ for which $h^n(c) = c$. The proof of these facts is left to the reader.

A piecewise C^1-map is *strictly expanding* if there is a positive δ such that

$$|h'(x)| \geq 1 + \delta ,$$

for all $x \in [0,1]$ where $h'(x)$ is defined. For uniformly expanding maps, all fixed points lie in different monotone equivalence classes, hence

$$\hat{\zeta}_h(s) = \zeta_h(s) \tag{2.8}$$

holds for such functions h.

Proof of Theorem 1.2 assuming Theorem 1.1. For a pinned unimodal map Propositions 2.1 and 2.2 combine to give the formula

$$\hat{\zeta}_h(z) = \frac{1-z}{z(1-z^p)} \left(L_h(z) - \frac{1}{1-z} \right) , \tag{2.9}$$

in which $p \geq 1$ is the minimal value of p such that

$$h^p(c) = c , \tag{2.10}$$

with $p = +\infty$ if no such value exists, and with the convention that $z^\infty \equiv 0$. Now tent maps for $1 < \beta \leq 2$ are strictly expanding piecewise C^1 maps, so we have

$$\zeta_{t_\beta}(z) = \hat{\zeta}_{t_\beta}(z). \tag{2.11}$$

Combining (2.9) with $h = t_\beta$ and (2.11), and substituting the result into Theorem 1.1 yields Theorem 1.2. ∎

3. Peak Function and Symbolic Dynamics

The *peak function* g_β is defined by

$$g_\beta(x) := \begin{cases} \beta x + 2 - \beta & \text{for} \quad 0 \leq x \leq 1 - \frac{1}{\beta} , \\ -\beta x + \beta & \text{for} \quad 1 - \frac{1}{\beta} \leq x \leq 1 . \end{cases} \tag{3.1}$$

It is pictured in Figure 4; its name describes its graph.

To prove Theorem 1.1 we establish a functional relation between the lap counting function of the tent map t_β to that of the symmetric linear mod one transformation f_β. This is accomplished by relating both of these to the lap counting function of the peak function. All three functions are piecewise linear with pieces of slope $\pm\beta$.

In this section we relate t_β and f_β to g_β, and give some basic facts on the standard symbolic dynamics of iterating these maps.

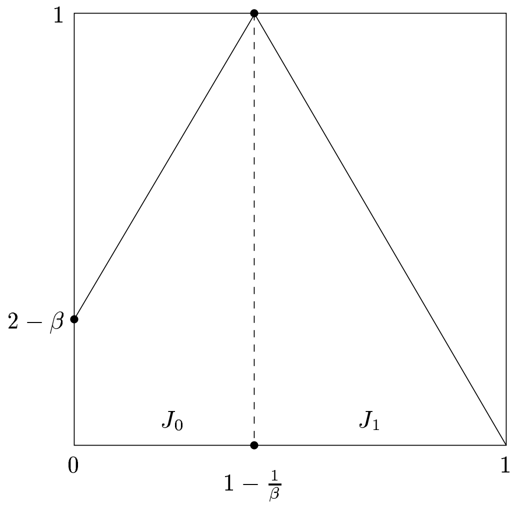

Figure 4: Peak Function g_β

3.1. Relation of f_β and g_β

Consider the map

$$\psi(x) = \begin{cases} 2x & \text{for} \quad 0 \le x \le 1/2 \ , \\ 2(1-x) & \text{for} \quad 1/2 \le x \le 1 \ . \end{cases} \tag{3.2}$$

A computation gives the semiconjugacy

$$\psi \circ f_\beta = g_\beta \circ \psi \ , \tag{3.3}$$

which establishes that the peak function g_β is a factor of f_β.

3.2. Relation of t_β and g_β

The closed interval $J = [\beta(1-\frac{\beta}{2}), \frac{\beta}{2}]$ contains the turning point $\frac{1}{2}$ of t_β in its interior, and t_β maps J into J, see Figure 5.

The restriction $t_\beta\Big|_J$ of t_β on J is conjugate to g_β by the linear rescaling

$$\phi(x) = \frac{x - \beta(1-\frac{\beta}{2})}{\frac{\beta(\beta-1)}{2}} \tag{3.4}$$

which maps J onto $I = [0,1]$, thus

$$g_\beta = \phi \circ t_\beta\Big|_J \circ \phi^{-1} \ . \tag{3.5}$$

The region J contains essentially all the non-wandering dynamics of t_β. That is, we show that, aside from the endpoints 0 and 1, the orbit of any point $\{f_\beta^n(x) : n \ge 0\}$ eventually enters and stays in J. Set

$$\beta_k = \beta^{1-k}\left(1 - \frac{\beta}{2}\right) \quad \text{for} \quad k \ge -1 \ , \tag{3.6}$$

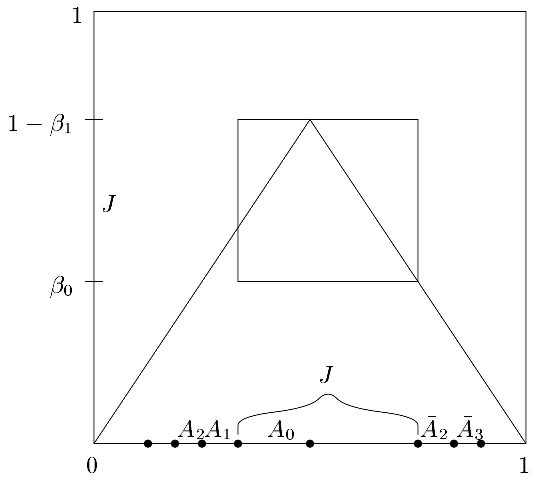

Figure 5: Invariant Subinterval $J = [\beta_0, 1-\beta_1]$ of Tent Map

and note that $J = [\beta_0, 1-\beta_1]$. Now define the intervals

$$A_k = [\beta_k, \beta_{k-1}],\ k \geq 0, \quad \text{and} \quad \bar{A}_k = [1-\beta_{k-1}, 1-\beta_k],\ k \geq 2. \tag{3.7}$$

They are depicted in Figure 5. The intervals $\{A_k\}_{k\geq 1}$ and $\{\bar{A}_k\}_{k\geq 2}$ together with J have mutually disjoint interiors and cover the open unit interval. The map t_β is a homeomorphism from A_k onto A_{k-1} for $k \geq 1$, and is a homeomorphism from $\bar{A}_k$ onto A_{k-1}, for $k \geq 2$. Furthermore $A_0 \subset J$. From these facts, we conclude that the orbit of any point $x \neq 0,\ 1$ eventually enters and stays in J.

3.3. Symbolic Dynamics

We attach a symbolic dynamics to f_β and g_β by applying labels 0, 1 to the two subintervals on which each of the maps is linear. This symbolic dynamics allows us to define lap numbers for the discontinuous map f_β, as was done in [12]–[14].

For f_β we use the open intervals $I_0 = (0, \frac{1}{2})$ and $I_1 = (\frac{1}{2}, 1)$ to define a symbolic dynamics. For any finite sequence $(a_0, a_1, ..., a_{n-1})$ of zeros and ones we set

$$I_{a_0a_1\ldots a_{n-1}} = \{0 < x < 1 : f_\beta^k \in I_{a_k},\quad 0 \leq k \leq n-1\},$$

and call $(a_0, a_1, ..., a_{n-1})$ *f-admissible* if $I_{a_0a_1\ldots a_{n-1}} \neq \varnothing$.

Lemma 3.1. *For $(a_0, \ldots a_{n-1})$ f-admissible, the set $I_{a_0a_1\ldots a_{n-1}}$ is a nonempty open interval. These open intervals are called n-th stage intervals for f_β and are denoted generically by $I^{(n)}$. The $I^{(n)}$'s are disjoint and $I = [0,1]$ is the union of their closures.*

Proof: Straightforward induction on n. ∎

For linear mod one transformations we define lap numbers using symbolic dynamics. The *lap number* $L_n(f_\beta)$ is the number of f-admissible sequences of length n, i.e. the number of n-th stage subintervals for f_β. This number generally agrees with the number of monotone linear segments in the graph of f_β^n, but for certain special β it can sometimes be larger, when two segments in adjoining $I^{(n)}$ "glue together." (An example of this phenomenon is given in

Figure 1b of [12].) Thus our definition sometimes differs from the geometric definition based on monotone pieces in the graph of f_β.

We have a similar symbolic dynamics describing iterates of the peak function g_β. We use the subintervals $J_0 = (0, \frac{\beta-1}{\beta})$ and $J_1 = (\frac{\beta-1}{\beta}, 1)$. For any finite sequence $(a_0, a_1, \ldots, a_n)$ of 0's and 1's we let $J_{a_0, a_1 \ldots a_{n-1}} = \{0 < x < 1; g^k(x) \in J_{a_k}, 0 \le k \le n-1\}$. We call $(a_0, a_1, \ldots, a_{n-1})$ *g-admissible* if $J_{a_0 a_1 \ldots a_{n-1}} \neq \phi$.

Lemma 3.2. *(i) For each $(a_0, a_1, \ldots, a_{n-1})$ that is g-admissible, the set $J_{a_0 a_1 \ldots a_{n-1}}$ is a nonempty open interval. These open intervals are called n-th stage intervals for g_β and are denoted generically by $J^{(n)}$. The $J^{(n)}$'s are disjoint and $I = [0,1]$ is the union of their closure.*

(ii) The restriction of g_β^n to any n-th stage subinterval $J^{(n)}$ is linear of slope $\pm\beta^n$. In two contiguous $J^{(n)}$'s, the slope of g_β^n has opposite signs.

Proof: Straightforward induction on n. ∎

In view of Lemma 3.2, we can interpret the lap number $L_n(g_\beta)$ both as the number of monotone pieces of g_β^n ("lap number" in the Milnor-Thurston sense) or as the number of g-admissible sequences of length n (symbolic "lap number").

4. Linear Mod One Transformation to Peak Function

For a symmetric linear mod one transformation we define the *lap counting function* by

$$L_{f_\beta}(z) := \sum_{n=0}^{\infty} L_n(f_\beta) z^n \ , \tag{4.1}$$

in which $L_n(f_\beta)$ are the lap numbers defined by the symbolic dynamics in §3.3, and $L_0(f_\beta) = 1$. We relate the lap-counting functions of f_β and g_β, as follows.

Theorem 4.1. *For $1 < \beta \le 2$, the lap-counting functions of the centrally symmetric linear mod one transformation f_β and the peak function g_β are related by*

$$L_{f_\beta}(z) = 1 + 2z L_{g_\beta}(z) \ . \tag{4.2}$$

Proof: We regard β as fixed and write f for f_β and g for g_β in what follows. Let $\mathcal{L}_n(f)$ denote the set of f-admissible sequences of length n, whose cardinality is $L_n(f)$. Similarly let $\mathcal{L}_n(g)$ denote the g-admissible sequences of length n, whose number is $L_n(g)$. We will show that

$$L_{n+1}(f) = 2L_n(g), \quad \text{for} \quad n \ge 0 \ . \tag{4.3}$$

Assuming this is done, multiplying (4.3) by z^n and summing over n yields (4.2).

To prove (4.3), we define a surjective map

$$\tau : \mathcal{L}_{n+1}(f) \to \mathcal{L}_n(g) \ , \tag{4.4}$$

and prove that it is two-to-one. To define τ we use the two-to-one map ψ given in (3.2). This map satisfies

$$\psi(I_{a_0 a_1}) = J_{a_0 + a_1} \quad \text{for} \quad a_0, a_1 \in \{0, 1\} \tag{4.5}$$

in which $a_0 + a_1$ is interpreted (mod 2). For $(a_0, \ldots, a_n) \in \mathcal{L}_{n+1}(f)$ we define $\tau(a_0, \ldots, a_n) = (b_0, b_1, \ldots, b_{n-1})$ by $b_i \in \{0, 1\}$ with

$$b_i \equiv a_i + a_{i+1} (\text{mod } 2) \quad \text{for} \quad 0 \le i \le n \; . \tag{4.6}$$

To prove that τ has the desired properties it suffices to show that

(i) $(b_0, b_1, \ldots, b_{n-1}) \in \mathcal{L}_n(g)$

(ii) For each $(b_0, \ldots, b_{n-1}) \in \mathcal{L}_n(g)$ there are exactly two $(a_0, \ldots, a_n) \in \mathcal{L}_{n+1}(f)$ with it as image under τ.

To prove (i), let $x \in I_{a_0 a_1 \cdots a_n}$ and set $y \in \psi(x)$. Then $f^k(x) \in I_{a_k}$, $f^{k+1}(x) \in I_{a_{k+1}}$, so that $f^k(x) \in I_{a_k a_{k+1}}$ for $0 \le k \le n-1$. Then (4.5) gives

$$g^k(y) = g^k \psi(x) = \psi f^k(x) \in \psi(I_{a_k a_{k+1}}) = J_{b_k} \quad \text{for} \quad 0 \le k \le n-1 \; . \tag{4.7}$$

Thus $(b_0, b_1, \ldots, b_{n-1}) \in \mathcal{L}_n(g)$.

To prove (ii), note that (4.6) shows that $(a_1, \ldots, a_n)$ are determined by a_0 and $(b_0, b_1, \ldots, b_{n-1})$. Since $a_0 = 0$ or 1, we conclude that there are at most two $(a_0, a_1, \ldots, a_n) \in \mathcal{L}_{n+1}(f)$ for which

$$\tau(a_0, a_1, \ldots, a_n) = (b_0, b_1, \ldots, b_{n-1}) \; . \tag{4.8}$$

To exhibit two such, suppose that $y \in J_{b_0 b_1 \cdots b_{n-1}}$. Then $y = \psi(x_0) = \psi(x_1)$ for some $x_0 \in I_0$ and $x_1 \in I_1$. We can choose y so that x_0 and x_1 are both in n-th stage subintervals for f, say $x_0 \in I_{a_0 a_1 \cdots a_n}$ and $x_1 \in I_{\bar{a}_0 \bar{a}_1 \cdots \bar{a}_n}$, by avoiding a finite set of bad points. By (i) we have

$$\tau(a_0, a_1, \ldots, a_n) = \tau(\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_n) = (b_0, b_1, \ldots, b_{n-1}) \; .$$

Since $a_0 = 0$ and $\bar{a}_0 = 1$ the sequences $(a_0, a_1, \ldots, a_n)$ and $(\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_n)$ are distinct, which proves (ii). ∎

5. Tent Map to Peak Function

We relate the lap-counting functions of t_β, and g_β, as follows.

Theorem 5.1. *For $1 < \beta \le 2$, the lap-counting functions of the tent map t_β and the peak function g_β are related by*

$$L_{t_\beta}(z) = \frac{2z^2}{1-z} L_{g_\beta}(z) + \frac{1+z}{1-z} \; . \tag{5.1}$$

We treat β as fixed and write $f = f_\beta$ and $g = g_\beta$ in what follows. We use a detailed analysis of the iteration of t_β on the intervals A_k and $\bar{A}_k$ introduced in §3.2. We define refined lap numbers $L_n(t, A_k)$ and $L_n(t, \bar{A}_k)$ and prove a set of preliminary lemmas about their properties.

For A, a closed subinterval of $I = [0, 1]$, we let $L_n(g, A)$ denote the number of monotone pieces of the graph of g^n that occur for $x \in A$. In particular $L_n(g, I) = L_n(g)$. We define $L_n(t, A)$ similarly.

For any $0 \le x \le 1$, we set

$$ord_g(x) := \begin{cases} \min\{n \ge 0 \quad \text{and} \quad g^n(x) = \frac{\beta-1}{\beta}\} \; , \\ \infty, \qquad \text{if no such } n \text{ exists} \; . \end{cases} \tag{5.2}$$

We define similarly

$$ord_t(x) := \begin{cases} \min\{n : n \geq 0 \quad \text{and} \quad t^n(x) = \frac{1}{2}\} \\ \infty, \quad \text{if no such } n \text{ exists} . \end{cases} \tag{5.3}$$

For all but a countable number of points, $ord_g(x) = \infty$ and $ord_t(x) = \infty$.

Lemma 5.1. *For $0 < x < 1$ and $n \geq 1$, g^n has a continuous derivative at x for all $n \leq ord_g(x)$ and a discontinuous derivative at x for all $n > ord_g x$. A similar statement holds for t^n.*

Proof: An easy induction on n. ∎

This lemma can be rephrased geometrically. The graph of g^n consists of line segments of slope $\pm\beta^n$. The point $(x, g^n(x))$ is interior to one of these segments precisely when $n \leq ord_g(x)$.

We now define the characteristic function

$$\chi[n \leq m] = \begin{cases} 1 & \text{if} \quad n \leq m \ , \\ 0 & \text{if} \quad n > m \ . \end{cases} \tag{5.4}$$

Lemma 5.2. *If $0 = w_0 < w_1 < \cdots < w_{r+1} = 1$ and $A_i = [w_i, w_{i+1}]$ for $0 \leq i \leq r$, then*

$$L_n(g) = \sum_{i=0}^{r} L_n(g, A_i) - \sum_{i=1}^{r} \chi[n \leq ord_g(w_i)] \ . \tag{5.5}$$

A similar formula holds for $L_n(t)$.

Proof: The graph of g^n consists of line segments of slope $\pm\beta^n$, and $L_n(g)$ counts the number of these line segments , which we call *total line segments.* Now $L_n(g, A_i)$ counts line segments of g^n over the domain A_i, and all of these are total except possibly the rightmost and leftmost ones. The rightmost segment of A_{i-1} and the leftmost segment of A_i form a single total line segment precisely when $(w_i, f^n(w_i))$ is interior to some $I^{(n)}$. Lemma 5.1 states that this happens exactly when $n \leq ord_g(w_i)$. This justifies the formula (5.5). The proof for $L_n(t)$ is similar. ∎

We now use the intervals $A_k = [\beta_k, \beta_{k-1}]$ for $k \geq 0$ and $\bar{A}_k = [1 - \beta_{k-1}, 1 - \beta_k]$ for $k \geq 2$, in which $\beta_k = \beta^{1-k}(1 - \frac{\beta}{2})$ for $k \geq -1$, which were introduced in Sect. 3.2, see Figure 5.

Lemma 5.3. *For $n \geq 2$,*

$$\begin{aligned} L_n(t) \quad &= \quad L_n(g) + \sum_{k=1}^{n} L_n(t, A_k) + \sum_{k=2}^{n} L_n(t, \bar{A}_k) \\ &- \quad \sum_{k=0}^{n-1} \chi[n \leq ord_t(\beta_k)] - \sum_{k=1}^{n} \chi[n \leq ord_t(1 - \beta_k)] \ . \end{aligned} \tag{5.6}$$

Proof: Set $A'_n = [0, \beta_{n-1}]$ and $\bar{A}'_n = [1 - \beta_{n-1}, 1]$. The points β_{n-1}, $\beta_{n-2}, \ldots, \beta_1, \beta_0$ and $1 - \beta_1$, $1 - \beta_2, \ldots, 1 - \beta_{n-1}$ partition $[0, 1]$ from left to right into the $2n$ subintervals A'_n, A_{n-1}, $A_{n-2}, \ldots, A_1$, J, $\bar{A}_2$, $\bar{A}_3, \ldots, \bar{A}_{n-1}$, $\bar{A}'_n$. From Lemma 5.2 applied to t with these points we get

$$\begin{aligned} L_n(t) \quad &= \quad L_n(t, J) + L_n(t, A'_n) + L_n(t, \bar{A}'_n) \\ &+ \quad \sum_{k=1}^{n-1} L_n(t, A_k) + \sum_{k=2}^{n-1} L_n(t, \bar{A}_k) \\ &- \quad \sum_{k=0}^{n-1} \chi[n \leq ord_t(\beta_k)] - \sum_{k=1}^{n-1} \chi[n \leq ord_t(1 - \beta_k)] \ . \end{aligned} \tag{5.7}$$

The linear rescaling of $t\Big|_J$ to g given by (3.5) shows that

$$L_n(g) = L_n(t, J) \ . \tag{5.8}$$

The map t^n sends A'_n and $\bar{A}'_n$ homeomorphically to $[0, \beta_{-1}]$. Since $A_n \subseteq A'_n$ and $\bar{A}_n \subseteq \bar{A}'_n$, we conclude that

$$L_n(t, A'_n) = L_n(t, A_n) = 1, \tag{5.9}$$

and

$$L_n(t, \bar{A}'_n) = L_n(t, \bar{A}_n) = 1 \ . \tag{5.10}$$

Inserting (5.8)–(5.10) into (5.7) gives the formula (5.6). ∎

Lemma 5.4. *Let $B = [0, g(0)] = [0, 2-\beta]$. Then for $n \geq 0$,*

$$L_n(g, B) = 2L_n(g) - L_{n+1}(g) + \chi[n+1 \leq ord_g(0)] \ . \tag{5.11}$$

Proof: Set $\bar{B} = [g(0), 1] = [2-\beta, 1]$. By Lemma 5.2 applied to g with $r = 1$ and $w_1 = g(0)$, we have, for $n \geq 0$,

$$L_n(g) = L_n(g, B) + L_n(g, \bar{B}) - \chi(n \leq ord_g(g(0))) \ .$$

As $ord_g g(0) = ord_g(0) - 1$, we may rewrite this as

$$L_n(g) = L_n(g, B) + L_n(g, \bar{B}) - \chi(n+1 \leq ord_g(0)) \ . \tag{5.12}$$

Again by Lemma 5.2 we obtain for $n \geq 0$

$$L_{n+1}(g) = L_{n+1}(g, J_0) + L_{n+1}(g, J_1) \tag{5.13}$$

in which $J_0 = [0, \frac{\beta-1}{\beta}]$, $J_1 = [\frac{\beta-1}{\beta}, 1]$ and $ord_g(\frac{\beta-1}{\beta}) = 0$ so there is no third term on the right side of (5.13).

Now g is a homeomorphism of J_0 onto $\bar{B}$ and a homeomorphism from J_1 onto $I = [0,1]$, hence

$$\begin{aligned} L_{n+1}(g, J_0) &= L_n(g, \bar{B}) \ , \\ L_{n+1}(g, J_1) &= L_n(g) \ . \end{aligned}$$

These last two relations and (5.13) give

$$L_{n+1}(g) = L_n(g) + L_n(g, \bar{B}) \ . \tag{5.14}$$

Eliminating $L_n(g, \bar{B})$ from (5.12) and (5.14) yields the desired formula (5.11). ∎

Lemma 5.5. *For $n \geq 2$ and $0 \leq k \leq n$,*

$$L_n(t, A_k) = 2L_{n-k}(g) - L_{n-k+1}(g) + \chi[n-k+1 \leq ord_t(\beta_j)] \ . \tag{5.15}$$

For $n \geq 2$ and $2 \leq k \leq n$,

$$L_n(t, \bar{A}_k) = 2L_{n-k}(g) - L_{n-k+1}(g) + \chi[n-k+1 \leq ord_t(\beta_0)] \ . \tag{5.16}$$

Proof: Recall that $\beta_0 = \beta(1-\frac{\beta}{2})$ is the left endpoint of J and of $A_0 \subseteq J$. We first consider A_0. The conjugacy $\phi : I \to [0,1]$ in (3.5) has

$$\phi \circ t^n\Big|_J = g^n \circ \phi \tag{5.17}$$

and $\phi(A_0) = B$. These facts, together with Lemma 5.4, give

$$L_n(t, A_0) = L_n(g, B) = 2L_n(g) - L_{n+1}(g) - \chi[n+1 \le ord_g(0)] \ . \tag{5.18}$$

Also from (5.17) and $\phi(\beta_0) = 0$, $\phi(\frac{1}{2}) = \frac{\beta-1}{\beta}$ we obtain

$$ord_g(0) = ord_t(\beta_0) \ . \tag{5.19}$$

Inserting (5.19) into (5.18) establishes (5.15) for $k = 0$.

Next we consider all other A_k and $\bar{A}_k$. In Section 3.2 we observed that t^k on A_k is homeomorphism sending it onto A_0 for $k \ge 0$, and t^k on $\bar{A}_k$ is also a homeomorphism sending it onto A_0 for $k \ge 2$. Since $t^n = t^{n-k} \circ t^k$, we conclude that

$$\begin{aligned} L_n(t, A_k) &= L_{n-k}(t, A_0) \quad \text{for} \quad 0 \le k \le n \ , & (5.20)\\ L_n(t, \bar{A}_k) &= L_{n-k}(t, A_0) \quad \text{for} \quad 2 \le k \le n \ . & (5.21) \end{aligned}$$

Now (5.18) and (5.20) gives (5.15) for $1 \le k \le n$ and (5.18) and (5.21) give (5.16) for $2 \le k \le n$. ■

Proof of Theorem 5.1: From the formulas of Lemma 5.3 and Lemma 5.5 we obtain, for $n \ge 2$,

$$\begin{aligned} L_n(t) &= L_n(g) + \sum_{k=1}^{n} (2L_{n-k}(g) - L_{n-k+1}(g)) \\ &+ \sum_{k=2}^{n} (2L_{n-k}(g) - L_{n-k+1}(g)) \\ &+ \sum_{k=1}^{n} \chi[n-k+1 \le ord_t(\beta_0)] - \sum_{k=0}^{n-1} \chi[n \le ord_t(\beta_k)] \\ &+ \sum_{k=2}^{n} \chi[n-k+1 \le ord_t(\beta_0)] - \sum_{k=1}^{n-1} \chi[n \le ord_t(1-\beta_k)] \ . \end{aligned} \tag{5.22}$$

Since

$$t^k(\beta_k) = t^k(1-\beta_k) = \beta_0 \ ,$$

we have

$$ord_t(\beta_k) = ord_t(1-\beta_k) = k + ord_t(\beta_0) \ . \tag{5.23}$$

It follows that

$$\chi[n \le ord_t(\beta_k)] = \chi[n \le ord_t(1-\beta_k)] = \chi[n-k \le ord_t(\beta_0)] \ . \tag{5.24}$$

Substituting this into (5.22) yields

$$L_n(t) = L_n(g) + 2L_{n-1}(g) - L_n(g) + 2\sum_{k=2}^{n} (2L_{n-k}(g) - L_{n-k+1}(g)) \ , \tag{5.25}$$

which in turn simplifies to

$$L_n(t) = 2\left(\sum_{j=0}^{n-2} L_j(g)\right) + 2 \ . \tag{5.26}$$

Also, by inspection,

$$L_0(t) = L_0(g) = 1 \quad \text{and} \quad L_1(t) = L_1(g) = 2 \ . \tag{5.27}$$

We obtain (5.1) by multiplying (5.26) by z^n and summing over $n \geq 2$, and using (5.27) for the $n = 0$ and $n = 1$ terms. ∎

Remark. Theorem 5.1 directly follows from (5.27). In deriving this equation we had to introduce various χ-terms which appear in (5.22) but then cancel out in the final result. This cancellation seems remarkable, and suggests the possibility that there may exist some other, simpler way of looking at the problem.

6. Tent Map and Linear Mod One Transformation

The previous results now easily combine to relate the lap counting function of tent map to that of the symmetric linear mod one transformation. We then complete the proof of Theorem 1.1.

Theorem 6.1. *For $1 < \beta \leq 2$, the lap counting function of the tent map t_β and the symmetric linear mod one transformation f_β are related by*

$$L_{t_\beta}(z) = \frac{zL_{f_\beta}(z)}{1-z} + \frac{1}{1-z} \ . \tag{6.1}$$

Proof: This follows directly from Theorem 4.1 and 5.1. ∎

Corollary 6.1. *For $1 < \beta \leq 2$ the kneading determinant*

$$D_{t_\beta}(z) = (1-\beta z)\Phi_\beta(z) \ . \tag{6.2}$$

Proof: Comparing Proposition 2.1 with Theorem 6.1, we obtain

$$D_{t_\beta}(z) = \frac{1}{(1-z)L_{f_\beta}(z)}. \tag{6.3}$$

By [12, formula (2.24)], we have

$$L_{f_\beta}(z) = \frac{1}{(1-z)(1-\beta z)\Phi_\beta(z)}. \tag{6.4}$$

Now (6.2) follows from (6.3) and (6.4). ∎

Proof of Theorem 1.1. We use Theorem 6.1 and apply the formulas of Theorem 4.2 of [12] to evaluate $L_{f_\beta}(z)$. The symmetric linear mod one transformation f_β is $f_{\beta,1-\frac{\beta}{2}}$ in the terminology of [12]. For $1<\beta\le\sqrt{2}$ the point $(\beta,1-\frac{\beta}{2})$ lies in the "bubble" $R_{2,1}$, and the renormalization map $(\beta,\alpha)\to(\bar{\beta},\bar{\alpha})$ is $(\beta,1-\frac{\beta}{2})\to(\beta^2,1-\frac{\beta^2}{2})$. This gives the formula

$$L_{f_\beta}(z)=\frac{1+z}{1-z}L_{f_{\beta^2}}(z^2), \tag{6.5}$$

which was stated in the introduction. For $\sqrt{2}<\beta\le 2$ we have

$$L_{f_\beta}(z)=\frac{1}{(1-z)(1-\beta z)\Phi_\beta(z)}\ , \tag{6.6}$$

which appears as [12, formula (2.24)]. Iterating (6.5) k times for $2^{2^{-k-1}}<\beta\le 2^{2^{-k}}$ and applying (6.1) and (6.6) we obtain formula (1.12) of Theorem 1.1.

Finally for $2^{2^{-k-1}}<\beta\le 2^{2^{-k}}$, set $\beta^*=\beta^{2^k}$, so that $\sqrt{2}<\beta^*\le 2$. The point $(\beta^*,1-\frac{\beta^*}{2})$ lies outside all the "bubbles" $R_{N,K}$, hence, by Theorem 1.1 of Flatto and Lagarias [14], $\Phi_{\beta^*}(z)\ne 0$ for all $|z|\le 1/\beta^*$. Thus $\Phi_{\beta^*}(z^{2^k})\ne 0$ for all $|z|\le 1/\beta$. ∎

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