INTERFERENCE-MINIMIZING COLORINGS OF REGULAR GRAPHS

P. C. Fishburn, *J. H. Kim*, *J. C. Lagarias* and *P. E. Wright*

AT&T Research, Murray Hill, NJ 07974

(Revised: April, 1996)

Abstract

Communications problems that involve frequency interference, such as the channel assignment problem in the design of cellular telephone networks, can be cast as graph coloring problems in which the frequencies (colors) assigned to an edge's vertices interfere if they are too similar. The paper considers situations modeled by vertex-coloring d-regular graphs with n vertices using a color set $\{1, 2, \ldots, n\}$, where colors i and j are said to *interfere* if their circular distance $\min\{|i-j|, n-|i-j|\}$ does not exceed a given threshold value α. Given a d-regular graph G and threshold α, an interference-minimizing coloring is a coloring of vertices that minimizes the number of edges that interfere. Let $I_\alpha(G)$ denote the minimum number of interfering edges in such a coloring of G. For most triples (n, α, d) we determine the minimum value of $I_\alpha(G)$ over all d-regular graphs, and find graphs that attain it. In determining when this minimum value is 0, we prove that for $r \geq 3$ there exists a d-regular graph G on n vertices that is r-colorable whenever $d \leq (1 - \frac{1}{r})n - 1$ and nd is even. We also study the maximum value of $I_\alpha(G)$ over all d-regular graphs, and find graphs that attain this maximum in many cases.

1. Introduction

This paper is motivated by telecommunication problems such as the design of planar regions for cellular telephone networks and the assignment of allowable frequencies to the regions. In our graph abstraction, vertices are regions, edges are pairs of contiguous regions, and colors correspond to frequencies. We presume that every region has the same number d of neighbors, which leads to considering degree-regular graphs. Interference occurs between two regions if they are neighbors and their frequencies lie within an interference threshold. We adopt the simplifying assumption that the number of colors available equals the number n of regions, and let α denote the threshold parameter so that colors i and j in $\{1, 2, \ldots, n\}$ interfere precisely when their circularly-measured scalar distance is less than or equal to α. Precedents for the use of circularly-measured distance in graph coloring include Vince (1988) and Guichard and Krussel (1992).

Our formulation leads to several interesting graph-theoretic problems. One is to determine for any given d-regular graph G and threshold α the minimum number $I_\alpha(G)$ of interfering edges over the possible colorings of G. Another is: given parameters n, α, and d, determine the minimum and maximum values of $I_\alpha(G)$ and find graphs G that attain these values. We focus on the latter problem. More specifically, let $\mathcal{G}(n, d)$ denote the set of undirected d-regular graphs on n vertices, which have no loops or multiple edges, but may be disconnected. We wish to determine the (global) *minimum interference level* $\ell(n, \alpha, d)$, which is the minimum of $I_\alpha(G)$ over $\mathcal{G}(n, d)$. For comparison purposes, we also wish to determine the (global) *minimax interference level* $L(n, \alpha, d)$, which is the maximum of $I_\alpha(G)$ over $\mathcal{G}(n, d)$. This latter problem measures how badly off you would be if an adversary gets to choose $G \in \mathcal{G}(n, d)$, and you can then color G to minimize interference.

Our graph-theoretic model is an approximation to the frequency assignment problem for cellular networks studied in Benveniste et al. (1995). In that paper the network of cellular nodes is viewed as vertices of a hexagonal lattice Λ in $\mathbb{R}^2$, and the graph G is specified by a choice of sublattice Λ' of Λ, with $n = |V(G)|$ being the index of the sublattice Λ' in Λ. More precisely, the vertices of G are cosets of Λ/Λ' and we draw an edge between two cosets if the cosets are "close" in the sense that they contain vectors $\mathbf{v}$, $\mathbf{v}'$ respectively with $||\mathbf{v} - \mathbf{v}'|| < x$, where $|| \cdot ||$ is a given norm on $\mathbb{R}^2$ and x is a cutoff value. Such graphs[1] G are d-regular for

[1] The graph G represents a fundamental domain of Λ/Λ'. In the cellular terminology a fundamental domain for Λ/Λ' is called a "reuse group." More generally a "reuse group" is a collection of contiguous cells that exhausts all frequencies, with no two cells in the group using the same frequency.

some value of d; the usual nearest-neighbors case gives $d = 6$: see Bernstein et al. (1995). The frequency spectrum is also divided into cosets (modulo n), and nodes in the same coset (mod Λ') are assigned a fixed coset of frequencies (mod n). In cellular problems the graph G is fixed (depending on Λ'). Typical parameters under consideration are $10 \leq n \leq 30$, $d = 6$, and n/α about 2 or 3. From this standpoint the quantities $\ell(n, \alpha, d)$ and $L(n, \alpha, d)$ represent lower and upper bounds for attainable levels of interference.

Related coloring problems motivated by the channel assignment problem are studied in Hale (1980), Cozzens and Roberts (1982), Bonias (1991), Liu (1991), Tesman (1993), Griggs and Liu (1994), Raychaudhuri (1994), Troxell (1996) and Guichard (1996) among others. Roberts (1991) surveys the earlier part of this work. Factors that distinguish prior work from the present investigation include our focus on regular graphs and the inevitability of interference when certain relationships hold among n, α and d.

Our main results give near-optimal bounds for $\ell(n, \alpha, d)$ and $L(n, \alpha, d)$ and identify d-regular graphs and colorings that attain extremal values. Many interference-minimizing designs use only a fraction of the available colors or frequencies. The most common number of colors used in these optimal designs is

$$\gamma = \left\lfloor \frac{n}{\alpha + 1} \right\rfloor ,$$

which is the maximum number of mutually noninterfering colors from $\{1, 2, \ldots, n\}$ at threshold α. Detailed statements of theorems for $\ell(n, \alpha, d)$ and $L(n, \alpha, d)$ appear in Section 2. Proofs follow in Sections 3 to 7.

In the course of our analysis we derive a graph-theoretic result of interest in its own right, which is a condition for the existence of a d-regular graph having chromatic number $\leq r$.

THEOREM 1.1. *If $r \geq 3$, then $\mathcal{G}(n, d)$ contains an r-colorable graph if nd is even and*

$$d \leq \begin{cases} (1 - \frac{1}{r})n - 1 & \textit{if } r \textit{ divides } n + 1 , \\ (1 - \frac{1}{r})n & \textit{otherwise} . \end{cases}$$

This result is proved in Section 5, and the proof can be read independently of the rest of the paper. Note that if nd is odd then $\mathcal{G}(n, d)$ is the empty set.

We preface the results in the next section with a few comments to indicate where we are headed. The case $\alpha = 0$ corresponds to no interference because the number of available colors equals the number of vertices, and therefore $\ell(n, 0, d) = L(n, 0, d) = 0$. We assume that $\alpha \geq 1$ in the rest of the paper.

For degrees near 0 or n, namely $d = 0, 1, n-2$ or $n-1$, the set $\mathcal{G}(n,d)$ contains only one unlabelled graph, so these cases are essentially trivial. We note at the end of Section 4 that

$$\ell(n,\alpha,n-1) = L(n,\alpha,n-1) = \lfloor \frac{n}{\gamma} \rfloor \left(n - \frac{1}{2}\gamma(\lfloor \frac{n}{\gamma} \rfloor + 1) \right) . \tag{1.1}$$

Our first main result in the next section, Theorem 2.1, applies to degree 2 and shows that most values of ℓ and L for $d = 2$ equal 0. A notable exception is that $L(n, 2, 2)$ is approximately $n/3$.

Subsequent results focus on $d \geq 3$, where we use the maximum number of noninterfering colors γ to express the results. The case $\gamma = 1$ is trivial because then all colors interfere with each other, so that $\ell = L = \#(\text{edges of } G) = nd/2$. For $\gamma \geq 2$, $\ell(n,\alpha,d)$ for most values of (n,α,d) is approximately

$$\max\left(\frac{nd}{2} - \frac{n^2}{2}\left(\frac{\gamma-1}{\gamma}\right),\ 0 \right) .$$

Moreover, $L(n,\alpha,d) = 0$ whenever $\gamma > d$, whereas if n is much larger than d, and d is somewhat larger than γ, then $L(n,\alpha,d)$ is approximately $nd/(2\gamma)$.

Extremal graphs which attain $\ell(n,\alpha,d)$ when $\ell > 0$ are usually connected, and the associated coloring can often be achieved using γ noninterfering colors. On the other hand, graphs that attain $L(n,\alpha,d)$ when $L > 0$ are usually disconnected and contain many copies of the complete graph K_{d+1}. There are exceptions, however.

Our results imply that there is often a sizable gap between the values of ℓ and L. The smallest instance of $l < L$ occurs at $(n,\alpha,d) = (6,2,2)$ where $l = 0$ and $L = 2$. Figure 1.1 shows the two graphs in $\mathcal{G}(6,2)$ with interference-minimizing colorings for $\alpha = 2$.

Figure 1.1 about here

A qualitative comparison of the regions where ℓ and L equal 0 and are positive is given in Figure 1.2, where the coordinates are d/n and γ/n.

Figure 1.2 about here

2. Main Results

An undirected graph is *simple* if it has no loops or multiple edges. Let $\mathcal{G}(n,d)$ denote the set of d-regular graphs on n vertices which are simple but which are not necessarily connected. Let $[n] = \{1, 2, \ldots, n\}$ be a set of n colors with *circular distance measure*

$$D(i,j) = \min\{|i-j|, n-|i-j|\} ,$$

and let $\alpha \in \{0, 1, \ldots\}$ be the threshold-of-interference parameter. A *coloring* of the vertex set $V(G)$ of graph $G = (V(G), E(G))$ in $\mathcal{G}(n, d)$ is a map $f : V(G) \to [n]$. The *interference* $I_\alpha(G, f)$ *of coloring* f *of* G *at threshold* α is

$$I_\alpha(G, f) := |\{\{x, y\} \in E(G) : D(f(x), f(y)) \le \alpha\}| \ .$$

The minimum interference in G at threshold α is

$$I_\alpha(G) := \min_{f: V(G) \to [n]} I_\alpha(G, f) \ .$$

We study the (global) *minimum interference level*

$$\ell(n, \alpha, d) := \min_{G \in \mathcal{G}(n,d)} I_\alpha(G) \tag{2.1}$$

and the (global) *minimax interference level*

$$L(n, \alpha, d) := \max_{G \in \mathcal{G}(n,d)} I_\alpha(G) \ . \tag{2.2}$$

We first note restrictions on the parameter space. Since all graphs in $\mathcal{G}(n, d)$ have $\frac{nd}{2}$ edges, it follows that

$$n \text{ and } d \text{ cannot both be odd} \ . \tag{2.3}$$

We restrict attention to the threshold range

$$1 \le \alpha \le \frac{n}{2} - 1 \ , \tag{2.4}$$

because $\alpha \ge \frac{n}{2}$ implies that all colors interfere. Thus

$$\gamma := \lfloor \frac{n}{\alpha + 1} \rfloor \ge 2 \ . \tag{2.5}$$

Our first result concerns ℓ and L for degree 2.

THEOREM 2.1. *Let* $d = 2$.

(a) For all $\gamma \ge 2$,

$$\ell(n, \alpha, 2) = 0 \ . \tag{2.6}$$

(b) For all $\gamma \ge 3$,

$$L(n, \alpha, 2) = 0 \ . \tag{2.7}$$

(c) If $\gamma = 2$, *and* $n = 3M + j$ *with* $0 \le j \le 2$, *then*

$$L(n,\alpha,2)=\begin{cases} M & if\ j=0,\ or\ j=2\ with \\ & \alpha \ge (2n-4)/5 \\ M-1 & if\ j=1,\ or\ j=2\ with \\ & \alpha < (2n-4)/5\ . \end{cases} \tag{2.8}$$

This is proved in Section 3.

We now consider d in the range

$$3 \le d \le n-3$$

for the minimum interference level ℓ. The cases of $\gamma = 2$ and $\gamma \ge 3$ are treated separately. We obtain an almost complete answer for $\gamma = 2$.

THEOREM 2.2. *Suppose that* $\gamma = 2$.

(a) If n *is even, then*

$$\ell(n,\alpha,d)=0 \quad if \quad d \le \frac{n}{2}\ ,$$

and

$$\ell(n,\alpha,d)=\begin{cases} \frac{nd}{2}-\frac{n^2}{4} & if\ d>\frac{n}{2}\ and \\ & \frac{n}{2}\ is\ even,\ or\ \frac{n}{2}\ and\ d\ are\ both\ odd \\ \frac{nd}{2}-\frac{n^2}{4}+1 & if\ d>\frac{n}{2}\ and\ \frac{n}{2}\ is\ odd \\ & and\ d\ is\ even\ . \end{cases} \tag{2.9}$$

(b) If n *is odd, then*

$$\ell(n,\alpha,d)=0 \quad if \quad d<n-2\alpha\ ,$$

and

$$\ell(n,\alpha,d)=\frac{nd}{2}-\frac{n^2}{4}+\frac{1}{4} \quad if \quad d>\frac{n}{2}\ . \tag{2.10}$$

(c) If n *is odd and in the remaining range* $n-2\alpha \le d \le \frac{n}{2}$, *then* $\ell(n,\alpha,d) \le \frac{d}{2}$. *Furthermore:*

(i) $\ell(n,\alpha,d)=0$ *if there is an integer* $2s+1 \ge 5$ *such that*

$$\alpha \le \left(\frac{s}{2s+1}\right)n-1 \quad and \quad d \le \left(\frac{2}{2s+1}\right)n\ ;$$

(ii) $\ell(n,\alpha,d) \le \frac{d}{2}-1$ *if* $d \ge 8$, $\frac{d}{2}$ *is even, and there is an integer* $4s+1 \ge 5$ *such that*

$$\alpha \le \left(\frac{2s}{4s+1}\right)n-1 \quad and \quad d=\left(\frac{2}{4s+1}\right)(n+1)\ ;$$

(iii) $\ell(n,\alpha,d)=\frac{d}{2}$ *for* $\alpha=(n-3)/2$.

Case (c) above is the only case not completely settled. Instances of it are illustrated in Figure 2.1. The number beside each vertex clump gives the color assigned to those vertices, and the number on a line between noninterfering clumps is the number of edges between them. Case analyses, omitted here, show that no improvements are possible in part (c) of the theorem when $n \le 21$. Given $n \le 21$, (i) has three realizations, namely $\ell(15,5,6) = \ell(21,7,8) = \ell(21,8,6) = 0$, (ii) has only the realization at the bottom of Figure 2.1, and $\ell = d/2$ for all other cases.

Figure 2.1 about here

We remark that the bounds on $\ell(n,\alpha,d)$ for $d > \frac{n}{2}$ are obtained using a variant of Turan's theorem on extremal graphs (Turán, 1941; Bondy and Murty, 1976, p. 110). Theorem 2.2 is proved in Section 4.

We now consider the minimum interference level ℓ when $\gamma \ge 3$. To handle this case we use Theorem 1.1, which is proved in Section 5. Let p and q be the unique nonnegative integers that satisfy

$$n = p\gamma + q \quad \text{with} \quad 0 \le q < \gamma ,$$

that is

$$p = \lfloor \frac{n}{\gamma} \rfloor \text{ and } q = n - \gamma \lfloor \frac{n}{\gamma} \rfloor . \tag{2.11}$$

Our bounds for $\gamma \ge 3$ are given in the next two theorems for $q = 0$ and $q > 0$, respectively, and are proved in Section 6. The $q = 0$ case is somewhat simpler.

THEOREM 2.3. *Suppose that $\gamma \ge 3$ and that γ divides n, i.e. $q = 0$.*

(a) If $d \le n - p$, then

$$\ell(n,\alpha,d) = 0 .$$

(b) If $d > n - p$ then

$$\ell(n,\alpha,d) = \begin{cases} \frac{n(d-n+p)}{2} & \text{if } n-d \text{ is odd or if} \\ & n-d \text{ is even and } p \text{ is even} \\ \frac{n(d-n+p)}{2} + \frac{\gamma}{2} & \text{if } n-d \text{ is even and } p \text{ is odd} . \end{cases}$$

THEOREM 2.4 *Suppose that $\gamma \ge 3$ and γ doesn't divide n, i.e. $q \ge 1$.*

(a) If $d < n - p$, then

$$\ell(n,\alpha,d) = \begin{cases} 0 & \text{if } d < n-p-1, \text{ or } d = n-p-1 \\ & \text{and } q < \gamma - 1 \\ \frac{p}{2} & \text{if } d = n-p-1 \text{ and } q = \gamma - 1 . \end{cases}$$

(b) If $d \ge n - p$, then

$$\ell(n,\alpha,d) = \frac{n(d-n+p)}{2} + \theta$$

where

$$\theta = \begin{cases} \frac{q(p+1)}{2} & if\ n-d\ is\ odd \\ \frac{q(p+2)}{2} & if\ n-d\ is\ even\ and\ p\ is\ even \\ \frac{pq+\gamma}{2} & if\ n-d\ is\ even\ and\ p\ is\ odd\ . \end{cases} \tag{2.12}$$

We turn next to results for the minimax interference level L. We first distinguish cases where $L = 0$ from cases where $L > 0$.

THEOREM 2.5. *Suppose that* $3 \le d \le n-2$. *Then:*

(a) $L(n, \alpha, d) = 0$ *whenever* $\gamma > d$ *and also when* $\gamma = d$ *and* $n < 2(d+1)$*;*

(b) $L(n, \alpha, d) > 0$ *for* $\gamma \le d$ *whenever* $n \ge 2(\gamma+1)$.

The only cases in the parameter range $1 \le \alpha \le \frac{n}{2} - 1$ and $\gamma \ge 2$ not settled by this theorem are those with

$$\gamma = d - a \ \ and \ \ n = 2(d-a) \ or \ 2(d-a)+1, \ where \ \ a > 0\ . \tag{2.13}$$

Both $L = 0$ and $L > 0$ occur in this exceptional case, e.g. for $a = 1$, $L(8, 1, 5) = 0$ while $L(7, 1, 4) = 1$.

Our final main result provides bounds for L. Set

$$Q = d + 1 - \gamma \lfloor \frac{d+1}{\gamma} \rfloor$$

and

$$W = n - (d+1) \lfloor \frac{n}{d+1} \rfloor\ .$$

In view of Theorem 2.5 we consider only the range that $2 \le \gamma \le d$.

THEOREM 2.6. *Suppose that* $3 \le d \le n-1$ *and that* $2 \le \gamma \le d$. *Then*

$$L(n, \alpha, d) \ge \frac{1}{2\gamma} \lfloor \frac{n}{d+1} \rfloor ((d+1)(d+1-\gamma) + Q(\gamma - Q)) - \frac{1}{2} W(d+1-W)\ ,$$

and

$$L(n, \alpha, d) \le \frac{1}{2\gamma} \left(\frac{d}{n-1} \right) (n(n-\gamma) + q(\gamma - q))\ .$$

In the special case that $d+1$ divides n, these bounds can be written more simply as

$$\frac{nd}{2\gamma} - \frac{n(\gamma-1)}{2\gamma} + \frac{nQ(\gamma-Q)}{2\gamma(d+1)} \le L(n, \alpha, d) \le \frac{nd}{2\gamma} - \frac{nd(\gamma-1)}{2\gamma(n-1)} + \frac{dq(\gamma-q)}{2\gamma(n-1)}\ .$$

This applies in particular when $d = n-1$, in which case the upper and lower bounds coincide, yielding (1.1). If n is substantially larger than d, and d is somewhat larger than γ, then L is closely approximated by $\frac{nd}{2\gamma}$.

Theorems 2.5 and 2.6 are proved in Section 7.

3. Elementary Facts: Theorem 2.1

We derive general conditions that guarantee $\ell = 0$ or $L = 0$, and then analyze degree-2 graphs (Theorem 2.1).

LEMMA 3.1. *If* $1 \leq \alpha < \frac{n}{2}$*, then*

$$\text{(a)} \quad \ell(n, \alpha, d) = 0 \quad \textit{whenever} \quad d < n - 2\alpha \ , \tag{3.1}$$

and

$$\text{(b)} \quad \ell(n, \alpha, d) = 0 \quad \textit{whenever} \quad d \leq \frac{n}{2} \textit{ and } n \textit{ is even.} \tag{3.2}$$

PROOF. (a) Given $d < n - 2\alpha$, let $V(G) = \{1, 2, \ldots, n\}$ and consider the coloring $f(i) = i$ for every i. We construct a suitable G starting with the edge set

$$E = \{\{i, j\} : i, j \in [n], \ i \neq j, \text{ with } D(i, j) \geq (n + 1 - d)/2\} \ .$$

If n is odd, or if n is even and d is odd, let $E(G) = E$. Then every vertex has degree d and every edge has $D > \alpha$, so $\ell(n, \alpha, d) = 0$. If n and d are both even, so $\alpha \leq (n - d)/2 - 1$, let $E(G) = (E \cup \{\{i, j\} : D(i, j) = \frac{n-d}{2}\}) \setminus \{\{1, \frac{n}{2} + 1\}, \{2, \frac{n}{2} + 2\}, \ldots, \{\frac{n}{2}, n\}\}$. Again, every vertex has degree d and every edge has $D > \alpha$, so $\ell(n, \alpha, d) = 0$.

(b) Let χ_G denote the chromatic number of the graph G. The definition implies that:

$$\ell(n, \alpha, d) = 0 \ \text{ if } \ \chi_G \leq \gamma \text{ for some } G \in \mathcal{G}(n, d) \ . \tag{3.3}$$

If n is even and $d \leq \frac{n}{2}$, then $\mathcal{G}(n, d)$ contains a bipartite graph with $n/2$ vertices in each part, so $\chi_G = 2$, and (b) follows from (3.3), since $\gamma \geq 2$. ■

We remark that the construction in part (a) uses all n colors, and when $d \geq n - 2\alpha$ this same construction gives many interfering edges. It is natural to consider the opposite extreme, which is to use only a maximal set of $\gamma = \lfloor \frac{n}{\alpha+1} \rfloor$ noninterfering colors. This leads to part (b).

The restriction in part (b) that n be even is crucial, because no d-regular bipartite graph exists for odd n. Indeed there are exceptions where $\ell(n, \alpha, d) > 0$ for some $d < \frac{n}{2}$ with n odd: see Theorems 2.1 and 2.2. These exceptions occur when $\gamma = 2$, but are not an issue for $\gamma \geq 3$.

We obtain bounds on the minimax interference level L using the following well-known bound for the chromatic number χ_G of a graph G.

PROPOSITION 3.1. *For every finite simple graph* G*,*

$$\chi_G \leq \Delta_G + 1 \ , \tag{3.4}$$

where Δ_G *is the maximum degree of a vertex of* G. *Furthermore* $\chi_G \le \Delta_G$ *provided that no connected component of* G *is an odd cycle or a complete graph.*

PROOF. Brooks (1941); Bondy and Murty (1976, pp. 118 and 122). ∎

This result immediately yields the following condition for the minimax interference level $L = 0$.

LEMMA 3.2. *If* $1 \le \alpha < \frac{n}{2}$, *then*

$$L(n, \alpha, d) = 0 \text{ whenever } \gamma > d \ . \tag{3.5}$$

PROOF. The definition of $L(n, \alpha, d)$ gives

$$L(n, \alpha, d) = 0 \text{ if } \chi_G \le \gamma \text{ for every } G \in \mathcal{G}(n, d) \ . \tag{3.6}$$

Since $\Delta_G = d$ for a d-regular graph, (3.5) follows from Proposition 3.1 via (3.6). ∎

PROOF OF THEOREM 2.1. (a) Since $d = 2$, $\ell = 0$ follows from (3.2) if n is even, and from (3.1) if n is odd and $\alpha \le \frac{n}{2} - 1$.

(b) follows from Lemma 3.2.

(c) Given $d = 2$, every graph in $\mathcal{G}(n, 2)$ is a sum of vertex-disjoint cycles. Suppose $\gamma = 2$, so $n/3 - 1 < \alpha \le n/2 - 1$. Then an even cycle has minimum interference 0, a 3-cycle has minimum interference 1, and an odd cycle with five or more vertices has minimum interference 0 or 1. It follows that $L = M$ if $n = 3M$ (M 3-cycles), $L = M - 1$ if $n = 3M + 1$ ($M - 1$ 3-cycles, one 4-cycle), and $L \in \{M - 1, M\}$ if $n = 3M + 2$. The last case uses $M - 1$ 3-cycles and one 5-cycle. When the 5-cycle's vertices are colored successively as 1, $\alpha + 2$, $2\alpha + 3$, $n - 2\alpha - 1$ and $n - \alpha$, it has no interference if $[n - (2\alpha + 3)] + [n - 2\alpha - 1] > \alpha$, i.e., if $\alpha < (2n - 4)/5$, so $L = M - 1$ in this case. More generally, suppose one vertex of the 5-cycle is colored 1. Its neighbors must have colors in $[\alpha + 2, n - \alpha]$ to avoid interference. Then their uncolored neighbors, which are adjacent, must have colors in $[2\alpha + 3, \ldots, n, 1, \ldots, n - 2\alpha - 1]$ to avoid interference. This set has $\max D = [n + (n - 2\alpha - 1)] - (2\alpha + 3)$, which is $\le \alpha$ if $(2n - 4)/5 \le \alpha$. Hence $L = (M - 1) + 1$ for $n = 3M + 2$ if $(2n - 4)/5 \le \alpha$. ∎

4. Minimal Interference Level: Theorem 2.2

We prove Theorem 2.2 in this section. The ranges stated where $\ell(n, \alpha, d) = 0$ follow from Lemma 3.1, so the main content of parts (a) and (b) of Theorem 2.2 concerns the values

$\ell(n, \alpha, d)$ for $d > \frac{n}{2}$. To obtain these we use a variant of Turán's theorem (Turán, 1941; Bondy and Murty, 1976, p. 110), which we state as a lemma. An application of the lemma at the end of the section yields the exact value of $L(n, \alpha, n-1)$ as well as $\ell(n, \alpha, n-1)$. Recall that an *equi-t-partition* of a vertex set V is a partition $\{V_1, \ldots, V_t\}$ with $||V_i| - |V_j|| \leq 1$ for all $i, j \in \{1, \ldots, t\}$.

LEMMA 4.1. *The maximum number of noninterfering edges in the complete graph K_n with vertex set V and threshold parameter α is attained only by a coloring $f : V \to [n]$ that has $D(f(x),\ f(y)) > \alpha$ whenever x and y are in different parts of an equi-γ-partition of V.*

PROOF. Suppose that a coloring f of the complete graph K_n has f_i vertices of color i and $f_i f_j > 0$ for some $i \neq j$ with $D(i, j) \leq \alpha$. Let m_{ab} denote the number of vertices of colors other than a and b that interfere with a and not b. If all color-i vertices are recolored j, the net increase in interference is $f_i(m_{ji} - m_{ij})$; if all color-j vertices are recolored i, the net increase in interference if $f_j(m_{ij} - m_{ji})$. Hence at least one of the recolorings does not increase interference. Continuing this recoloring process implies that noninterference in K_n is maximized by a γ-partite partition of V such that $D(f(x),\ f(y)) > \alpha$ whenever x and y are in different parts of the partition. Turán's theorem then implies that maximum noninterference obtains only when the partition is an equi-γ-partition. ∎

We can assume without loss of generality that the coloring f found in Lemma 4.1 is constant on each part of an equi-γ-partition, with $f(V) = \{(i-1)(\alpha+1)+1 : i = 1, \ldots, \gamma\}$. If interfering edges are then dropped from K_n, we obtain a complete equi-γ-partite graph with zero interference and chromatic number γ. This graph is regular if and only if γ divides n and each part of the partition has n/γ vertices.

PROOF OF THEOREM 2.2. Throughout this proof $\gamma = 2$, so that

$$n/3 - 1 < \alpha \leq n/2 - 1 \ . \tag{4.1}$$

We consider first (a) and (b). The ranges given where $\ell(n, \alpha, d) = 0$ come from Lemma 3.1. So assume now that $d > \frac{n}{2}$. Let G_0 be a complete bipartite graph $\{A, B\}$ such that

$$|A| = \lceil \frac{n}{2} \rceil \ \text{ and } \ |B| = \lfloor \frac{n}{2} \rfloor \ .$$

Lemma 4.1 implies that two-coloring G_0 using noninterfering colors for A and B uniquely maximizes the number of edges with no interference when $\gamma = 2$. Therefore $\ell \geq nd/2 - |A||B|$.

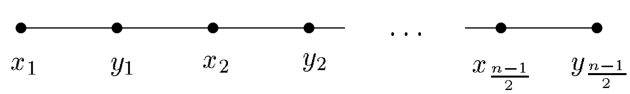
(a) Suppose n is even. If $n/2$ and d are odd, the number of edges needed within each part of G_0 to increase all degrees to d is $(n/2)(d - n/2)/2$, which is an integer since $d - n/2$ is even. It follows that if $n/2$ is even, or if $n/2$ and d are odd, then $\ell = (n/2)(d - n/2) = nd/2 - n^2/4$.

If instead $n/2$ is odd, and d is even, then $(n/2)(d - n/2)$ is odd, G_0 is not part of any graph in $\mathcal{G}(n, d)$, and $\ell > nd/2 - n^2/4$. We obtain $\ell = nd/2 - n^2/4 + 1$ by replacing G_0 with a complete bipartite graph G_1 with bipartition $\{A', B'\}$, $|A'| = n/2 + 1$ and $|B'| = n/2 - 1$. Beginning with G_1, each vertex in A' requires $d - n/2 + 1$ more degrees to have degree d, and each vertex in B' requires $d - n/2 - 1$ more edges added to have degree d. Both $d - n/2 + 1$ and $d - n/2 - 1$ are even, so edge additions as needed can be made within A' and B' to obtain $G \in \mathcal{G}(n, d)$. Therefore $\ell = nd/2 - (n/2 + 1)(n/2 - 1) = nd/2 - n^2/4 + 1$ in this case; and (2.9) is proved.

(b) Suppose n is odd, so d is even by (2.3). Beginning with G_0, each of the $(n + 1)/2$ vertices in A requires $d - (n - 1)/2$ more incident edges added to have degree d, and each of the $(n - 1)/2$ vertices in B requires $d - (n + 1)/2$ more incident edges added to have degree d. Each of $\{(n + 1)/2, d - (n - 1)/2\}$ and $\{(n - 1)/2, d - (n + 1)/2\}$ contains an even integer, so we can make the required additions of edges within A and B. Hence $\ell = nd/2 - [(n + 1)/2][(n - 1)/2] = nd/2 - (n^2 - 1)/4$, and (2.10) is proved.

It remains to prove (c), which has three parts (i)–(iii). Assume henceforth that n is odd and $n - 2\alpha < d < n/2$, with d even because n is odd. Augmented equi-bipartite graphs, illustrated at the top of Figure 2.1, show that $\ell \le d/2$ since they require $d/2$ edges within the $(n + 1)/2$-vertex part to obtain degree d for every vertex. Sometimes $\ell = d/2$. A case in point is $\alpha = (n - 3)/2$, the largest possible α for $\gamma = 2$ and odd n.

Suppose $\alpha = (n - 3)/2$. Then $d > n - 2\alpha = 3 \Rightarrow d \in \{4, 6, \ldots, n - 1\}$. Each vertex in the color set $[n]$ has exactly two others for which $D > \alpha$, and the graph of noninterfering colors is an n-cycle whose successive colors are $1, (n + 3)/2, 2, (n + 5)/2, 3, \ldots, (n + 1)/2$. If every color were assigned to some vertex in $G \in \mathcal{G}(n, d)$, there would be at least $n(d - 2)/2$ interference edges. But $n(d - 2)/2 > d/2$, so f must avoid at least one color to attain ℓ. Deletion of one color from the n-cycle of noninterfering colors breaks the cycle and leaves the noninterference graph



$(n - 1$ colors$)$.

Because all x_i colors interfere with each other, and all y_i colors interfere with each other, we can presume that f uses only one x_i and an adjacent y_j. This yields the augmented bipartite structure of the preceding paragraph, and it follows from maximization of between-parts edges that $\ell = d/2$. This completes the proof of (iii).

For (i) and (ii), assume $\alpha < (n-3)/2$ and consider an odd $r \geq 5$ sequence of colors c_1, $c_2, \ldots, c_r$ with $c_1 = 1$ and $D(c_{i+1}, c_i) \geq (\alpha+1)$ for $i = 1, \ldots, r-1$. The tightest such sequence has $c_i = (i-1)(\alpha+1)+1$ for $i = 2, \ldots, r$, where color $jn+k$, $1 \leq k \leq n$, is identical to color k. It follows that the final color c_r can be chosen not to interfere with $c_1 = 1 = jn+1$ if

$$\frac{1}{2}(r-1)n - (r-1)(\alpha+1) \geq (\alpha+1) \ ,$$

i.e., if

$$\alpha \leq \left(\frac{r-1}{2r}\right) n - 1 \quad \Longleftrightarrow \quad r \geq \frac{n}{n-2(\alpha+1)} \ . \tag{4.2}$$

We usually consider the smallest such odd $r \geq 5$ because this allows the $\ell = 0$ conclusion for the largest d values. Our approach, illustrated on the lower part of Figure 2.1, is to assign clumps of vertices to the c_i in such a way that all edges for $G \in \mathcal{G}(n,d)$ are between adjacent clumps on the noninterference color cycle $c_1, \ldots, c_r, c_1$.

Suppose (4.2) holds for a fixed odd $r \geq 5$. We assume that $r < n$ because the ensuing analysis requires this for $d \geq 3$. Let a and b be nonnegative integers that satisfy

$$n = ar + b, \quad 0 \leq b < r \ .$$

We prove (i), then conclude with (ii). The analysis for (i) splits into three cases depending on the parity of a and $\lfloor r/4 \rfloor$.

Case 1: a odd

Case 2: a even, $\lfloor r/4 \rfloor$ odd

Case 3: a even, $\lfloor r/4 \rfloor$ even.

Because n is odd, Case 1 requires b to be even and Cases 2 and 3 require b to be odd.

Case 1. Given an odd a, we partition the n vertices into b clumps of $a+1$ vertices each and $r-b$ clumps of a vertices each. The clumps are assigned to colors in the noninterference cycle $c_1, \ldots, c_r, c_1$ so that the clumps of each type are contiguous. Cases for $b = 0$ and $b = 4$ are illustrated at the top of Figure 4.1. We begin at the central (top) a clump and proceed symmetrically in both directions around the color cycle, assigning between-clumps edges as we go so that all vertices end up with degree $2a$. The required edges into the next clump encountered are distributed as equally as possible to the vertices in that clump. When we get into the clumps with $a+1$ vertices, the number of between-clumps edges needed will generally be less than the maximum possible number of $(a+1)^2$. Numbers of between-clumps edges used to get degree $2a$ for every vertex are shown on the noninterference lines between the c_i on Figure 4.1.

Figure 4.1 about here

The preceding construction yields $\ell(n, \alpha, d) = 0$ for $d = 2a = 2(n-b)/r$. If even d is less than $2a$, say $d = 2a'$ with $a' < a$, we modify the procedure by using fewer between-clumps edges for the required vertex degrees: clump sizes are unchanged. Because $n/r < d/2 = a$ yields the contradiction that $n < ra$, it follows for Case 1 that $\ell = 0$ if $d \leq (2/r)n$.

Case 2. With a even and $\lfloor r/4 \rfloor$ odd, we have b odd and $r \in \{5, 7, 13, 15, 21, 23, \ldots\}$. In this case we assign $a-1$ vertices to c_1 and proceed in each direction around the c_i cycle, assigning a, $a+1$, a, $a-1$, a, $a+1$, a, $a-1, \ldots, a^0$, a^* vertices to the next $(r-1)/2$ c_i in order. The penultimate number a^0 equals a if $r \in \{5, 13, 21, \ldots\}$ and is $a+1$ if $r \in \{7, 15, 23, \ldots\}$. The ultimate number a^* in chosen so that there are $[n-(a-1)]/2$ vertices (excluding the $a-1$ for c_1) on each side of the color cycle. If $a^0 = a$ then $a^* = a + (b+1)/2$, and if $a^0 = a+1$ then $a^* = a + (b-1)/2$. The two cases are shown on the lower left of Figure 4.1 with numbers of between-clumps edges that give degree $d = 2a$ for every vertex. If even d is less than $2a$, fewer edges are used, as needed, down the two sides. As in Case 1, we get $\ell = 0$ if $d \leq (2/r)n$.

Case 3. With a even and $\lfloor r/4 \rfloor$ even, we have b odd and $r \in \{9, 11, 17, 19, 25, 27, \ldots\}$. Here we assign $a+1$ vertices to c_1 and proceed with a, $a-1$, a, $a+1$, a, $a-1$, a, $a+1, \ldots, a^0$, a^* vertices assigned to the next $(r-1)/2$ c_i in each direction away from c_1. We get $a^0 = a$ and $a^* = a + (b+1)/2$ if $r \in \{9, 17, 25, \ldots\}$, and $a^0 = a+1$ and $a^* = a + (b-1)/2$ if $r \in \{11, 19, 27, \ldots\}$. The two cases are shown on the lower right of Figure 4.1. As before, $\ell = 0$ if $d \leq (2/r)n$.

This completes the proof of (i), after defining s by $r = 2s+1$. We have also checked that the construction used here cannot yield $l = 0$ unless the conditions of (i) hold.

There is however one other set of circumstances where this construction yields a value of

$\ell < \frac{d}{2}$ for some $d > \frac{2}{r}n$, and these circumstances are exactly the hypotheses of (ii), namely:

$$\left\{\begin{array}{lcl} d & \geq & 8, \;\; d/2 \;\text{ is even} \\ r & = & 4s+1 \;\; \text{for some integer } s \geq 1 \\ n & = & r(d/2)-1 \;\; (n \text{ is odd since } d/2 \text{ is even}) \\ \alpha & & \text{satisfies (4.2)} \; . \end{array}\right. \tag{4.3}$$

In this case we partition the vertices into $(r+1)/2$ clumps of $d/2-1$ vertices each and $(r-1)/2$ clumps of $d/2+1$ vertices each. The clumps (-1 for $d/2-1$, $+1$ for $d/2+1$) are arranged around the noninterference color cycle c_1, $c_2, \ldots, c_r$ as $-1, -1, -1, +1, +1, -1, -1, +1, +1$, $\ldots, -1, -1, +1, +1$. We use all possible between-clumps edges. This gives degree d for every vertex except those in the c_2 clump, which has $2(d/2-1)^2 = d^2/2 - 2d + 2$ incoming edges from c_1 and c_3. The degree total for c_2 should be $d(d/2-1) = d^2/2 - d$, so we need to add $(d-2)/2 = d/2-1$ edges within c_2 to get degree d for each c_2 vertex. Prior to the additions, each c_2 vertex has degree $d-2$ by our equalization construction, so the additions can be made by a complete cycle within the clump. It follows that $\ell \leq d/2-1$, proving (ii). ■

We conclude this section by noting that the modified Turán's theorem (Lemma 4.1) easily allows us to completely settle the case of degree $d = n-1$.

Corollary 4.2. *For* $d = n-1$,

$$\ell(n,\alpha,n-1) = L(n,\alpha,n-1) = \lfloor \frac{n}{\gamma} \rfloor \left(n - \frac{1}{2}\gamma \left(\lfloor \frac{n}{\gamma} \rfloor + 1 \right) \right) \; . \tag{4.4}$$

Proof. Write

$$n = p\gamma + q \; , \quad 0 \leq q < \gamma \; ,$$

so $p = \lfloor \frac{n}{\gamma} \rfloor$. An equi-$\gamma$-partition of an n vertex set has

$$\left\{\begin{array}{l} q \text{ parts, each with } p + 1 \text{ vertices,} \\ \\ \gamma - q \text{ parts, each with } p \text{ vertices.} \end{array}\right.$$

Now the unique graph $G \in \mathcal{G}(n, n-1)$ is K_n, so applying Lemma 4.1, we have

$$\ell(n,\alpha,n-1) = L(n,\alpha,n-1) = q \binom{p+1}{2} + (\gamma - q)\binom{p}{2} \; ,$$

which is (4.4). ■

5. Chromatic Number Bound: Theorem 1.1

This section gives a self-contained proof of Theorem 1.1. We first recall two preliminary facts, stated as propositions.

PROPOSITION 5.1. (Dirac (1942)) *Let G be a simple graph. If every vertex of G is of degree at least $|V(G)|/2$, then G is Hamiltonian, that is, G has a cycle of length $|V(G)|$.*

PROOF. See Bondy and Murty (1976), p. 54. ■

Recall that a *matching* in a simple graph G is a subset of mutually vertex-disjoint edges of G. A matching is *perfect* if every vertex in G is on some edge of the matching. The following is a consequence of a well-known theorem of Hall (1935).

PROPOSITION 5.2. (Marriage Theorem) *If G is a d-regular bipartite graph with $d > 0$, then G has a perfect matching.*

PROOF. See Bondy and Murty (1976), p. 73. ■

We study the function $\phi(n,d;r)$ defined by

$$\phi(n,d;r) = \begin{cases} 1 & \text{if there exists an } n\text{-vertex } d\text{-regular } r\text{-colorable graph,} \\ 0 & \text{otherwise .} \end{cases}$$

When $\phi(n,d;r) = 1$ we let $G(n,d;r)$ denote such a d-regular r-colorable (that is, r-partite) graph having n vertices. We consider only values in which nd is even.

Our first observation is that because an r-colorable graph is also $(r+1)$-colorable,

$$\phi(n,d;r_1) \le \phi(n,d;r_2) \text{ if } r_1 < r_2 \ . \tag{5.1}$$

The purpose of the next two lemmas is to prove that $\phi(n,d;r)$ is monotone when $r \ge 3$ is held fixed and d varies over values where nd is even.

LEMMA 5.1 *(a) If $d \le \frac{n}{2}$ and if either $r \ge 3$ or $r = 2$ and n is even, then*

$$\phi(n,d;r) = 1 \ . \tag{5.2}$$

(b) If $d \ge \frac{n}{2}$, then

$$\phi(n,d;r) = 1 \ \ \textit{implies} \ \ \phi(n,d-2;r) = 1 \ . \tag{5.3}$$

If in addition n is even, then

$$\phi(n,d;r) = 1 \ \ \textit{implies} \ \ \phi(n,d-1;r) = 1 \ . \tag{5.4}$$

PROOF. (a) Suppose that n is even. The inequality (5.1) implies that it is enough to show

$$\phi(n,d;2) = 1 \text{ for } d \le \frac{n}{2}, \ n \text{ even} \ . \tag{5.5}$$

We use reverse induction on $d \leq n/2$. For the base case $d = n/2$, the complete equi-2-partite graph gives $\phi(n, n/2; 2) = 1$. For the induction step, suppose we know that $\phi(n, d; 2) = 1$. Then a d-regular bipartite graph $G(n, d; 2)$ exists, and by Proposition 5.2 it has a perfect matching M. Remove all edges in M from G to obtain a $(d-1)$-regular bipartite graph $G(n, d-1; 2)$. Hence $\phi(n, d-1, 2) = 1$.

Suppose n is odd. Then (5.1) implies that it is enough to show

$$\phi(n, d; 3) = 1 \text{ for } d \leq \frac{n}{2},\ n \text{ odd} . \tag{5.6}$$

Now d must be even by (2.3), and $d \leq (n-1)/2$. Because $n-1$ is even, we have $\phi(n-1, d; 2) = 1$ by (5.5). Consider $G := G(n-1, d; 2)$ with $|V(G)| = n-1$. By Proposition 5.2 we may find a perfect matching of G, say $M = \{\{x_1, y_1\}, \ldots, \{x_k, y_k\}\}$, with $k = (n-1)/2 \geq d/2$. Remove from G the edges $\{x_1, y_1\}, \ldots, \{x_{\frac{d}{2}}, y_{\frac{d}{2}}\}$, and add to G a new vertex z and the edges $\{z, x_i\}$ and $\{z, y_i\}$ for $1 \leq i \leq d/2$. Then it is easy to see that the resulting graph is a d-regular 3-partite graph with n vertices, which proves (5.6).

(b) Let $G = G(n, d; r)$, which exists by hypothesis. Since $d \geq \frac{n}{2}$, Proposition 5.1 guarantees that G has a Hamilton cycle C. Removing all edges from C yields a $G(n, d-2; r)$, so $\phi(n, d-2; r) = 1$. If moreover n is even, then C has even length and we get a perfect matching M by taking alternate edges in C. Removing all edges in M from G yields a $G(n, d-1; r)$, so $\phi(n, d-1; r) = 1$ in this case. ∎

Lemma 5.2. *If $r \geq 3$, then*

$$\phi(n, d_1; r) \geq \phi(n, d_2; r) \quad \mathit{if} \quad d_1 < d_2 , \tag{5.7}$$

provided that nd_1 and nd_2 are both even.

Proof. Suppose $d_1 \leq n/2$. Then by Lemma 5.1(a), $\phi(n, d_1; r) = 1$ for all $r \geq 3$ and we are done.

Suppose $d_1 > n/2$. For even n, Lemma 5.1(b) used inductively on decreasing d gives

$$\phi(n, d_2; r) = 1 \Rightarrow \phi(n, d_2 - 1; r) = 1 \Rightarrow \ldots \Rightarrow \phi(n, d_1; r) = 1 .$$

For odd n, since nd_1 and nd_2 are both even, both d_1 and d_2 must be even. Now Lemma 5.1(b) gives

$$\phi(n, d_2; r) = 1 \Rightarrow \phi(n, d_2 - 2; r) = 1 \Rightarrow \ldots \Rightarrow \phi(n, d_1; r) = 1 ,$$

so (5.7) follows. ∎

PROOF OF THEOREM 1.1. To commence the proof, we define p and q by

$$n = pr + q \quad \text{with} \quad 0 \le q < r \ , \tag{5.8}$$

that is $p = \lfloor \frac{n}{r} \rfloor \ge 1$. Note that r divides $n+1$ if and only if $q = r-1$. In terms of p and q the assertions of the theorem then become:

(i) If $q = 0$, then $\phi(n,d;r) = 1$ if $d \le n - p$.

(ii) If $1 \le q \le r-2$, then $\phi(n,d;r) = 1$ if $d \le n-p-1$.

(iii) If $q = r-1$ then $\phi(n,d;r) = 1$ if $d \le n-p-2$.

To prove (i)-(iii), we use the complete equi-r-partite graph $G^r(n)$ defined as follows. The graph $G^r(n)$ has vertices $V = \{v_1, v_2, \ldots, v_n\}$ and for $1 \le j \le r$ we define the vertex sets

$$X_j = \{v_i : i \equiv j \pmod r\} \ . \tag{5.9}$$

The edge set of $G^r(n)$ is

$$E(G^r(n)) = \{\{v_i, v_j\} : i \not\equiv j (\text{mod } r)\} \ .$$

Here $\{X_1, \ldots, X_r\}$ is an equi-r-partition of V with

$$|X_1| = |X_2| = \ldots = |X_q| = p+1, \ |X_{q+1}| = \ldots = |X_r| = p \ . \tag{5.10}$$

For $1 \le a \le b \le r$ we let $G^r_{a,b}$ denote the induced subgraph of $G^r(n)$ on the vertex set

$$V_{a,b} := \cup_{j=a}^{b} X_j \ .$$

To prove (i), if $q = 0$ then $G^r(n)$ is an $(n-p)$-regular graph, hence

$$\phi(n, n-p; r) = 1 \ . \tag{5.11}$$

Lemma 5.2 implies $\phi(n,d;r) = 1$ if $d \le n - p$, and (i) follows.

To prove (ii), let $H = G^r_{q+1,r}$. Then (5.10) shows that H is a $p(r-q-1)$-regular graph having $p(r-q)$ vertices. Now $r - q \ge 2$ implies that H has degree $p(r-q-1)$, which is greater than half its vertices, so H has a Hamilton cycle C by Proposition 5.1.

If $p(r-q)$ is even, then H has a perfect matching M obtained by taking every other edge in C. Removing all edges in M from $G^r(n)$, the resulting graph is $(n-p-1)$-regular, hence $\phi(n, n-p-1; r) = 1$. Lemma 5.2 then completes the proof of (ii).

If $p(r-q)$ is odd, then p is odd, hence so is

$$n = pr + q = (p+1)q + p(r-q) \ .$$

Then $n-p-1$ is also odd, so $d=n-p-1$ is forbidden by (2.3). Thus it suffices to show that $\phi(n, n-p-2; r)=1$ in this case, for then Lemma 5.2 gives $\phi(n, d; r)=1$ for $d \leq n-p-2$.

Let $H' := G^r_{1,q}$. Then H' is a $(p+1)(q-1)$-regular graph with $(p+1)q$ vertices. If $q>1$ then

$$(p+1)(q-1) \geq (p+1)q/2 \ ,$$

hence H' is Hamiltonian. Since $(p+1)q$ is even, H' has a perfect matching M'. Removing all edges in $M' \cup C$ from $G^r(n)$, the resulting graph is $(n-p-2)$-regular, hence $\phi(n, n-p-2; r)=1$.

Suppose $q=1$. Notice that since $p(r-1)$ is odd, $r \neq 3$, hence $r \geq 4$. Let H'' be the induced subgraph of $G^r(n)$ on the set

$$\{v_{jr+2} \in X_2 : (p+1)/2 \leq j \leq p\} \cup \bigcup_{j=3}^{r} X_j$$

and

$$E := \{\{v_{ir+1}, v_{jr+2}\} : 0 \leq j \leq (p-1)/2,\ i=2j \text{ or } i=2j+1\} \ .$$

Then the number of vertices of H'' is $p(r-2)+(p-1)/2$ and the minimum degree of H'' is $p(r-3)+(p-1)/2$. Since

$$p(r-3)+(p-1)/2 \geq \frac{1}{2}(p(r-2)+(p-1)/2) \text{ for } r \geq 4 \ ,$$

Proposition 5.1 implies that H'' has a Hamiltonian cycle C''. By removing all edges in $C'' \cup E$ from $G^r(n)$ we have an $(n-p-2)$-regular graph, hence $\phi(n, n-p-2; r)=1$.

To prove (iii) we proceed by induction on r, with an induction step from r to $r+2$. There are two base cases, $r=3$ and $r=4$.

Base Case $r=3$. We have $q=2$, so $n=3p+2$. Let

$$E_1 = \{\{v_{3i}, v_{3i-2}\} : i=1,2,\ldots,p\} \text{ and } E_2 = \{\{v_{3i}, v_{3i-1}\} : i=1,2,\ldots p\} \ .$$

Consider the graph G obtained by removing from $G^3(n)$ all edges in $E_1 \cup E_2 \cup \{v_{3p+1}, v_{3p+2}\}$. Then it is easy to see that G is $(n-p-2)$-regular, so $\phi(n, n-p-2; r)=1$. Now Lemma 5.2 gives $\phi(n, d; r)=1$ for $d \leq n-p-2$.

Base Case $r=4$. We have $q=3$, and $n=4p+3$. Suppose first that p is odd. We relabel the vertices of $G^4(n)$ so that the sets X_j in (5.9) become

$$X_j = \{w_i : i \equiv j \pmod 3\} \ \text{ for } j=1,2,3, \text{ while } \quad X_4 = \{u_i : 1 \leq i \leq p\} \ . \tag{5.12}$$

Let H be the subgraph of $G^4(n)$ induced on the vertex set $\{w_j : 2p+1 \le j \le 3p+3\}$. Then $|V(H)| = p+3$ is even and H is Hamiltonian. Thus H has a perfect matching, call it M. Consider the edge set

$$E = \{\{u_i, w_j\} : 1 \le i \le p,\ j = 2i-1 \text{ or } 2i\} ,$$

and form a graph G by removing all edges in $E \cup M$ from $G^4(n)$. Then G is an $(n-p-2)$-regular subgraph of $G^4(n)$, hence $\phi(n, n-p-2; r) = 1$, and $\phi(n, d; r) = 1$ for $d \le n-p-2$ by Lemma 5.2.

Suppose now that p is even. Then $n = 4p+3$ is odd and $n-p-2$ is also odd, so $d = n-p-2$ is forbidden by (2.3). It suffices therefore to show that $\phi(n, n-p-3; r) = 1$ in this case, for then Lemma 5.2 gives $\phi(n, d; r) = 1$ for $d \le n-p-3$, hence also for $d \le n-p-2$. We use the vertex labelling (5.12), and let H be the subgraph of $G^4(n)$ induced on $\{w_j : 1 \le j \le 3p\}$. Then $|V(H)| = 3p$ is even, and H is Hamiltonian, so H has a perfect matching M. Consider the edge set

$$E = \{\{u_i, w_j\} : 1 \le i \le p,\ j = 3i-2,\ 3i-1 \text{ or } 3i\} \cup \{\{w_{3p+1}, w_{3p+2}\},\ \{w_{3p+2}, w_{3p+3}\},\ \{w_{3p+3},\ w_{3p+1}\}\} .$$

Form a graph G by removing $E \cup M$ from $G^4(n)$. It is an $(n-p-3)$-regular graph, whence $\phi(n, n-p-3; r) = 1$.

Induction Step. Fix $r \ge 5$ and define

$$d_0 := d_0(n, r) = \max\{d : d \le n-p-2,\ nd \text{ is even}\} ,$$

so $d_0 = n-p-2$ or $n-p-3$. It is enough to show that $\phi(n, d_0; r) = 1$, for Lemma 5.2 then yields $\phi(n, d; r) = 1$ for $d \le n-p-2, nd$ even.

To do this, set

$$n' = n - 2(p+1) = p(r-2) + q - 2 ,$$

where $q = r-1$ so $q-2 > 0$. Then $d_1 = d_0(n, r) - n'$ has $0 \le d_1 \le p$, and furthermore we easily check that

$$d' := d_0(n', r-2) = d_0(n, r) - 2(p+1) . \qquad (5.13)$$

Take $r' = r-2$, whence $q' = q-2 = r'-1$. We may apply the induction hypothesis at $r' = r-2$ to conclude that there exists a d'-regular $(r-2)$-partite graph $G = G(n', d'; r-2)$. Let H be a d_1-regular bipartite graph with $2(p+1)$ vertices disjoint from those of G; such a graph H

exists by Lemma 5.1(a). Take the disjoint union of G and H and add in all edges between $V(G)$ and $V(H)$ to obtain a new graph G' on n vertices which is $d_0(n,r)$-regular, according to (5.13). Thus $\phi(n, d_0; r) = 1$, completing the induction step for (iii). ■

6. Minimal Interference Level: Theorems 2.3 and 2.4

In this section we study the range $\gamma \geq 3$ and prove Theorems 2.3 and 2.4. The cases where $\ell(n, \alpha, d) = 0$, i.e. for d smaller than about $n - p$, follow from Theorem 1.1 applied with $r = \gamma$. For the remaining cases, the harder step in the proofs is obtaining the (exact) lower bounds for $\ell(n, \alpha, d)$. The upper bounds are obtained by explicit construction.

We proceed to derive a lower bound for $\ell(n, \alpha, d)$ stated as Lemma 6.2 below. Let G be any d-regular graph on n-vertices, let $f : V(G) \to \{1, 2, \ldots, n\}$ be a given coloring of G, and let α also be given. We begin by partitioning the n colors into γ groups $\{\tilde{A}_i : 1 \leq i \leq \gamma\}$, such that each group $\tilde{A}_i$ consists of consecutive colors and the groups $\tilde{A}_1, \ldots, \tilde{A}_\gamma$ are themselves consecutively arranged with respect to the cyclic ordering of colors (mod n), with all groups but $\tilde{A}_1$ containing exactly $\alpha + 1$ colors, and $\tilde{A}_1$ contains the remaining $\alpha + 1 + m$ colors. Here m is given by

$$n = \gamma(\alpha + 1) + m, \text{ with } 0 \leq m < \alpha + 1 \ , \tag{6.1}$$

and such a partition is completely determined by the choice of $\tilde{A}_1 = \{i, i+1, \ldots, i+\alpha+1+m\}$. We now choose $\tilde{A}_1$ so as to minimize the number of vertices v in G that are assigned colors $f(v)$ in $\tilde{A}_1$. After doing this, we have the freedom to cyclically relabel the colors (via the map $\phi_\ell(j) = j + \ell (\text{mod } n)$) without affecting which edges have vertex colors that interfere. We use this freedom to specify that

$$\tilde{A}_1 := \{-m, -m+1, \ldots, \alpha - 1, \alpha\} \ ,$$

in which case

$$\tilde{A}_i := \{j : (i-1)(\alpha+1) \leq j < i(\alpha+1)\} \quad \text{for} \quad 2 \leq i \leq \gamma \ :$$

see Figure 6.1. Notice that for $2 \leq i \leq \gamma$ any two colors in $\tilde{A}_i$ interfere with each other.

Figure 6.1 about here

This partition of the colors induces a corresponding partition of the vertices of G into the color classes

$$A_i := \{v \in G : f(v) \in \tilde{A}_i\}, \ 1 \leq i \leq \gamma \ . \tag{6.2}$$

Now set

$$a_i := |A_i| \ .$$

We now count the edges in G and in its complement $\bar{G} = K_n - G$ in various ways. For any two subsets V and W of vertices, let $e(V,W)$ count the number of edges between vertices in V and those in W, and let $\bar{V} := V(G) \setminus V$. Let $\bar{a}_{i,j}$ count the number of edges between A_i and A_j that are *not* in G, which is

$$\bar{a}_{i,j} := a_i a_j - e(A_i, A_j) \ , \quad 1 \leq i, \ j \leq \gamma \ .$$

Along with this we define

$$\bar{a}_i := \Sigma_{j \neq i} \bar{a}_{i,j} = a_i(n - a_i) - e(A_i, \bar{A}_i) \ , \quad 1 \leq i \leq \gamma \ .$$

The d-regularity of G then yields

$$e(A_i, A_i) = \frac{1}{2}(da_i - e(A_i, \bar{A}_i)) = \frac{1}{2}(a_i(d + a_i - n) + \bar{a}_i) \ . \tag{6.3}$$

The *potential interfering edge set* $B_{i,j}$ between vertices in A_i and those in A_j is

$$B_{i,j} := B_{i,j}(G,f) = \{\{v,w\} \in E(K_n) : v \in A_i, w \in A_j, \text{with } D(f(v), f(w)) \leq \alpha\} \ .$$

The *actual interfering edge set* is $E(G) \cap B_{i,j}$ and we set

$$c_{i,j} := |E(G) \cap B_{i,j}| \ .$$

We clearly have

$$\bar{a}_{i,j} + c_{i,j} \geq |B_{i,j}| \ . \tag{6.4}$$

Finally, let δ^* and δ count the potential and actual non-interfering edges in A_1, respectively, i.e.

$$\begin{aligned} \delta^* &:= |\{\{v,w\} \in E(K_n) : v, w \in A_1 \text{ and } D(f(v), f(w)) \geq \alpha + 1\}| \ , \\ \delta &:= |\{\{v,w\} \in E(G) : v, w \in A_1 \text{ and } D(f(v), f(w)) \geq \alpha + 1\}| \ . \end{aligned}$$

Certainly $\delta^* \geq \delta$. Since all edges between the vertices in the same component A_i interfere, except for δ edges in A_1, we obtain the bound

$$\begin{aligned} I_\alpha G, f) &= \Sigma_{i<j} c_{i,j} + \Sigma_{i=1}^{\gamma} e(A_i, A_i) - \delta \\ &\geq \Sigma_{i<j} c_{i,j} + \Sigma_{i=1}^{\gamma} e(A_i, A_i) - \delta^* \ . \end{aligned} \tag{6.5}$$

To bound this further, we need the following bounds for edges connecting a vertex in the color set $\tilde{A}_1$ to a vertex in its two neighboring color sets $\tilde{A}_2$ and $\tilde{A}_\gamma$.

LEMMA 6.1. *We have*

$$\bar{a}_{1,2} + c_{1,2} \geq \delta^* \ , \tag{6.6}$$

and

$$\bar{a}_{1,\gamma} + c_{1,\gamma} \geq \delta^* \ . \tag{6.7}$$

PROOF. We start with (6.6). By (6.4) it is enough to show that

$$|B_{1,2}| \geq \delta^* \ .$$

It suffices to show for fixed $v \in A_1$ with $\alpha + 1 - m \leq f(v) \leq \alpha$ that

$$|\{w \in A_2 : D(f(v), f(w)) \leq m\}| \geq |\{w \in A_1 : D(f(v), f(w)) \geq \alpha + 1\}| \ , \tag{6.8}$$

because, using $\alpha \geq m$, this implies that, for sums over $v \in A_1$ with $\alpha + 1 - m \leq f(v) \leq \alpha$,

$$\begin{aligned} |B_{1,2}| &\geq \Sigma_v |\{w \in A_2 : D(f(v), f(w)) \leq m\}| \\ &\geq \Sigma_v |\{w \in A_1 : D(f(v), f(w)) \geq \alpha + 1\}| = \delta^* \ . \end{aligned}$$

To prove (6.8), given $v \in A_1$ with $\alpha + 1 - m \leq f(v) \leq \alpha$, we define the vertex set

$$A' := \{w \in V(G) : f(w) \in \{f(v) - \alpha, f(v) - \alpha + 1, \ldots, f(v) + m\}\} \subseteq A_1 \cup A_2 \ .$$

This is a set of $\alpha + 1 + m$ consecutive colors, hence $|A'| \geq |A_1|$ by the minimizing property of the color set $\tilde{A}_1$. Now $\alpha + 1 - m \leq f(v) \leq \alpha$ implies that

$$A_1 \cap A' = \{w \in V(G) : f(w) \in \{f(v) - \alpha, \ldots, \alpha\}\} \ .$$

Thus

$$|A' \setminus (A_1 \cap A')| \geq |A_1 \setminus (A_1 \cap A')|$$

which is exactly (6.8). Thus (6.6) follows.

The proof of (6.7) is analogous. ■

To state the lower bound lemma, recall that the quantities p and q are defined by

$$n = p\gamma + q \text{ with } 0 \leq q < \gamma \ ,$$

so $p = \lfloor \frac{n}{\gamma} \rfloor$.

LEMMA 6.2. *If $d \geq n - p$ then*

$$\ell(n, \alpha, d) \geq q\lceil (p+1)(d+p+1-n)/2 \rceil + (\gamma - q)\lceil p(d+p-n)/2 \rceil \ . \tag{6.9}$$

PROOF. We derive this result from the general bound

$$I_\alpha(G, f) \geq \Sigma_{i=1}^{\gamma} \lceil a_i(d + a_i - n)/2 \rceil \ , \tag{6.10}$$

where $a_i = |A_i|$ for the vertex partition (6.2). To establish (6.10), we first note that Lemma 6.1 yields

$$\frac{1}{2}(\bar{a}_{1,2} + \bar{a}_{1,\gamma}) + c_{1,2} + c_{1,\gamma} \geq \delta^* \ .$$

Together with (6.3), this yields

$$\begin{aligned} e(A_1, A_1) - \delta^* + c_{1,2} + c_{1,\gamma} \ &\geq \ e(A_1, A_1) - \tfrac{1}{2}(\bar{a}_{1,2} + \bar{a}_{1,\gamma}) \\ &\geq \ \tfrac{1}{2} a_1(d - a_1 - n) \ . \end{aligned}$$

Since the left side of this inequality is an integer,

$$e(A_1, A_1) - \delta^* + c_{1,2} + c_{1,\gamma} \geq \lceil a_1(d - a_1 - n)/2 \rceil \ .$$

However, (6.3) also gives

$$e(A_i, A_i) \geq \lceil a_i(d + a_i - n)/2 \rceil \ , \quad \text{for } 2 \leq i \leq \gamma \ .$$

Substituting these bounds in (6.5) yields (6.10).

To derive (6.9), we minimize the right side of (6.10) over all possible values: $a_i \geq 0$ subject to $\Sigma_{i=1}^{\gamma} a_i = n$. It is easy to verify that this occurs when all the a_i's are as equal as possible, i.e.

$$\begin{cases} q \text{ of the } a_i \text{ take the value } p+1, \\ \gamma - q \text{ of the } a_i \text{ take the value } p \ . \end{cases} \tag{6.11}$$

Thus

$$I_\alpha(G, f) \geq q\lceil (p+1)(d+p+1-n)/2 \rceil + (\gamma - q)\lceil p(d+p-n)/2 \rceil \ ,$$

which gives (6.9). ■

PROOF OF THEOREM 2.3. (a) This bound follows from Theorem 1.1, taking $r = \gamma$ noting that $q = 0$ guarantees that r doesn't divide $n + 1$.

(b) For $d > n - p$ we first establish the lower bounds

$$\ell(n, \alpha, d) \geq \frac{n(d-n+p)}{2} + \mu \tag{6.12}$$

where

$$\mu = \begin{cases} 0 & \text{if } n-d \text{ is odd or if } n-d \text{ is even} \\ & \text{and } p \text{ is even,} \\ \frac{\gamma}{2} & \text{if } n-d \text{ is even and } p \text{ is odd ,} \end{cases} \tag{6.13}$$

using Lemma 6.2. The case $q = 0$ is $n = p\gamma$, so (6.9) simplifies to

$$\begin{aligned} \ell(n, \alpha, d) &\geq \gamma \lceil p(d-n+p)/2 \rceil \\ &= \tfrac{n}{p} \lceil p(d-n+p)/2 \rceil \end{aligned}$$

Now (6.12) follows on determining the cases for which $p(d-n+p)$ is odd.

To show that this bound is attained, we simply construct the graph G with the coloring f that makes (6.11) hold. The constructions are easy and are left to the reader. ■

Proof of Theorem 2.4. (a) The bounds where $\ell(n, \alpha, d) = 0$ follow from Theorem 1.1 with $r = \gamma$.

There remains the case in which $d = n - p - 1$ and $q = \gamma - 1$, i.e. when $n = \lfloor \frac{n}{\gamma} \rfloor \gamma + \gamma - 1$ (where Theorem 1.1 does not apply). We must show that

$$\ell(n, \alpha, n-p-1) = \frac{p}{2} .$$

For the upper bound $\ell \leq \frac{p}{2}$, it suffices to construct an appropriate graph. Note first that p must be even since if p is odd then $n = p\gamma + \gamma - 1 \equiv (p+1)\gamma - 1$ is odd and $d = n - p - 1$ is also odd, contradicting (2.3). Now consider the equi-γ-partite graph $G^\gamma(n)$ defined in the proof of Theorem 1.1. We take a perfect matching M from the induced subgraph of $G^\gamma(n)$ on the vertex set $(X_{\gamma-1} \setminus \{v_n\}) \cup X_\gamma$. We remove all the edges in M from $G^\gamma(n)$ and add the edges $\{v_{\gamma-1}, v_{2\gamma-1}\}, \{v_{3\gamma-1}, v_{4\gamma-1}\}, \ldots, \{v_{(p-1)\gamma-1}, v_{p\gamma-1}\}$. Then it is straightforward to check that the resulting graph G is $(n-p-1)$-regular and it clearly has exactly $\frac{p}{2}$ interfering edges when the sets X_i are colored with γ mutually noninterfering colors.

To show the lower bound $\ell \geq \frac{p}{2}$, let G be an $(n-p-1)$-regular graph and f an n-coloring of $V(G)$ such that

$$I_\alpha(G, f) = \ell(n, \alpha, n-p-1) .$$

Take the partition $\{A_i : 1 \leq i \leq \gamma\}$ of $V(G)$ associated to f constructed at the beginning of this section. We consider cases.

Case (i). $a_1 \geq p+2$.

The minimality property of A_1 implies that, for all $v \in V(G)$,

$$|\{w \in V(G) : f(w) = f(v) + j \pmod n \text{ with } -m \leq j \leq \alpha\}| \geq p+2 .$$

Since $d = n-p-1$, for each $v \in V(G)$ there exists $w \in V(G) \setminus \{v\}$ such that $|f(v) - f(w)| \leq \alpha$. Thus $I_\alpha(G,f) \geq n/2 > p/2$.

Case (ii). $a_i \geq p+2$ *for some* $2 \leq i \leq \gamma$.

Here the equality in (6.5) combined with $e(A_1, A_1) \geq \delta$ yields

$$I_\alpha(G,f) \geq \Sigma_{i=2}^{\gamma} e(A_i, A_i) . \tag{6.14}$$

Using (6.3) we then have

$$I_\alpha(G,f) \geq e(A_i, A_i) \geq \frac{1}{2} a_i (d + a_i - n) \geq \frac{p+2}{2} > \frac{p}{2} .$$

Case (iii). All $a_i \leq p+1$.

Since $n = (p+1)\gamma - 1$, this case requires that $\gamma - 1$ of the a_i equal $p+1$ and one a_i equals p.

Suppose first that $a_1 = p$. Observe that (6.14) and (6.3) yield

$$\begin{aligned} I_\alpha(G,f) &\geq \Sigma_{i=2}^{\gamma} e(A_i, A_i) \\ &\geq \tfrac{1}{2} \Sigma_{i=2}^{\gamma} \bar{a}_i \geq \tfrac{1}{2} \Sigma_{i=2}^{\gamma} \bar{a}_{i,1} . \end{aligned} \tag{6.15}$$

Now (6.2) and $a_1 = p$ give

$$\begin{aligned} \Sigma_{i=2}^{\gamma} \bar{a}_{i,1} &= \bar{a}_1 = (a_1(n - a_1) - e(A_1, \bar{A}_1)) \\ &\geq p(n-p) - pd = p . \end{aligned}$$

Substituting this in (6.15) gives $I_\alpha(G,f) \geq \frac{p}{2}$.

Suppose finally that $a_1 = p+1$. Since $\gamma \geq 3$, and only one $a_i = p$, either $a_2 = p+1$ or $a_\gamma = p+1$ or both. We treat only the case that $a_2 = p+1$, since the argument for $a_\gamma = p+1$ is similar. Let $a_{i_0} = p$. Now by (6.5) and (6.3)

$$\begin{aligned} I_\alpha(G,f) &\geq \tfrac{1}{2}(\bar{a}_1 + \bar{a}_2) + c_{1,2} - \delta^* + \tfrac{1}{2} \textstyle\sum_{\substack{i=3 \\ i \neq i_0}}^{\gamma} \bar{a}_i \\ &\geq \tfrac{1}{2}(2\bar{a}_{1,2} + \bar{a}_{1,i_0} + \bar{a}_{2,i_0}) + c_{1,2} - \delta^* + \tfrac{1}{2} \textstyle\sum_{\substack{i=3 \\ i \neq i_0}}^{\gamma} \bar{a}_i . \end{aligned}$$

Lemma 6.1 gives $\bar{a}_{1,2} + c_{1,2} \geq \delta^*$, hence

$$\begin{aligned} I_\alpha(G,f) &\geq \tfrac{1}{2}(\Sigma_{i \neq i_0} \bar{a}_{i,i_0}) = \tfrac{1}{2} \bar{a}_{i_0} \\ &\geq \tfrac{1}{2}(p(n-p) - pd) = \tfrac{p}{2} , \end{aligned}$$

completing case (iii).

(b) We start from the formula (6.9) of Lemma 6.2, which gives a lower bound. We claim that equality occurs. This formula of $\ell(n, \alpha, d)$ splits into several cases, according to when $(p+1)(d+p+1-n)/2$ and $p(d+p-n)/2$ are integers or half-integers, and consideration of the parities of $n-d$ and p leads to the formulas for θ in (2.12).

For the upper bound, obtaining equality in the formula for $\ell(n, \alpha, d)$ requires (6.11) to hold, and this easily determines the construction of a suitable graph G and a coloring f. We omit the details. ∎

7. Minimax Interference Level: Theorems 2.5 and 2.6

We conclude by proving the bounds for $L(n, \alpha, d)$ stated in Section 2.

Proof of Theorem 2.5. To show part (a), the condition $L(n, \alpha, d) = 0$ certainly holds if the chromatic number $\chi_G \leq \gamma$ for all $G \in \mathcal{G}(n, d)$. This holds for $\gamma > d$ by Brooks' theorem (Proposition 3.1). For the case $\gamma = d \geq 3$ we use the strong version of Brooks' theorem, which states that $\chi_G \leq \Delta_G$ if no component of G is an odd cycle or a complete subgraph. Here $\Delta_G = d$, and $d \geq 3$ implies there are no odd cycles, while the condition $n < 2(d+1)$ prohibits any connected component being the complete subgraph K_d, for any other components must be d-regular but have at most d vertices, a contradiction.

To show part (b), suppose that $n \geq 2(d+1)$. Let $G \in \mathcal{G}(n, d)$ consist of a complete graph K_{d+1} plus a d-regular graph G' on the other $n-(d+1) \geq d+1$ vertices. If d is odd then n is even, so that $n-(d+1)$ is even, and the existence of G' is assured by a theorem of Erdös and Gallai (1960) for simple graphs with specified degree sequences. If $\gamma \leq d$, at least two vertices of K_{d+1} interfere, so $L > 0$.

Suppose that $n = 2(d+1-a)$, $a \geq 1$. This implies $d \geq 2a$ because we presume that $n \geq d+2$. Let G consist of two disjoint copies of K_{d+1-a}, adding edges between them that increase every degree to d. Each vertex requires a such edges, and this is feasible because $d+1-a > a$. If $\gamma \leq d-a$, at least two vertices of K_{d+1-a} interfere, so $L > 0$.

Suppose finally that $n = 2(d+1-a)+1$, $a \geq 1$. Then n is odd, so d must be even. Moreover, $n \geq d+2 \Rightarrow n \geq d+3 \Rightarrow d \geq 2a$. Let G consist of two disjoint graphs $G_1 = K_{d+1-a}$ and $G_2 = K_{d+2-a}$ with edge additions and deletions as follows. Add a edges from each G_1 vertex to G_2 vertices in as equal a way as possible for resulting vertex degrees in G_2. Then each vertex in G_1 has degree d, x vertices in G_2 have degree $d+1$, and y vertices in G_2 have degree d, where

$$x + y = d + 2 - a \ ,$$

$$xa + y(a-1) = a(d+1-a) \ .$$

These equations imply that $x = d + 2 - 2a > 0$, so x is even. We then remove $x/2$ edges within G_2 so that all vertices have degree d. We thus arrive at a graph $G \in \mathcal{G}(n,d)$. If $\gamma \le d - a$ then at least two vertices in G_1 interfere, so $L > 0$. Thus part (b) holds. ■

Proof of Theorem 2.6. Suppose that $3 \le d \le n-1$ and that $\gamma \le d$. Let P, Q, U and W be nonnegative integers that satisfy

$$\begin{aligned} d+1 &= P\gamma + Q, \quad 0 \le Q < \gamma \ ; \\ n &= (d+1)U + W, \quad 0 \le W < d+1 \ . \end{aligned}$$

To derive the upper bound on L in Theorem 2.6, let G be any graph in $\mathcal{G}(n,d)$. Let S denote the family of all partitions of the vertex set of G into γ groups, with q groups of size $p+1$ and $\gamma - q$ groups of size p. We adopt a probability model for S that assigns probability $1/|S|$ to each partition. Whichever partition obtains, we use γ mutually noninterfering colors for the γ groups in the partition. Suppose $\{u,v\}$ is an edge in G. The probability that u and v lie in the same part of a member of S, so that $\{u,v\}$ is an interference edge, is

$$\frac{q\binom{p+1}{2} + (\gamma - q)\binom{p}{2}}{\binom{n}{2}} = \frac{n(n-\gamma) + q(\gamma - q)}{\gamma n(n-1)} \ .$$

The expected number $E[I]$ of interference edges is $nd/2$ times this amount, i.e.,

$$E[I] = \frac{d(n(n-\gamma) + q(\gamma - q))}{2\gamma(n-1)} \ ,$$

so some member of S has a coloring that gives less than or equal to $E[I]$ edges whose vertices interfere. This is true for every $G \in \mathcal{G}(n,d)$. Therefore we get the upper bound

$$L(n,\alpha,d) \le \frac{d}{2\gamma(n-1)}[n(n-\gamma) + q(\gamma - q)] \ .$$

For the lower bound, assume initially that $(d+1)$ divides n, so $W = 0$ and $U = n/(d+1)$. Let G consist of U disjoint copies of K_{d+1}. Then $L(n,\alpha,d) \ge UL(d+1,\alpha)$, where $L(d+1,\alpha)$ is the minimum number of interfering edges in K_{d+1} for an $f : V_{d+1} \to [n]$. The analysis in Lemma 4.1 shows that $L(d+1,\alpha)$ is attained by an equi-γ-partition of V_{d+1} with f constant in each part. Since an equi-γ-partition of V_{d+1} has

$$\begin{cases} Q \text{ groups of } P+1 \text{ vertices each,} \\ \gamma - Q \text{ groups of } P \text{ vertices each,} \end{cases}$$

we have

$$\begin{aligned} L(d+1,\alpha) &= [Q(P+1)P + (\gamma - Q)P(P-1)]/2 \\ &= \frac{(d+1)(d+1-\gamma) + Q(\gamma - Q)}{2\gamma} \ . \end{aligned}$$

Since $L(n,\alpha,d) \geq UL(d+1,\alpha)$, this gives

$$L(n,\alpha,d) \geq \frac{n(d+1-\gamma)}{2\gamma} + \frac{nQ(\gamma - Q)}{2\gamma(d+1)} \ ,$$

when $d+1$ divides n.

Suppose $(d+1) \nmid n$ with $n = (d+1)U + W$, where $U = \left\lfloor \frac{n}{d+1} \right\rfloor$ and $0 < W \leq d$. To form G we begin with U disjoint copies of K_{d+1} and a disjoint K_W. Each vertex in K_W needs $d-(W-1)$ more incident edges, so we add a total of $W(d+1-W)$ edges between K_W and the K_{d+1} in such a way that $W(d+1-w)/2$ edges can be removed from within the K_{d+1} to end up with degree d for every vertex. Note that $W(d+1-W)$ is even, for otherwise both n and d would be odd. We ignore possible interference within K_W and allow for the possibility that every edge removed from the K_{d+1} is an interference edge to get the lower bound

$$\begin{aligned} L(n,\alpha,d) &\geq UL(d+1,\alpha) - W(d+1-W)/2 \\ &= \frac{\left\lfloor \frac{n}{d+1} \right\rfloor}{2\gamma}[(d+1)(d+1-\gamma) + Q(\gamma - Q)] - \frac{W(d+1-W)}{2} \ . \end{aligned}$$

■

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Figure 1.1.

Figure 1.2. Zero and positive regions.

Figure 2.1. $\gamma = 2$, n odd, $n - 2\alpha < d < n/2$.

Figure 4.1.

Figure 6.1. Color set partition.