Multivariate Descartes' Rules of Signs and Sturmfels's Challenge Problem

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Abstract

Descartes' rule of signs bounds the number of real roots of a polynomial $f(x)$ in terms of the sign pattern of its nonzero monomials. Recently I. Itenberg and M.-F. Roy conjectured a multidimensional generalization of Descartes' rule of signs, which, if true, would be best possible. B. Sturmfels circulated a special case of this conjecture as a challenge problem. This paper describes the Itenberg-Roy conjecture and solves Sturmfels's challenge problem.

Multivariate Descartes' Rules of Signs and Sturmfels's Challenge Problem

Jeffrey C. Lagarias *and* *Thomas J. Richardson*

Descartes' rule of signs bounds the number of positive real zeros $n^+(f)$ of a polynomial $f(x)$ in one variable. If

$$f(x) = \sum_{j=1}^{r} c_j x^{m_j} ,$$

with $0 \le m_1 < m_2 < \ldots < m_r$ and with all coefficients $c_j \ne 0$, then the number of positive real zeros of f is upper bounded by the number of sign changes $N^+(f)$ between consecutive coefficients c_j when taken in order of increasing j, see [2], [5, Chapter 6]. There is a similar upper bound $N^-(f)$ for the number of negative real zeros of f which follows by applying this bound to the polynomial $f(-x)$. Together these give an upper bound $N(f) = N^+(f) + N^-(f)$ for the total number of zeros in $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. For any set $\{m_i : 1 \le i \le r\}$ one can construct polynomials $f(x)$ in which the upper bounds $N^+(f)$ and $N(f)$ both hold with equality.

It immediately follows from Descartes' rule of signs that the number of nonzero real roots $n(f)$ of f is bounded in terms of the number of monomials r appearing in f, namely

$$n(f) \le 2(r-1) .$$

This bound does not depend on the degrees of these monomials. In contrast, the number of nonzero complex roots of f is exactly $m_r - m_1$, by the fundamental theorem of algebra.

In the 1970's A. G. Kushnirenko raised the problem of bounding the number of real roots in $(\mathbb{R}^*)^n$ of a system of multivariate polynomials purely in terms of the monomials appearing in the various polynomials. Call a system $F = \{f_1, \ldots, f_n\}$ of n polynomials in n variables

$$f_i(x_1, x_2, \ldots, x_n) = 0 , \quad 1 \le i \le n$$

a *regular system* if its zeros in $\mathbb{C}^n$ are isolated and nondegenerate, that is, if the system F is a complete intersection. (Here *nondegenerate* means that the Jacobian determinant of the system is nonvanishing at each zero.) Let $\mathcal{M}_i$ denote the set of monomials appearing in $f_i(x_1, \ldots x_n)$. Kushnirenko [10, p. 123] conjectured that the number of roots of a complete intersection $F(\mathbf{x}) =$

$\mathbf{0}$ in $(\mathbb{R}^*)^n$ can be bounded purely in terms of the cardinality $S_i := |\mathcal{M}_i|$, by

$$\prod_{i=1}^{n}(S_i - 1) \ .$$

This conjecture remains open. In 1980 A. G. Khovanskii [10] obtained a weaker upper bound depending only on the S_i, namely

$$(n+2)^S 2^{S(S+1)/2} \ ,$$

in which S is the total number of distinct monomials appearing in all the $p_i(x_1, \ldots, x_n)$, so that $S \leq \sum_{i=1}^{n} S_i$. Bounds on the number of real zeros have applications in computational complexity theory, see Risler [14].

In studying these questions it is natural to look for a multivariate analogue of Descartes' rule of signs. Consider, for a regular system with real coefficients, its sets of monomials $\mathcal{M} := \{\mathcal{M}_i : 1 \leq i \leq n\}$ and the pattern of signs $\boldsymbol{\epsilon} = \{\epsilon_i : 1 \leq i \leq n\}$ attached to its coefficients; each $\epsilon_i : \mathcal{M}_i \to \{+1, -1\}$. Set $\tilde{\Delta}_i = (\mathcal{M}_i, \epsilon_i)$, and let $N^+(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n)$ and $N(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n)$ denote the maximal possible number of roots in the positive orthant $(\mathbb{R}^+)^n$ and in $(\mathbb{R}^*)^n$, respectively. In view of Khovanskii's bound these numbers are well-defined and finite. A multivariate Descartes' rule would be algebraic formulae for the numbers $N^+(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n)$ and $N(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n)$. Recent developments in real algebraic geometry due to O. Viro [18] have suggested a possible answer. Sturmfels [17] developed formulas for Viro's method applied to regular systems. Using these, I. Itenberg and M.-F. Roy [9] produced conjectural explicit combinatorial formulae for the values of $N^+(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n)$ and $N(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n)$. These formulae are known to be correct in a few special cases. To raise awareness of the Itenberg-Roy conjecture B. Sturmfels proposed as a challenge problem a special case, and offered a \$500 reward for its solution, see Figure 4.

In this paper we give an overview of the Itenberg-Roy conjecture and then present a solution to Sturmfels's challenge problem. The combinatorial invariants in the Itenberg-Roy conjecture involve triangulations of the Newton polytopes Δ_i attached to the polynomials $f_i(x_1, \ldots, x_n)$. We first describe Newton polytopes, and their relation to the complex zeros in $(\mathbb{C}^*)^n$ of regular systems. In the following section we describe the Itenberg-Roy conjecture in the special case $n = 2$ (to reduce notation), and in the final section we solve Sturmfels's problem.

Fewnomials and their Complex Zeros

Polynomials described in terms of their constituent monomials are called *fewnomials* or *sparse polynomials*, see [6], [11]. (The term *fewnomial* is due to Kushnirenko.). A monomial $x^{\mathbf{m}} = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ is indexed by the integer vector $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$. The *sparse representation* of a polynomial $f(\mathbf{x})$ is the data $\{c(\mathbf{m}) : \mathbf{m} \in \mathcal{M}\}$, where

$$f(\mathbf{x}) = \sum_{\mathbf{m} \in \mathcal{M}} c(\mathbf{m}) x^{\mathbf{m}} ,$$

and all coefficients $c(\mathbf{m}) \neq 0$ for $\mathbf{m} \in \mathcal{M}$. The set $\mathcal{M} := \mathcal{M}[f] \subseteq \mathbb{Z}^n_{\geq 0}$ indexes the monomials present in $f(\mathbf{x})$.

A fundamental invariant of a fewnomial is its *Newton polytope* $\Delta = \Delta(\mathcal{M})$, which is the convex hull of the points of $\mathcal{M}$. This polytope is a *lattice polytope* in $\mathbb{R}^n$, i.e. all its vertices are in $\mathbb{Z}^n$. Figure 1 pictures the Newton polytopes Δ_1, Δ_2 for the system

$$\begin{aligned} f_1(x_1, x_2) &= 1 + x_1 x_2 + 3 x_1^2 x_2^2 , \\ f_2(x_1, x_2) &= 1 - x_1 + 2 x_1 x_2 . \end{aligned}$$

This example illustrates that Newton polytopes may be of lower dimension than n.

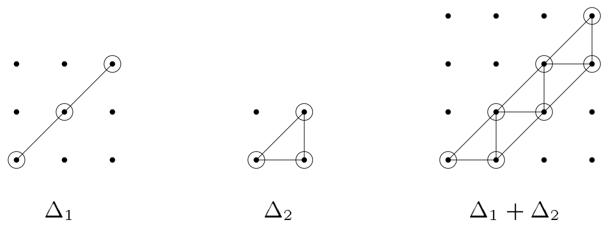

Figure 1: Newton polytopes

In 1975 Bernstein [1] and Kushnirenko [12] obtained upper bounds for the number of solutions in $(\mathbb{C}^*)^n$ of regular systems in terms of the mixed volumes of their associated Newton polytopes. Mixed volumes can be defined in terms of volumes of Minkowski sums of polytopes. The *Minkowski sum* $\Delta_1 + \Delta_2$ of two polytopes is

$$\Delta_1 + \Delta_2 := \{\mathbf{x}_1 + \mathbf{x}_2 : \mathbf{x}_1 \in \Delta_1, \mathbf{x}_2 \in \Delta_2\} ,$$

Figure 1 pictures the Minkowski sum Δ_1 and Δ_2 for the example above. The Minkowski sum of n polytopes is defined similarly. (This definition makes sense for arbitrary convex bodies Δ_1, Δ_2, see Schneider [13].) We define the *combinatorial mixed volume* $V(\Delta_1, \ldots, \Delta_n)$ by the formula

$$V(\Delta_1, \Delta_2, \ldots, \Delta_n) := \sum_{k=1}^{n} (-1)^{n+k} \sum_{i_1 < i_2 < \ldots < i_k} vol_n(\Delta_{i_1} + \ldots + \Delta_{i_k}) \ , \tag{1}$$

in which $vol_n(\cdot)$ denotes n-dimensional volume. In particular $V(\Delta, \Delta, \ldots, \Delta) = n!\, vol_n(\Delta)$. The combinatorial mixed volume differs by a factor $n!$ from the usual Minkowski definition of mixed volume, see [13, Lemma 5.1.3].

Theorem (Bernstein). *The number of isolated zeros in $(\mathbb{C}^*)^n$ of a system of n polynomials with complex coefficients*

$$f_i(x_1, \ldots, x_n) = 0, \quad 1 \leq i \leq n \tag{2}$$

is at most the combinatorial mixed volume $V(\Delta_1, \Delta_2, \ldots, \Delta_n)$, where Δ_i is the Newton polytope of $f_i(x_1, \ldots, x_n)$.

Bernstein [1] also observed that a system of "general position" polynomials (2) having Newton polyhedra $\Delta_1, \ldots, \Delta_n$ is a regular system and has *exactly* $V(\Delta_1, \ldots, \Delta_n)$ isolated zeros in $(\mathbb{C}^*)^n$. In particular $V(\Delta_1, \ldots, \Delta_n)$ is an integer.[1]

One important special case is where each $f_i(x_1, \ldots, x_n)$ is a dense polynomial of total degree d_i, in which $\mathcal{M}_i = \{\mathbf{m} : \sum_{j=1}^{n} m_j \leq d_i\}$. In this case a calculation reveals that

$$V(\Delta_1, \ldots, \Delta_n) = d_1 d_2 \ldots d_n \ ,$$

which is the upper bound given by Bezout's theorem for the number of zeros of the system (2) in $\mathbb{C}^n$.

In the two-dimensional case, the combinatorial mixed volume formula (1) becomes

$$V(\Delta_1, \Delta_2) = \text{area}(\Delta_1 + \Delta_2) - \text{area}(\Delta_1) - \text{area}(\Delta_2) \ .$$

For the example in Figure 1, we have $\text{area}(\Delta_1) = 0$, $\text{area}(\Delta_2) = \frac{1}{2}$, $\text{area}(\Delta_1 + \Delta_2) = \frac{5}{2}$, so that the Newton polytope bound $V(\Delta_1, \Delta_2) = 2$. It is easy to check that this system has no zeros with $x_1 = 0$ or $x_2 = 0$, hence all zeros in $\mathbb{C}^2$ are in $(\mathbb{C}^*)^2$. The Newton polytope bound for this case strictly improves on the Bezout bound, which is 8.

[1] Each Minkowski sum $\Delta_{i_1} + \Delta_{i_2} + \ldots + \Delta_{i_k}$ on the right side of (1) is a lattice polytope, hence its n-dimensional volume is of the form $\frac{k}{n!}$ for some integer k. This however only guarantees that $n! V(\Delta_1, \ldots, \Delta_n)$ is an integer.

The bound of Theorem 2.1 can be extended to a bound on isolated zeros in $\mathbb{C}^n$, see Rojas [15] and Huber and Sturmfels [7].

Multivariate Descartes' Rule

For comparison purposes, we reformulate Descartes' rule of signs for a univariate polynomial

$$f(x) = \sum_{j=1}^{r} c_j x^{m_r} ,$$

in terms of its signed Newton polytope $\tilde{\Delta}$, which consists of the Newton polytope $\Delta = [m_1, m_j]$, plus all its points $\{m_i : 1 \leq i \leq r\} \subseteq \mathbb{Z}_{\geq 0}$, with a sign $\{\epsilon(m_i) = sign(c_i) : 1 \leq i \leq r\}$. The line segment Δ is subdivided into $r-1$ subintervals $\{J_i = [m_i, m_{i+1}] : 1 \leq i \leq r-1\}$, and each subinterval is assigned a *weight* counting the sign change, namely

$$wt^+(J_i) := \begin{cases} 1 & \text{if} \quad \epsilon(m_i) \neq \epsilon(m_{i+1}) , \\ 0 & \text{if} \quad \epsilon(m_i) = \epsilon(m_{i+1}) , \end{cases}$$

which is

$$wt^+(J_i) = \frac{1}{2}(1 + sign(c_i c_{i+1})) .$$

Then

$$N^+(f) := \sum_{i=1}^{r-1} wt^+(J_i) .$$

The number of negative real zeros $n^-(f)$ is bounded similarly with a bound $N^-(f)$ obtained from the function $f(-x)$, which uses the weights

$$wt^-(J_i) := \frac{1}{2}(1 + (-1)^{m_i + m_{i+1}} sign(c_i c_{i+1})) .$$

Now set $wt(J_i) = wt^+(J_i) + wt^-(J_i)$. We obtain the upper bound $N(f)$ for the total number of zeros in $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ given by

$$N(f) := \sum_{i=1}^{r-1} wt(J_i) .$$

Each weight $wt(J_i)$ takes on a value 0, 1 or 2.

Itenberg and Roy actually determined lower bounds for the number of isolated zeros of special systems of polynomials in $\mathbb{R}^n$. This they did using the method of Viro [19], as extended to apply to complete intersections (see above) by B. Sturmfels [17]. They considered systems

$$f_i(x_1, \ldots, x_n, t) = \sum_{\mathbf{m} \in \mathcal{M}_i} c(\mathbf{m}) t^{\omega_i(\mathbf{m})} \mathbf{x}^{\mathbf{m}} , \quad 1 \leq i \leq n . \tag{3}$$

in which t is a parameter such that t is positive and sufficiently close to 0, and

$$\omega_i : \mathcal{M}_i \to \mathbb{R}^+ , \quad 1 \le i \le n ,$$

are essentially arbitrary functions.

A *signed Newton polytope* $\tilde{\Delta}$ is a set of points $\mathcal{M} \subseteq \mathbb{Z}^n_{\ge 0}$ whose convex hull is Δ, together with an assignment of signs to the points of $\mathcal{M}$, i.e. a function

$$\epsilon : \mathcal{M} \to \{+1, -1\} .$$

Itenberg and Roy obtained lower bounds $n^+(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n, \omega)$ and $n(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n, \omega)$ for the number of roots of (3) in $(\mathbb{R}^+)^n$ and $(\mathbb{R}^*)^n$, respectively, which apply when t is positive and sufficiently close to 0. They then define

$$\begin{aligned} n^+(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n) &:= \max_\omega [n^+(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n, \omega)] , \\ n(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n) &:= \max_\omega [n(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n, \omega)] . \end{aligned}$$

These quantities can be determined by a finite calculation, described below. By definition these quantities are lower bounds for $N^+(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n)$ and $N(\Delta_1, \ldots, \Delta_n)$, i.e.

$$\begin{aligned} N^+(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n) &\ge n^+(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n) , \\ N(\Delta_1, \ldots, \Delta_n) &\ge n(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n) . \end{aligned}$$

The multivariate Descartes' rule conjectured by Itenberg and Roy is that equality always holds.

Itenberg-Roy Conjecture. *For any set of signed Newton polytopes in $\mathbb{R}^n$,*

$$\begin{aligned} N^+(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n) &= n^+(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n) \\ N(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n) &= n(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n) . \end{aligned}$$

This conjecture is based on the belief that the Viro construction includes extremal systems. It is known that there are arrangements of zeros not produced by any Viro construction (3) in the limit as parameter values $t \to 0^+$, in some analogous problems, see Itenberg [8, Section 7].

It remains to describe the formulae for $n^+(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n, \omega)$ and $n(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n, \omega)$. For simplicity we suppose that the dimension $n = 2$. The functions $\omega = (\omega_1, \omega_2)$ are required to be "sufficiently generic" in a sense explained below. First, the function ω_1 induces a polygonal subdivision τ_1 of Δ_1 as follows. Look at the convex hull Γ_1 of the graph of ω_1 on Δ_1, i.e., of the points $\{(\mathbf{m}, \omega_1(\mathbf{m})) : \mathbf{m} \in \mathcal{M}_1\}$ in $\mathbb{R}^3$. This is a polytope Γ_1 in $\mathbb{R}^3$, and its orthogonal projection onto its first two coordinates has image Δ_1. The orthogonal projection of the lower convex hull of Γ_1 onto Δ_1 gives the polygonal subdivision. Here the *lower convex hull* of Γ_1 consists of the two dimensional faces F of Γ_1 such that if $(x_1, x_2, t_3) \in F$ then $(x_1, x_2, t) \notin \Gamma_1$ if $t < t_3$. The function ω_2 similarly induces a polygonal subdivision τ_2 of Δ_2. Together they induce a polygonal subdivision τ_ω of $\Delta_1 + \Delta_2$ using the function

$$\omega_{1,2} : \mathcal{M}_1 + \mathcal{M}_2 \to \mathbb{R} \ ,$$

defined by

$$\omega_{1,2}(\mathbf{m}) := \min\{\omega_1(\mathbf{m}_1) + \omega_2(\mathbf{m}_2) : \mathbf{m}_1 + \mathbf{m}_2 = \mathbf{m},\ m_1 \in \mathcal{M}_1,\ m_2 \in \mathcal{M}_2\} \ .$$

Each two-dimensional face F of τ_ω has a unique representation

$$F := F_1 + F_2 \ ,$$

with F_1 a face of τ_1 and F_2 a face of τ_2. We say that the function $\omega = (\omega_1, \omega_2)$ is *sufficiently generic* if the equality

$$\dim(F) = \dim(F_1) + \dim(F_2) \ .$$

holds for all two-dimensional faces F of τ_ω. This condition holds for almost all functions ω.

In this polygonal subdivision an especially important role is played by the mixed cells. A *mixed cell* F of τ_ω is a cell $F = F_1 + F_2$ whose representation [2] has

$$\dim(F_1) = \dim(F_2) = 1 \ .$$

Note that the mixed cells and the polygonal subdivision τ_ω are completely determined by the monomial set $\mathcal{M}$ and the functions ω_1 and ω_2. The signs $\boldsymbol{\epsilon} = (\epsilon_1, ..., \epsilon_n)$ attached to $\mathcal{M}$ will enter in computing $n^+(\tilde{\Delta}_1, ..., \tilde{\Delta}_n, \omega)$ and $n(\tilde{\Delta}_1, ..., \tilde{\Delta}_n, \omega)$ as a sum of contributions from the mixed cells.

[2] In the n-dimensional case a *mixed cell* is a cell $F = F_1 + F_2 + \ldots + F_n$ of $\Delta_1 + \Delta_2 + \ldots + \Delta_n$ such that $\dim(F_1) = \dim(F_2) = \ldots = \dim(F_n) = 1$.

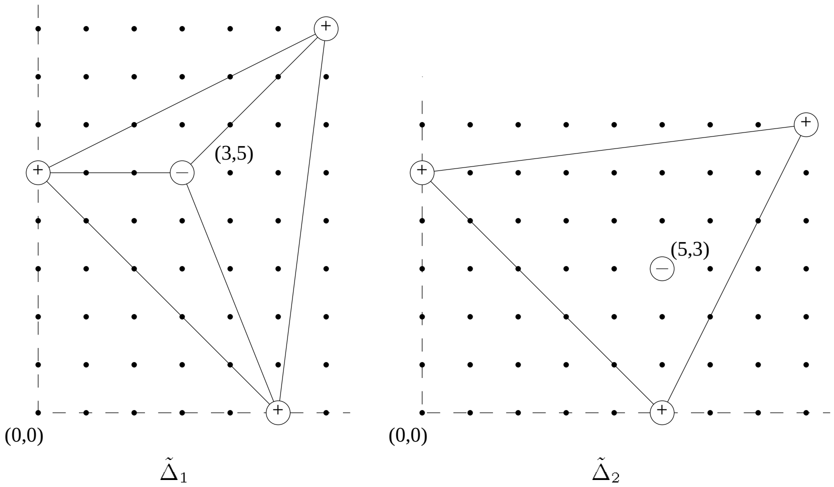

Figure 2: Signed Newton polytopes and subdivisions τ_1, τ_2.

As an example, consider the signed Newton polytopes given by

$$\mathcal{M}_1 := \{(0,5),(5,0),(3,5),(6,8)\} \quad \text{and} \quad \mathcal{M}_2 := \{(0,5),(5,0),(5,3),(8,6)\}$$

with sign patterns

$$\epsilon_1 = \{1,1,-1,1\} \qquad\qquad \text{and} \qquad\qquad \epsilon_2 = \{1,1,-1,1\}\ ,$$

together with the functions ω_1 and ω_2 on $\mathcal{M}_1$ and $\mathcal{M}_2$ with values

$$\omega_1 := \{5,6,2,7\} \qquad\qquad \text{and} \qquad\qquad \omega_2 := \{1,4,10,7\}$$

corresponding to the vertices above. The resulting signed Newton polytopes $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ and the polyhedral subdivisions τ_1 and τ_2 of Δ_1 and Δ_2 are indicated in Figure 2.

The polyhedral subdivision τ_ω induced by $\omega_{1,2}$ on $\Delta_1 + \Delta_2$ is pictured in Figure 3 and the values of $\omega_{1,2}(\cdot)$ are indicated. Note that for the points $\mathbf{x} = (5,5)$ and $\mathbf{x} = (11,11)$, we have

$$\begin{aligned} \omega_{1,2}(5,5) &= \min(\omega_1(0,5) + \omega_2(5,0), \omega_1(5,0) + \omega_2(0,5)) \\ &= \min(9,7) = 7\ , \end{aligned}$$

$$\begin{aligned} \omega_{1,2}(11,11) &= \min(\omega_1(3,5) + \omega_2(8,6), \omega_1(6,8) + \omega_2(5,3)) \\ &= \min(17,9) = 9\ . \end{aligned}$$

There are three mixed cells in τ_ω, which are the three shaded parallelograms in Figure 3. The function $\omega = (\omega_1, \omega_2)$ is sufficiently generic.

The general formulae for $n^+(\tilde{\Delta}_1, \tilde{\Delta}_2, \omega)$ and $n(\tilde{\Delta}_1, \tilde{\Delta}_2, \omega)$ consist of a sum of contributions from each mixed cell F of weights $wt^+(F)$ and $wt(F)$, respectively.[3] Given a two-dimensional mixed cell, its decomposition is into an edge F_1 of τ_1 with vertices $\mathbf{m}_1^{(1)} = (m_{11}^{(1)}, m_{12}^{(1)})$ and $\mathbf{m}_2^{(1)} = (m_{21}^{(1)}, m_{22}^{(1)})$ in $\mathcal{M}_1$ and an edge F_2 of τ_2 with vertices $\mathbf{m}_1^{(2)}, \mathbf{m}_2^{(2)}$ in $\mathcal{M}_2$. The edge F_1 is *alternating* if the signs of its endpoints don't agree, that is, if $\epsilon_1(\mathbf{m}_1^{(1)}) \neq \epsilon_1(\mathbf{m}_2^{(1)})$. A mixed cell is *alternating* if all its edges are alternating. The weight $\omega t^+(F)$ for a mixed cell F is defined by

$$wt^+(F) := \begin{cases} 1 & \text{if } F \text{ is alternating ,} \\ 0 & \text{otherwise .} \end{cases}$$

The weight $wt(F)$ for a mixed cell F in $\mathbb{R}^n$ is defined by

$$wt(F) := \sum_{\delta \in \{+1,-1\}^n} wt^+(F_\delta) \ ,$$

in which $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$, and the system $F_\delta(\mathbf{x})$ is the system obtained from $F(\mathbf{x})$ by replacing each variable x_i by $\delta_i x_i$. This has the effect of changing the sign pattern $\epsilon_i(m_1, m_2)$ of $\mathbf{m} = (m_1, m_2)$ to

$$\tilde{\epsilon}_i(m_1, m_2) = \epsilon_i(m_1, m_2)(\delta_1)^{m_1}(\delta_2)^{m_2} \ .$$

Itenberg and Roy derived a combinatorial formula for $wt(F)$ which shows that it can only take the values 0, 1, or a power of 2, up to 2^n.

For the example in Figure 3, there is a mixed cell $F = F_1 + F_2$ in τ_ω where F_1 has vertices $\mathbf{m}_1^{(1)} = (3,5), \mathbf{m}_2^{(1)} = (6,8)$, and F_2 has vertices $\mathbf{m}_1^{(2)} = (0,5)$, $\mathbf{m}_2^{(2)} = (8,6)$. For this mixed cell, F_1 is not alternating and F_2 is alternating, hence $wt^+(F) = 0$. The other two mixed cells in Figure 3 also have $wt^+(F) = 0$, hence

$$n^+(\tilde{\Delta}_1, \tilde{\Delta}_2, \omega) = 0 \ .$$

Further computation yields $wt(F) = 1$ and

$$n(\tilde{\Delta}_1, \tilde{\Delta}_2, \omega) = 3 \ .$$

[3] Here $wt^+(F)$ and $wt(F)$ depend on the subdivision τ_ω and the sign patterns $\epsilon_1, \ldots, \epsilon_n$.

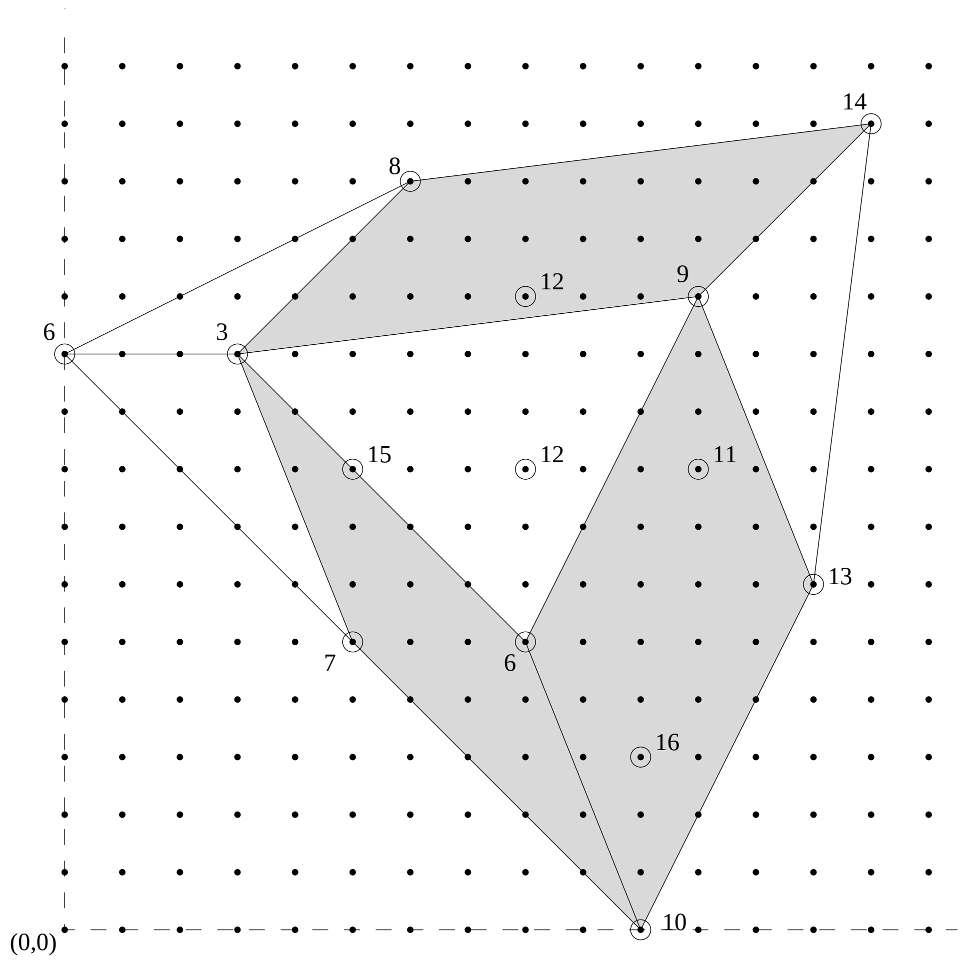

Figure 3: Polyhedral Subdivision of $\Delta_1 + \Delta_2$.(Function values of $\omega_{1,2}$ are indicated.)

For example, if $\delta = (-1,-1)$ then the mixed cell F above has $\tilde{\epsilon}_1(\mathbf{m}_1^{(1)}) = (-1)(-1)^3(-1)^5 = -1$, $\tilde{\epsilon}_1(\mathbf{m}_1^{(2)}) = 1$, $\tilde{\epsilon}_2(\mathbf{m}_2^{(1)}) = 1(1)^0(-1)^5 = -1$, $\tilde{\epsilon}_2(\mathbf{m}_2^{(2)}) = 1$, hence F is alternating.

The calculations above depend on the signed Newton polytopes, and the dependence on ω is completely specified by the polygonal subdivisions τ_1, τ_2 and τ_ω. Consequently only finitely many cases must be examined to determine $n^+(\tilde{\Delta}_1, \tilde{\Delta}_2)$ and $n(\tilde{\Delta}_1, \tilde{\Delta}_2)$.

The Itenberg-Roy conjecture is true for $n = 1$ by Descartes' rule of signs. Itenberg and Roy observed that it is true for $N(\Delta_1, \ldots, \Delta_n)$ whenever $n(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n) = V(\Delta_1, \ldots, \Delta_n)$ by Bernstein's Theorem. The combinatorial mixed volume $V(\Delta_1, \ldots, \Delta_n)$ is always equal to the sum of the volumes of the mixed cells in any subdivision τ_ω of $\Delta_1+\Delta_2+\ldots+\Delta_n$, i.e.,

$$V(\Delta_1,\Delta_2,\ldots,\Delta_n)=\sum_{\substack{F \text{ a mixed cell}\\ \text{of } \tau_\omega}} vol_n(F)\ ,$$

see [6]. In the example τ_ω in Figure 3, the three mixed cells have areas $15, 21$ and 27, and $V(\Delta_1,\Delta_2)=63$. Itenberg and Roy observed that any mixed cell F of a subdivision which has $vol(F)=1$ necessarily has $wt(F)=1$. It follows that if $(\tilde\Delta_1,\ldots,\tilde\Delta_n)$ has a sufficiently generic function $\omega=(\omega_1,\ldots,\omega_n)$ such that all the mixed cells of τ_ω have volume 1, then

$$n(\tilde\Delta_1,\tilde\Delta_2,\ldots,\tilde\Delta_n)=V(\Delta_1,\Delta_2,\ldots,\Delta_n)\ ,$$

and the Itenberg-Roy conjecture holds.

Sturmfels's Challenge Problem

B. Sturmfels offered a reward for resolving the Itenberg-Roy conjecture for the system

$$\begin{aligned} x^5 &= a_1\, y^5 + a_2\, x^3y^5 + a_3\, x^6y^8 \\ y^5 &= b_1\, x^5 + b_2\, x^5y^3 + b_3\, x^8y^6 \end{aligned} \tag{4}$$

with $a_1,a_2,a_3,b_1,b_2,b_3>0$. Here the Newton polytope bound of Bernstein's Theorem for the number of roots in $(\mathbb{C}^*)^2$ is 63, and the combinatorial bound for roots in $(\mathbb{R}^*)^2$ is 3. Its signed Newton polytopes $(\tilde\Delta_1,\tilde\Delta_2)$ appear in the example in the preceding section. The system (4) was found by computer search to be one that exhibits a particularly striking difference between the Newton polytope bound and the combinatorial bound.

Sturmfels circulated his challenge problem at Oberwolfach and other places as an advertisement for the Itenberg-Roy conjecture, see Figure 4. (The system (4) arises by removing a factor of x and a factor of y from his first and second equations, respectively.)

We solve Sturmfels's problem using ad hoc techniques involving Descartes' rule of signs in one variable. Perhaps this solution contains the seed of some idea relevant to the general case; however this is not obvious. It does verify the Itenberg-Roy conjecture in a nontrivial case.

We will actually prove the Itenberg-Roy conjecture for $N(\tilde\Delta_1,\tilde\Delta_2)$ for all sign combinations of the coefficients for which the conjecture predicts that the system (4) has at most three solutions in $(\mathbb{R}^*)^2$. These consist of all positive coefficients; of $a_2<0$, $b_2<0$ with all other

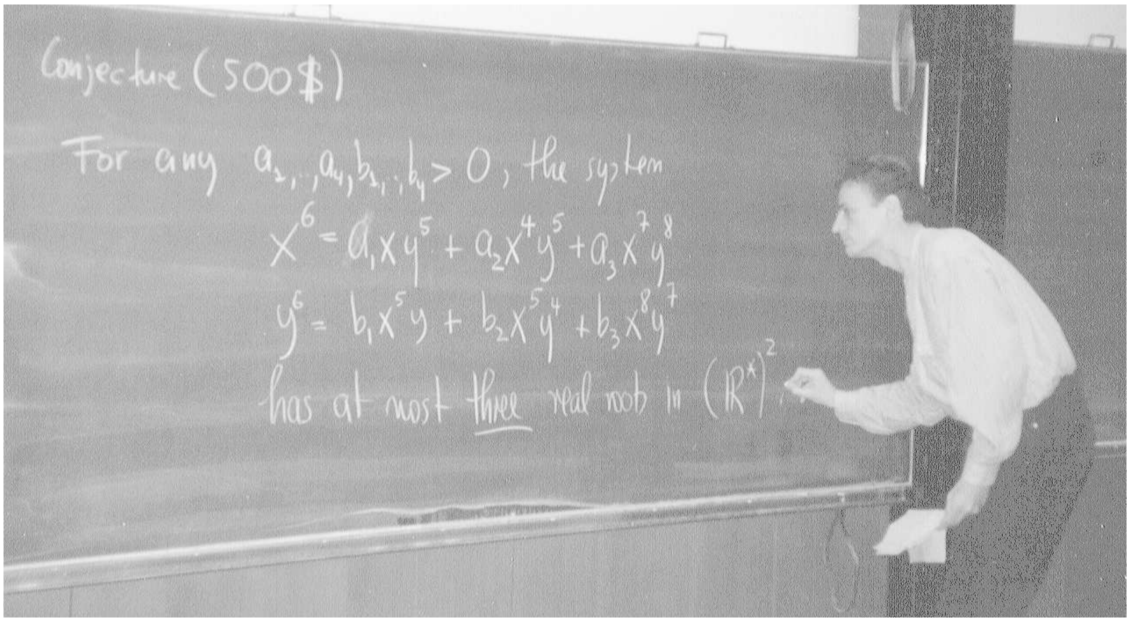

Figure 4: B. Sturmfels at Mathematisches Forschungsinstitut Oberwolfach. (Photo courtesy of G. Ziegler.)

coefficients positive; and in all other cases resulting from these two by sign changes of the variables $x \to -x$ or $y \to -y$.

Theorem. *For $a_1, a_3, b_1, b_3 > 0$, $a_2 b_2 > 0$, the system* (4) *has at most 3 roots in* $(\mathbb{R}^*)^2$.

Proof. Step 1: Without loss of generality we can assume $a_2 = b_2$. If $a_2 \neq b_2$ then we obtain an equivalent system in which $a_2 = b_2$ as follows. Let $\gamma := (b_2/a_2)^{1/7}$. Substitute $y \to \gamma y$ into and multiply through the second equation in by γ^{-5}.

Step 2: Let $x = y/m$, and write the equations in the equivalent form

$$m = a_1\, m^6 + a_2\, m^3 y^3 + a_3\, y^9 \tag{5}$$

$$m^8 = b_1\, m^3 + a_2\, m^3 y^3 + b_3\, y^9\,. \tag{6}$$

This birational transformation preserves roots in $(\mathbb{R}^*)^2$.

Step 3: Eliminate $m^3 y^3$ and y^9 respectively from these to obtain the following pair of equations:

$$(a_3 - b_3)\, y^9 = m - a_1\, m^6 - (m^8 - b_1\, m^3) \tag{7}$$

$$\left(1-\frac{a_3}{b_3}\right) a_2\, m^3 y^3 \;=\; m - a_1\, m^6 - \frac{a_3\,(m^8 - b_1\, m^3)}{b_3}\,, \tag{8}$$

If $a_3 = b_3$ then (7) reduces to

$$m^8 + a_1\, m^6 - b_1\, m^3 - m = 0.$$

By Descartes' rule of signs, this has at most one root for $m > 0$ and no roots for $m < 0$. It is clear that one positive root exists, and for this single positive root there is a unique real value of y that solves (5) (uniqueness here follows from Descartes' rule of signs). Thus, there is one solution to the original system in this case.

Henceforth we assume $a_3 \neq b_3$, and set

$$r_3 := a_3/b_3 \neq 1.$$

In this case (7) and (8) are equivalent to (5) and (6). Note that when $a_3 \neq b_3$, then for fixed m there is a unique real value of y satisfying (7), and similarly for (8).

Step 4: Eliminate y by cubing both sides of (8) and dividing them into the respective sides of (7). Then multiply through by m^9 to obtain

$$-b_3\,(1-r_3)^{-2}\, a_2{}^{-3} = \frac{m^7\,(1 - a_1\, m^5 - m^7 + b_1\, m^2)}{(1 - a_1\, m^5 - r_3\,(m^7 - b_1\, m^2))^3}\,.$$

Note that the right-hand side does not depend on a_2. Thus, the left-hand side can be set to any non-zero constant independent of the right-hand side. It is notationally more convenient to replace m with $-m$. Thus, we are left to consider

$$\alpha = \frac{m^7\,(1 + a_1\, m^5 + m^7 + b_1\, m^2)}{(1 + a_1\, m^5 + r_3\,(m^7 + b_1\, m^2))^3}\,,$$

where α can be any non-zero constant. Let us write this equation as

$$f(m) := m^7\,\frac{u(m)}{v(m)^3} = \alpha\,, \tag{9}$$

where

$$\begin{aligned} u(m) &= 1 + a_1\, m^5 + m^7 + b_1\, m^2 \\ v(m) &= 1 + a_1\, m^5 + r_3(m^7 + b_1\, m^2)\,. \end{aligned}$$

By Descartes' rule of signs, u and v both have at most one root in $m<0$. It is clear that both have at least one negative root, hence, both have exactly one negative root. We will let m_u and m_v denote the negative root of u and v respectively.

Step 5: Here we consider the range $m>0$. If $\alpha<0$ then there are no solutions to (9) with $m>0$ since $f(m)>0$ in this case. We claim that, if $\alpha>0$, then there are at most two solutions to (9) with $m>0$. Note that $f(0)=0$ and $f(+\infty)=0$. Since $v(m)\geq 1$ for $m\geq 0$ the claim follows if we show that f has a single critical point (necessarily a maxima) for $m>0$, i.e., $f(m)$ is unimodal.

Differentiating $m^7 u(m) v(m)^{-3}$ with respect to m, we obtain

$$f'(m) = m^6\,\frac{g(m)}{v(m)^4}\,,$$

where

$$\begin{aligned}
g(m) \;&:=\; (7\,u(m) + m\,u'(m))\,v(m) - 3\,u(m)\,m\,v'(m)\\
&=\; -7\,r_3\,m^{14} - (a_1 + 9\,a_1\,r_3)\,m^{12} - 3\,a_1{}^2 m^{10} - 4\,r_3\,b_1\,m^9 +\\
&\quad + (-6\,b_1\,a_1 + 14 - 14\,r_3 + 6\,a_1\,r_3\,b_1)\,m^7 +\\
&\quad +4\,a_1\,m^5 + 3\,b_1{}^2\,r_3\,m^4 + (r_3\,b_1 + 9\,b_1)\,m^2 + 7\,.
\end{aligned}$$

The coefficients of g for terms of degree greater than 7 are negative and the coefficients for terms of degree less than 7 are positive. Thus, Descartes' rule of signs implies that $g(m)$ has at most one positive root, hence, f has at most one positive critical point.

Step 6: We consider the remaining range $m<0$. We claim that for $m<0$ equation (9) has at most one solution if $\alpha>0$ and at most three solutions if $\alpha<0$. This will complete the proof.

We will first show that $f'(m)=0$ can have at most one solution in $m<0$. Note that here $f'(m)=0$ implies $g(m)=0$. Now,

$$\begin{aligned}
m\,g'(m) - 7g(m) \;&=\; -r_3\,m^4\left(49\,m^{10} + 8\,b_1\,m^5 + 9\,b_1{}^2\right)\\
&\quad -(9\,a_1{}^2 m^{10} + 8\,a_1\,m^5 + 49)\\
&\quad -5\,(a_1 + 9\,a_1\,r_3)\,m^{12} - 5\,(r_3\,b_1 + 9\,b_1)\,m^2\\
&<\; 0\,.
\end{aligned}$$

The inequality follows by noting that $49\,m^{10} + 8\,b_1\,m^5 + 9\,b_1{}^2$ is positive everywhere, being quadratic in m^5 without real roots, and that a similar statement applies to $9\,a_1{}^2 m^{10} + 8\,a_1\,m^5 +$

49. We conclude that if $m < 0$ and $g(m) = 0$, then $g'(m) > 0$. Since $g(0) > 0$ and $g(-\infty) = -\infty$, we see that $g(m) = 0$ has exactly one real solution with $m < 0$. Let m_g be this root of g. Note that $g(m) > 0$ for $m \in (m_g, 0)$, and $g(m) < 0$ for $m \in (-\infty, m_g)$.

It is easy to see that $m_u = m_v$ if and only if $a_1 b_1 = 1$. Assume $a_1 b_1 = 1$; then, since $u(m) = v(m) = 0$ implies $g(m) = 0$, we have $m_u = m_v = m_g = -b_1^{1/5}$. We obtain the reduction

$$f(m) = m^7 \frac{1 + b_1\, m^2}{(1 + b_1\, r_3\, m^2)v(m)^2} .$$

We see that $f(m)$ is monotonically decreasing on $(-\infty, m_v)$ and monotonically increasing on $(m_v, 0)$, tending to 0 from below at both endpoints and tending to $-\infty$ from both sides at m_g. We conclude in this case that, for $m < 0$, (9) has no solutions for $\alpha > 0$ and exactly two solutions for $\alpha < 0$.

Next, assume $m_u > m_v$. The argument here is similar to the one above. In this case, we have $m_g \in (-\infty, m_u)$ is the sole critical point of f in $m < 0$. On $(-\infty, m_u)$ we have $f < 0$ and unimodal, tending to zero at the endpoints. On (m_u, m_v) we have $f > 0$ and $f' > 0$ with $f(m_u) = 0$ and $f(m_v-) = +\infty$. On $(m_v, 0)$ we have $f < 0$ and $f' > 0$, with $f(0) = 0$ and $f(m_v+) = -\infty$. We conclude that (9) has exactly one negative solution if $\alpha > 0$ and at most three negative solutions if $\alpha < 0$.

Finally, assume $m_u < m_v$. Since $f(m_u) = f(0) = 0$ and $f(m) < 0$ for $m \in (m_u, 0)$, we observe that $f'(m) = 0$ has a solution in $(m_u, 0)$. We conclude that m_g, the sole negative critical point of f, is in $(m_u, 0)$. Thus, on $(m_u, 0)$ we see that f is unimodal and negative, and tends to zero at the endpoints. On (m_v, m_u) we have $f > 0$ and $f' < 0$, while $f(m_v+) = +\infty$ and $f(m_u) = 0$. On $(-\infty, m_v)$ we have $f < 0$ and $f' < 0$, while $f(m_v-) = -\infty$ and $f(-\infty) = 0$. We conclude that (9) has exactly one negative solution if $\alpha > 0$ and at most three negative solutions if $\alpha < 0$. This completes the proof. $\square$

Conclusion

Sturmfels paid us the \$500.

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