Geometric Models for Quasicrystals
I. Delone Sets of Finite Type

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(May 4, 1998 revision)

ABSTRACT

This paper studies three classes of discrete sets X in $\mathbb{R}^n$ which have a weak translational order imposed by increasingly strong restrictions on their sets of interpoint vectors $X - X$. A *finitely generated Delone set* is one such that the abelian group $[X - X]$ generated by $X - X$ is finitely generated, so that $[X - X]$ is a lattice or a quasilattice. For such sets the abelian group $[X]$ is finitely generated, and by choosing a basis of $[X]$ one obtains a homomorphism $\phi : [X] \to \mathbb{Z}^s$. A *Delone set of finite type* is a Delone set X such that $X - X$ is a discrete closed set. A *Meyer set* is a Delone set X such that $X - X$ is a Delone set.

Delone sets of finite type form a natural class for modelling quasicrystalline structures, because the property of being a Delone set of finite type is determined by "local rules." That is, a Delone set X is of finite type if and only if it has a 20 finite number of neighborhoods of radius $2R$, up to translation, where R is the relative denseness constant of X. Delone sets of finite type are also characterized as those finitely generated Delone sets such that the map ϕ satisfies the Lipschitz-type condition $||\phi(\mathbf{x}) - \phi(\mathbf{x}')|| < C||\mathbf{x} - \mathbf{x}'||$ for $\mathbf{x}, \mathbf{x}' \in X$, where the norms $|| \cdot ||$ are Euclidean norms on $\mathbb{R}^s$ and $\mathbb{R}^n$, respectively. Meyer sets are characterized as the subclass of Delone sets of finite type for which there is a linear map $\tilde{L} : \mathbb{R}^n \to \mathbb{R}^s$ and a constant C such that $||\phi(\mathbf{x}) - \tilde{L}(\mathbf{x})|| \le C$ for all $\mathbf{x} \in X$.

Suppose that X is a Delone set with an inflation symmetry, which is a real number $\eta > 1$ such that $\eta X \subseteq X$. If X is a finitely generated Delone set, then η must be an algebraic integer; if X is a Delone set of finite type then in addition all algebraic conjugates $|\eta'| \le \eta$; and if X is a Meyer set then all algebraic conjugates $|\eta'| \le 1$.

Keywords: quasicrystal, Delone set, Meyer set, lattice, quasilattice

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1. Introduction

The discovery of quasicrystalline materials in 1984 has generated much theoretical and experimental work.([10], [30],[31], [57], [78]). These are materials which have X-ray diffraction spectra that have sharp spots ("Bragg peaks") indicating long-range order of their atomic structure under translations, but which exhibit forbidden symmetries indicating that they cannot have crystallographic atomic structure. The list of quasicrystalline materials now inclues a few types of thermodynamically stable "perfect quasicrystals" which have structure as uniform as crystals, as well as a much larger variety of quasicrystalline structures with less perfect order ([21]). There are a number of different structural models proposed for quasicrystalline materials, ranging from random tiling models ([23], [24]) to models based on cut-and-project sets ([38],[44], [45]). Indeed such structures can exhibit diffraction patterns of approximately the right kind ([12], [25], [30], [54]). This series of papers is motivated by the problem of finding atomic models for such materials, that can describe both thermodynamically stable quasicrystals and more disordered quasicrystalline structures. In particular, we consider sets with weaker kinds of long-range order under translation that would not necessarily be manifested by 'Bragg peaks' in their diffraction spectrum.

The concept of a Delone set is a mathematical abstraction for the positions of atoms in a solid state material.

Definition 1.1 A discrete set X is a *Delone set* (or *Delaunay set* or *(r, R)-set*) if it satisfies the two conditions:

(i) *Uniform Discreteness.* There is a value r such that each open ball of radius r contains at most one point of X.

(ii) *Relative Denseness.* There is a value R such that each closed ball of radius R contains at least one point of X.

The maximal value of r is the packing radius for X using equal spheres centered at its points, and the minimal value of R is the covering radius of X using equal spheres centered at its points to cover $\mathbb{R}^n$.

The notion of Delone sets as the fundamental object of study in crystallography was proposed by the Russian school in the 1930's, see Delone et al. [8], [9], Galiulin [20] and Senechal [72, p. 25]. Delone set models for quasicrystals have been studied by a number of authors, e.g. Danzer [5, p. 571], Klitzing and Baake [36].

In this paper we consider three classes of discrete geometric structures in $\mathbb{R}^n$, which consist of Delone sets X with some translational order imposed by restrictions on their set of interpoint vectors $X - X$. It is natural to consider restrictions on the set of interpoint vectors $X - X$, first because the diffraction pattern is determined by the set $X - X$, and second because restrictions on local arrangements of atoms appear to be imposed in some materials by local energy minimization[1].

The three classes of Delone sets that we study are specified by increasingly strong restrictions on their sets of interpoint vectors $X - X$, as follows.

Definition 1.2 Let X be a Delone set.

(i) X is a *finitely generated Delone set* if the abelian group

$$[X - X] = \mathbb{Z}[\mathbf{x} - \mathbf{x}' : \text{ all } \mathbf{x}, \mathbf{x}' \in X] \tag{1.1}$$

is finitely generated.

(ii) X is a *Delone set of finite type* if $X - X$ is a discrete closed subset of $\mathbb{R}^n$, i. e. the intersection of $X - X$ with any closed ball is a finite set.

(iii) X is a *Meyer set* if $X - X$ is a Delone subset of $\mathbb{R}^n$.

We show that class (i) includes class (ii), and class (ii) obviously includes class (iii). The terminology "Delone set of finite type" is motivated by the fact that Delone sets of finite type

[1] The resultant of the forces acting on each atom must be zero (local stability). The electromagnetic force is long-range, but the dominant effects on one atom come from nearby atoms due to approximate cancellation of forces from atoms further away.

have only finitely many types of local neighborhoods of radius $2R$ under translations, see §2. More precisely (ii) defines Delone *sets of finite type under translations*, to distinguish them from the larger class of *Delone sets of finite type under isometries* considered in [11] and [40]. The class of Meyer sets was introduced by Meyer [54] under the name "quasicrystal"; they are now generally called Meyer sets, see [39], [55], [56]. The definition (iii) of a Meyer set is not the usual one, but in [39] it is shown equivalent to Meyer's original definition that $X - X \subseteq X + F$, for some finite set F. Meyer [54] observes that all cut-and-project sets are Meyer sets, and presents evidence that they form a suitable class to be termed quasicrystals. We argue here that the more inclusive class of Delone sets of finite type is the natural general class of sets to consider in making models for the atomic structure of quasicrystalline materials. However both of these classes include sets that are not quasicrystalline in the sense of possessing a well defined diffraction spectrum.Various examples that illustrate these points appear in §5.

A Delone set X is finitely generated if and only if the abelian group

$$[X] := \mathbb{Z}[\mathbf{x} : \mathbf{x} \in X] \tag{1.2}$$

is finitely generated. We view this property as a weak form of long-range order under translations[2] of the elements of X. A finitely generated abelian group $[X]$ that spans $\mathbb{R}^n$ is called a *quasilattice*; it is called a *lattice* if it is discretely embedded in $\mathbb{R}^n$. Quasilattices have been introduced as a unifying concept in studying the symmetries of quasicrystals, see Le et al. [43], Mermin [51] and Piunikhin [61]. The *rank* of X is the number of generators of $[X]$ as a free abelian group. (All finitely generated subgroups of $\mathbb{R}^n$ are free.)

Definition 1.3 Let X be a finitely generated Delone set with $rank(X) = s$, and choose a basis of $[X]$, say

$$[X] = \mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s] \ . \tag{1.3}$$

The *address map* $\phi : [X] \to \mathbb{Z}^s$ associated to this basis is

$$\phi(\sum_{i=1}^{s} n_i \mathbf{v}_i) = (n_1, n_2, \ldots, n_s) \ . \tag{1.4}$$

The address map is specified by the choice of basis of $[X]$, so it is determined up to left-multiplication by an element of $GL(s, \mathbb{Z})$. The structure of a finitely generated Delone set is to some extent analyzable by studying its image in $\mathbb{R}^s$ under the address map. Using the address

[2] See the note on terminology at the end of the introduction.

map the points of X can be coordinatized as a subset Y of the integer lattice $\mathbb{Z}^s$ regarded as embedded in $\mathbb{R}^s$, with $s = rank(X)$, and there is a linear projection $\psi : \mathbb{R}^s \to \mathbb{R}^n$ with $\psi(Y) = X$. The address map describes X using s "*internal dimensions.*" In this coordinatization all s dimensions are on an equal footing; there seems no natural way to select n of these dimensions as "physical dimensions."

The class of finitely generated Delone sets is the largest class of sets on which an address map is well-defined. It seems to be too large a class to be interesting, but membership in it does put algebraic restrictions on inflation symmetries, as described in §4. Various interesting subclasses of such sets are obtained by putting restrictions on the address map which require that points that are close in X have addresses which are not too far apart in the address space $\mathbb{R}^s$. The most important subclass of such sets seems to be Delone sets of finite type,which we show are characterized by a Lipschitz-type condition on the address map.

The class of Delone sets of finite type appears to be a natural unifying concept in classifying quasicrystalline structures. In §2 show that Delone sets of finite type form the largest class of Delone sets that can be described by "local rules under translations". That is, we show that a Delone set X is of finite type if and only if X has only a finite number of different local neighborhoods of radius $2R$ around its points $\mathbf{x} \in X$, up to translation (Theorem 2.1). This result shows that the property of being a Delone set of finite type is enforceable by local rules under translations, in the sense of part II, see also Le et al. [43] and Levitov [46]. It also follows that a Delone set X of finite type in $\mathbb{R}^n$ is always finitely generated, and $rank(X)$ is bounded by the number of points in $X - X$ of length at most $2R$.

One way to model the atomic structure of quasicrystalline materials which have X-ray diffraction patterns which exhibit crystallographically forbidden symmetries is through the use of sets described using a finite number of extra "internal dimensions," cf. Mermin [51]. Certain cut-and-project sets provide examples; they are obtained as a projections of part of a higher-dimensional lattice, see section 3. To get a Delone set in $\mathbb{R}^3$ having a diffraction pattern having "Bragg peaks" located at points with icosahedral symmetry, one can project from a suitable six dimensional lattice. The "local rules under translations" result above for Delone sets of finite type shows that finiteness of the number of "internal dimensions" describing a set follows as a consequence of finiteness of the number of allowable local configurations under translations. In part II we strengthen this result by showing that finiteness of the number of "internal dimensions" can also be forced by local rules under isometries, for a large class of structures.

Tiling models of quasicrystals naturally produce Delone sets of finite type. Consider a tiling of $\mathbb{R}^n$ which consists of translates of finitely many types of prototiles. We associate a Delone set X to the tiling by marking a finite set of points in each prototile ("decorations"), see for example Klitzing and Baake [36]. If there are only a finite number of different ways two prototiles may be adjacent in this tiling, then X is necessarily a Delone set of finite type, and conversely (Corollary 2.1). In particular, random tiling models of quasicrystals in the sense of Henley [24, Section 2.3] yield Delone sets of finite type.

In §2 we give several characterizations of Delone sets of finite type. We show that they are characterized by either of the following conditions:

(1) The address map ϕ satisfies the Lipschitz-type condition

$$||\phi(\mathbf{x}) - \phi(\mathbf{x}')|| \le C||\mathbf{x} - \mathbf{x}'|| \quad \text{for} \quad \mathbf{x}, \mathbf{x}' \in X \ ,$$

where the norms are Euclidean norms on $\mathbb{R}^s$ and $\mathbb{R}^n$, respectively.

(2) The tiling of $\mathbb{R}^n$ by the Voronoi domains induced by X (with the points of X marked) contains finitely many translation-equivalence classes of marked Voronoi domains.

(3) The Delaunay tiling of $\mathbb{R}^n$ induced by X contains finitely many translation-equivalence classes of Delaunay domains.

In §2 we also show that Delone sets of finite type in $\mathbb{R}^n$ have bounds on their number of distinct local configurations under translations. Let $N_X^*(T)$ be the atlas counting function which counts the number of translation-inequivalent patches $(X - \mathbf{x}) \cap B(\mathbf{0}; T)$ of radius T over all $\mathbf{x} \in X$. The *configurational entropy* $H_c(X)$ of a set X in $\mathbb{R}^n$ is defined by

$$H_c(X) := \limsup_{T\to\infty} \frac{\log N_X^*(T)}{T^n} \ , \tag{1.5}$$

in analogy with definitions in [42]. We show that any Delone set of finite type X has finite configurational entropy, i.e. there exists a constant C depending on X such that

$$N_X^*(T) \le \exp(CT^n) \ , \tag{1.6}$$

This is best possible; in §3 we give an example of a Meyer set X in $\mathbb{R}^n$ with positive configurational entropy

$$N_X^*(T) \ge \exp(C'T^n) \quad \text{as} \quad T\to\infty \ ,$$

where C' is a positive constant.

In §3 we study Meyer sets viewed as a subclass of Delone sets of finite type. These sets have been extensively studied and characterized by Meyer [54] and Moody [55], [56]. We show that they are exactly those Delone sets X of finite type whose address map is an approximate linear mapping on X. That is, they are those sets X such that there exists a one-to-one linear map $\tilde{L} : \mathbb{R}^n \to \mathbb{R}^s$ such that

$$||\phi(\mathbf{x}) - \tilde{L}(\mathbf{x})|| \le C \ , \quad \text{all } \mathbf{x} \in X \ . \tag{1.7}$$

Meyer previously showed that a set X is a Meyer set if and only if *every* homomorphism $\psi : [X] \to \mathbb{R}^d$ for $1 \le d < \infty$ is an approximate linear mapping on X, see Moody [29, Theorem 9.1]. Our small addition here is to observe that it suffices to check this property for the single map ϕ. The criterion (1.7) is useful in determining whether a set X is a Meyer set, and we apply it in §5. We also show that every Meyer set X is contained in some cut-and-project set Y of dimension at most $rank(X)$.

We introduce a relation of *local derivability* between Delone sets which parallels a notion for tilings introduced by Baake, Schlottman and Jarvis [3]. Let Y be a Delone set which is locally derivable from X. We show that if X is a Delone set of finite type then so is Y (Corollary 2.1c), and if X is a Meyer set then so is Y (Corollary 3.1b).

In §4 we study Delone sets X that have an *inflation symmetry* η, which is a real number $\eta > 1$ such that $\eta X \subseteq X$. This is not a true symmetry, since $X \backslash \eta X$ contains infinitely many points. Delone sets having an inflation symmetry can be obtained from self-similar tiling models of quasicrystals as sets of "control points," see [19] [33], [54], [75]. We show that:

(1) If X is a finitely-generated Delone set, then η is an algebraic integer.

(2) If X is a Delone set of finite type, then η is an algebraic integer all of whose algebraic conjugates η' satisfy $|\eta'| \le \eta$.

(3) If X is a Meyer set, then η is an algebraic integer all of whose algebraic conjugates η' satisfy $|\eta'| \le 1$.

The result (3) was originally proved by Meyer [13, Theorem 6]. These results are apparently best possible in the sense that if η has the stated property then there exists an appropriate Delone set X having the symmetry $\eta X \subseteq X$. This has been demonstrated for cases (1) and (3), and has been checked for some extremal examples in case (2).

In §5 we study one-dimensional Delone sets of finite type. Such sets correspond exactly to tilings of the line $\mathbb{R}$ using a finite set of prototiles which are intervals of different lengths. We study special examples that are constructed using one-dimensional symbolic dynamical systems. These examples include some Meyer sets which might not be considered "quasicrystals." We also indicate the possible existence of non-Meyer sets which might be considered "quasicrystalline" in the sense of having some discrete component in their diffraction spectrum.

The deepest problem about quasicrystalline materials concerns finding physical mechanisms that lead to noncrystallographic ordered structures[3]. This paper does not address this question. However, we note that many recent studies of quasicrystalline structure propose "clusters" or "local rules" as representing locally energy-minimizing finite arrangements of atoms at low temperature, and then propose decorations or "matching rules" for how clusters may overlap or abut each other as producing quasicrystalline structures, see Gähler et al. [18], Henley [24, Section 2.2], Ingersent [29, Section 6] and Levitov [46]. The resulting structures are Delone sets of finite type.

The viewpoint of this paper is that Delone sets of finite type form a suitable class of sets that encompass all structures which might reasonably be called quasicrystalline. The narrower class of Meyer sets has previously been proposed as such a class (Meyer [29], Moody [30]–[32]). However self-affine tilings of $\mathbb{R}^n$ which have an inflation matrix whose largest eigenvalue is not a Pisot or Salem number correspond (by picking "control points" in tiles) to Delone sets of finite type which are not Meyer sets. There exist such sets that have *long-range order under translations* in the sense of having a well-defined diffraction measure. That is, they have a unique autocorrelation measure in the sense of Hof [25], [27]; the diffraction measure is the Fourier transform of the autocorrelation measure. These diffraction measures are not known to contain any discrete component except at $\mathbf{0}$. It may be that there exist Delone sets of finite type which are not Meyer sets which have the quasicrystalline property of having some 'Bragg peaks' in their diffraction patterns; this is currently an open question, cf. §5.

Various special cases of the mathematical framework developed here are implicit in previous work. The notion of finite type for tilings appears under the name "finite number of local patterns" in Solomyak [75, p. 699] who cites earlier references. E. A. Robinson [68] defines a notion of "tiling of finite type" which corresponds to a tiling which satisfies some finite

[3]Even the problem of giving a physical explanation for crystallographic order is not resolved, see Radin [64], [66].

set of "local rules" under translations. Analogues of the address map have been previously introduced in special cases for tiling models for quasicrystals, in studying local rules for such tilings. For example, it appears as the "lifting function" in Levitov [46], see also [32] and [43]. For cut-and-project sets it appears as "perp-space coordinates" in Henley [24, Section 3.1.1]. L. Danzer [5] has long advocated the study of sets describable by local conditions. Danzer and Dolbilin [7] recently introduced a notion of Delone graph which is analogous to "local rules under isometries." They prove an analogue of Lemma 2.2, which is further sharpened in Danzer [6].

Note on Terminology. The concept of "quasicrystal" does not have a generally agreed on definition in the literature. In the physics literature([78]) it typically refers to materials whose X-ray diffraction patterns exhibit some "Bragg peaks" with non-crystallographic symmetries; thus crystals are not considered quasicrystals. In this paper we do not define the term "quasicrystal" but the various classes of sets we define are inclusive: we view ideal crystals as a special kind of "quasicrystal". In the physics literature "long-range order" refers to the presence of Bragg peaks in a diffraction spectrum. Analogously to this we consider a Delone set X to have *strong long-range order under translations* if $[X]$ is a quasilattice and X has a well-defined diffraction measure in the sense of Hof [25], [27] which contains some discrete spectrum whose points span $\mathbb{R}^n$. We consider a Delone set X to have *long-range order under translations* if $[X]$ is a quasilattice and X has a well-defined diffraction measure, i.e. X has a unique autocorrelation measure. Finally, we consider a Delone set X to have *weak long-range order under translations* if $[X]$ is a quasilattice, i.e. if X is a finitely generated Delone set; this is a mathematical notion of long-range order that may not be easily detectable physically.

Notation. $||\cdot||$ denotes the Euclidean norm on $\mathbb{R}^n$.
$B(\mathbf{x};T)$ is the closed ball of radius T around $\mathbf{x}\in\mathbb{R}^n$, i.e. $B(\mathbf{x};T) := \{\mathbf{y} : ||\mathbf{y}-\mathbf{x}|| \le T\}$. The symbol C designates a positive constant, which may be different at different occurrences.

2. Delone Sets of Finite Type

A Delone set X is of *finite type* if $X - X$ is a discrete closed set, i.e. each ball $B(\mathbf{0};T)$ contains only finitely many points of $X - X$. In this section we characterize Delone sets of finite type in several ways and establish some of their properties.

Lemma 2.1 *If X is a Delone set of finite type then any Delone set Y with $Y \subseteq X$ is a Delone set of finite type.*

Proof. The set $Y - Y \subseteq X - X$, hence each ball $B(\mathbf{0}; T)$ contains finitely many points of $Y - Y$. $\square$

It is useful to have a quantitative measure of the discreteness of the closed set $X - X$.

Definition 2.1 For a Delone set X, its *distance counting function* $D_X(T)$ is

$$D_X(T) := \#\{\mathbf{y} \in X - X : ||\mathbf{y}|| \leq T\} . \tag{2.1}$$

A Delone set X is of finite type if and only if $D_X(T)$ is finite for all $T > 0$.

We now prove a "local rules" criterion for a Delone set to be of finite type, which also establishes that such sets are finitely generated.

Theorem 2.1 *If X is a Delone set with relative denseness constant R. Then X is a Delone set of finite type if and only if*

$$D_X(2R) = \#\{(X - X) \cap B(\mathbf{0}; 2R)\} \quad is \quad finite\ . \tag{2.2}$$

Any Delone set X of finite type is finitely generated, with

$$rank\ (X) \leq D_X(2R)\ . \tag{2.3}$$

To prove Theorem 2.1 we use the following "stepping-stone" property of Delone sets.

Lemma 2.2 *Let X be a Delone set in $\mathbb{R}^n$ with parameters (r,R). Given any two points $\mathbf{x}, \mathbf{x}' \in X$ there exists a chain of points*

$$\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m = \mathbf{x}' \tag{2.4}$$

in X such that

(i) $||\mathbf{x}_i - \mathbf{x}_{i-1}|| \leq 2R$, *for* $1 \leq i \leq m$.

(ii) $m \leq \frac{4R}{r^2}||\mathbf{x} - \mathbf{x}'||$.

Proof. We construct the sequence recursively as follows. Given $\mathbf{x}_i$, if $||\mathbf{x}_i - \mathbf{x}'|| \le 2R$ set $m = i+1$ and stop. Otherwise consider the closed ball

$$B_i := B(\mathbf{x}_i + R\frac{\mathbf{x}' - \mathbf{x}_i}{||\mathbf{x}' - \mathbf{x}_i||};\ R)$$

of radius R which has the property that $\mathbf{x}_i$ is on its boundary, and $\mathbf{x}_i$ is the furthest point in this ball from $\mathbf{x}'$. This ball B_i necessarily contains another point of X. To show this we consider for $\epsilon > 0$ the translated ball

$$B_i(\epsilon) := B(\mathbf{x}_i + (R+\epsilon)\frac{\mathbf{x}' - \mathbf{x}_i}{||\mathbf{x}' - \mathbf{x}_i||}\ ;\ \ R)\ .$$

It contains some point of X by the Delone property, call it $\mathbf{x}(\epsilon)$, and $\mathbf{x}(\epsilon) \ne \mathbf{x}$ since $\mathbf{x}_i \notin B_i(\epsilon)$ for $\epsilon > 0$. Since X is discrete, some point occurs infinitely often among the points $\{\mathbf{x}(\frac{1}{n}) : n = 1, 2, \ldots\}$, call it $\mathbf{x}_{i+1}$, and it is in B_i because its distance from B_i is less than $\frac{1}{n}$ for infinitely many n. Thus (i) holds. Next, the Delone property says that

$$||\mathbf{x}_{i+1} - \mathbf{x}_i|| \ge r\ .$$

But $\mathbf{x}_i$ is the furthest point in the ball B_i from $\mathbf{x}'$, so this condition slices off a spherical cap off B_i, and there is a constant C such that

$$||\mathbf{x}_{i+1} - \mathbf{x}'|| \le ||\mathbf{x}_i - \mathbf{x}'|| - C\ .$$

One can take $C = \frac{r^2}{4R}$, so (ii) follows. □

Proof of Theorem 2.1. If X is a Delone set of finite type then (2.2) certainly holds.

To show the converse, suppose that (2.2) holds. Given $\mathbf{x}, \mathbf{x}' \in X$, by Lemma 2.2 there exists a chain of points (2.4) in X connecting $\mathbf{x}$ to $\mathbf{x}'$ that satisfies conditions (i) and (ii). These conditions together imply that $(X - X) \cap B(\mathbf{0}; T)$ is finite for any fixed T. To see this, note that the chain (2.4) connecting two points $\mathbf{x}, \mathbf{x}' \in X$ with $||\mathbf{x} - \mathbf{x}'|| \le T$ has at most CT steps by (ii), and there are at most $D_X(2R)$ choices for each step $\mathbf{x}_{i+1} - \mathbf{x}_i$ in the chain, by (i). Thus $D_X(T) = \#\{(X - X) \cap B(\mathbf{0}, T)\}$ is finite for all T, hence X is of finite type.

Finally, the existence for all $\mathbf{x}' \in X$ of the chain (2.4) satisfying (i), (ii) shows that

$$[X] := \mathbb{Z}[X] \subseteq \mathbb{Z}[\mathbf{x}_0 \cup [((X - X)\backslash\{\mathbf{0}\}) \cap B(\mathbf{0}; 2R)]\ .$$

Thus (2.3) follows. □

The finiteness criterion of Theorem 2.1 carries over to finiteness of local neighborhoods of the set X under translations.

Definition 2.2 (i) The *T-neighborhood* of a point $\mathbf{x} \in X$ is the set

$$\mathcal{P}_X(\mathbf{x}, T) := X \cap B(\mathbf{x}; T) = \{\mathbf{x}' \in X : \|\mathbf{x}' - \mathbf{x}\| \leq T\} \ . \tag{2.5}$$

A *T-patch* of X is a T-neighborhood of X regarded as movable under translations.

(ii) A *centered T-patch* $\mathcal{P}$ is a T-neighborhood translated to the origin, i.e.

$$\mathcal{P} := \mathcal{P}_X(\mathbf{x}; T) - \mathbf{x} \ . \tag{2.6}$$

Definition 2.3 (i) The *T-atlas* $\mathcal{A}_X(T)$ of a Delone set X is the set of all centered T-patches of X, i.e.

$$\mathcal{A}_X(T) := \{X \cap B(\mathbf{0}; T) - \mathbf{x} : \mathbf{x} \in X\} \ . \tag{2.7}$$

(ii) The *atlas counting function* $N_X^*(T)$ counts the number of different centered T-patches, i.e.

$$N_X^*(T) := |\mathcal{A}_X(T)| \ , \quad for \quad T > 0. \tag{2.8}$$

Corollary 2.1a *Let X be a Delone set with relative denseness constant R. Then X is a Delone set of finite type if and only if it has finitely many different centered $2R$-patches, i.e. $N_X^*(2R) < \infty$.*

Proof. This follows from Theorem 2.1, since if X has finitely many different $2R$-patches then $D_X(2R)$ is finite, while if $D_X(2R)$ is finite, then there are finitely many choices for points within distance $2R$ of a fixed point $\mathbf{x}$. $\square$

Delone sets of finite type arise from tilings of $\mathbb{R}^n$ by translates of a finite set of prototiles. We suppose that each prototile is simply connected, is the closure of its interior and has a boundary of measure zero. To any such tiling $\mathcal{T}$ we can associate a Delone set (in many different ways) by marking a finite set of points in each prototile, and taking X to be the set of marked points in the tiling. We say that such a tiling $\mathcal{T}$ has a *finite number of local patches* if it contains only finitely many translation-inequivalent arrangements of neighbors of each prototile. We have:

Corollary 2.1b *Let $\mathcal{T}$ be a tiling of $\mathbb{R}^n$ by translations of a finite set of prototiles. Suppose that each prototile is decorated by a finite set of points, and let $X = X(\mathcal{T})$ be the point set associated to $\mathcal{T}$ by these decorations. If $\mathcal{T}$ has a finite number of local patches then $X(\mathcal{T})$ is a Delone set of finite type, and conversely.*

Proof. This follows from Theorem 2.1 because there are only finitely many translation-inequivalent ways to tile neighborhoods of diameter $2R$. $\square$

Baake, Schlottman and Jarvis [3], (see also [1], [2]) introduced a notion of "local derivability" between tilings , which has the following analogue for Delone sets.

Definition 2.4 (i)Let X and Y both be Delone sets which are relatively dense with constant R. We say that Y is *locally derivable* from X if there is a constant R' such that for each $\mathbf{x} \in X$ the set of points of $Y \cap B(\mathbf{x}; 2R)$ is completely determined by the centered R'-patch of X around $\mathbf{x}$. That is, the point set $\mathcal{P}_{\mathbf{x}}(Y; 2R) = (Y \cap B(\mathbf{x}; R)) - \mathbf{x}$ is a function of the centered patch $\mathcal{P}_{\mathbf{x}}(X, R') = (X \cap B(\mathbf{x}; R')) - \mathbf{x}$.

(ii) We say that X and Y are *mutually locally derivable* if each is locally derivable from the other.

Corollary 2.1c *If X is a Delone set of finite type and Y is a Delone set locally derivable from X, then Y is a Delone set of finite type.*

Proof. By hypothesis $D_X(R')$ is finite. Since the translation classes of R'-patches of X determine the $2R$-patches of Y, we have $D_Y(2R)$ is finite, and Theorem 2.1 applies. $\square$

Our next object is to derive several equivalent criteria for a set X to be a Delone set of finite type. To formulate these, we make some definitions.

Definition 2.5 A Delone set X has the *locally finite atlas property under translations* if $N_X^*(T)$ is finite for all $T > 0$.

Definition 2.6 (i) Given a discrete set X in $\mathbb{R}^n$ and a point $\mathbf{x} \in X$ the *Voronoi domain* $V_X(\mathbf{x})$ at $\mathbf{x}$ is the set of all points in $\mathbb{R}^n$ as close to $\mathbf{x}$ as to any other point of X, namely

$$V_X(\mathbf{x}) = \{\mathbf{y} : ||\mathbf{y} - \mathbf{x}|| \le ||\mathbf{y} - \mathbf{x}'|| \ , \quad \text{all} \ \ \mathbf{x}' \in X\} \ .$$

A *marked Voronoi domain* is a Voronoi domain together with a specific point in it viewed as a marked point.

(ii) The set of Voronoi domains of a discrete set X form a tiling of $\mathbb{R}^n$ called the *Voronoi tesselation* of $\mathbb{R}^n$ induced by X. A *marked Voronoi tesselation* consists of the Voronoi tesselation induced by a discrete set X with the points of X marked.

If X is a Delone set then each Voronoi domain $V_X(\mathbf{x})$ is a convex polytope contained in $B(\mathbf{x}; 2R)$. (For a general discrete set X the set $V_X(\mathbf{x})$ is a convex polytope, possibly unbounded.) The set X cannot always be uniquely reconstructed from its Voronoi tesselation. It is uniquely determined by the induced Voronoi tesselation plus a single Voronoi domain marked with a point of X; this determines the marking in all other Voronoi domains by the points of X.

Definition 2.7 (i) Given a Delone set X in $\mathbb{R}^n$ a finite subset Y of X determines an *empty sphere* if all points of Y lie on a sphere in $\mathbb{R}^n$, there are no points of X in the interior of this sphere, and the convex hull of Y has nonempty interior. We call the convex hull of all points of X lying on this sphere the *Delaunay domain* determined by Y in X. It is a convex polytope.

(ii) The set of Delaunay domains of a Delone set X give a tiling of $\mathbb{R}^n$ called the *Delaunay tesselation* of $\mathbb{R}^n$ induced by X.

The Delaunay tesselation was introduced by Delone [9] in 1934. He showed that for a Delone set X in "general position" all Delaunay domains are simplices; thus the Delaunay tesselation is often called the *Delaunay triangulation* in the computational geometry literature[4], cf. Mulmuley [59, p. 49]. A proof that Delaunay domains give a facet-to-facet tiling of $\mathbb{R}^n$ can be found in FortuneFor95, and in the"generic case" in Rogers [69, Chapter 8]. The set X is uniquely reconstructible from its associated Delaunay tesselation, because X is the set of all vertices of all Delaunay domains. For general references on Voronoi and Delaunay tesselations, see Fortune [14], [15].

Theorem 2.2 *For a Delone set X in $\mathbb{R}^n$, the following properties are equivalent.*

(i) X is a Delone set of finite type.

(ii) X has the locally finite atlas property under translations.

(iii) The marked Voronoi tesselation of $\mathbb{R}^n$ induced by X has finitely many translation-inequivalent marked Voronoi domains.

(iv) The Delaunay tesselation of $\mathbb{R}^n$ induced by X has finitely many translation-inequivalent Delaunay domains.

[4]The spelling *Delaunay triangulation* is standard in the computational geometry literature. Here "general position" means that there do not exist n+2 points of X on any sphere of diameter at most $2R$. Rogers [69, Chapter 8] requires instead that the set X have no $n+2$ equidistant points; in this case a Delaunay domain as we have defined it can be triangulated in a canonical way to obtain his "Delaunay triangulation."

(v) X is finitely generated and any address map $\phi : [X] \to \mathbb{Z}^s$ has

$$||\phi(\mathbf{x}) - \phi(\mathbf{x}')|| \le C_0 ||\mathbf{x} - \mathbf{x}'|| \ , \quad \textit{all } \mathbf{x}, \mathbf{x}' \in X \ , \tag{2.9}$$

for some constant C_0 depending on ϕ.

Proof. (i) $\Rightarrow$ (ii). Since $(X - X) \cap B(\mathbf{0}; T)$ is finite, and for $\mathbf{x} \in X$ all points in

$$\mathcal{P}(\mathbf{x}; T) := (X - \mathbf{x}) \cap B(0; T) \tag{2.10}$$

lie in this set, there are only finitely many choices for $\mathcal{P}(\mathbf{x}; T)$. In fact

$$N_X^*(T) \le 2^{D_X(T)} \ . \tag{2.11}$$

(ii) $\Rightarrow$ (iii). A marked Voronoi domain $V_X(\mathbf{x})$ is completely determined by the patch

$$\mathcal{P}(\mathbf{x}; 2R) := (X - \mathbf{x}) \cap B(\mathbf{0}; 2R) \ .$$

For any point $\mathbf{y}$ with $||\mathbf{x} - \mathbf{y}|| > 2R$ is within distance $2R$ of some point $\mathbf{x}' \in X$ hence cannot be in $V_X(\mathbf{x})$. The locally finite atlas property says that there are only $N_X^*(2R) < \infty$ such patches, hence there are at most $N_X^*(2R)$ possible translation-inequivalent Voronoi domains.

(iii) $\Rightarrow$ (i). The Delone property of X guarantees that the ball $B(\mathbf{x}; \frac{1}{2}r)$ is contained in the Voronoi domain $V_X(\mathbf{x})$. The facets of each marked Voronoi domain determine the marking in each neighboring Voronoi domain, namely drop a perpendicular from $\mathbf{x}$ to the hyperplane determined by the facet and extend it an equal distance to the other side. By hypothesis there are only a finite number of choices for each neighboring domain, up to translation. Continue adding marked Voronoi domains in this way to fill up the entire ball of radius $2R$ around $\mathbf{x}$. One has to add at most $\frac{2R}{r}$ concentric layers of Voronoi domains around $\mathbf{x}$ to do this. There are only finitely many possible extensions, and this locates all possible locations for elements of X within distance $2R$ of X. Since this construction is translation-invariant, we conclude that $(X - X) \cap B(\mathbf{0}; 2R)$ is finite. Now (i) follows by Theorem 2.1.

(i) $\Rightarrow$ (iv). A Delaunay domain including a given point $\mathbf{x} \in X$ is completely determined by all points within distance $2R$ of X, since the largest empty sphere touching X has diameter at most $2R$. By hypothesis (i) there are only finitely many translation-inequivalent arrangements of such points.

(iv) $\Rightarrow$ (i). The Delaunay tesselation of X uniquely determines X. Given any two elements of X within a distance $2R$, there is a path of edges in the Delaunay tesselation connecting them which remains in a ball of radius $4R$ about one of the points. Since a Delaunay tesselation meets vertex-to-vertex, hypothesis (iv) implies that there are only finitely many ways to tile a ball of radius $4R$. Thus $D_X(2R)$ is finite, and Theorem 2.1 implies that X is a Delone set of finite type.

(i) $\Rightarrow$ (v). A Delone set X of finite type is finitely generated by Theorem 2.1. To show the other part, suppose that $\phi : [X] \to \mathbb{Z}^s$ is an address map and set

$$C_2 := \max\{||\phi(\mathbf{y})|| : \mathbf{y} \in (X - X) \cap B(0; 2R)\} \ .$$

Given $\mathbf{x}, \mathbf{x}' \in X$ by Lemma 2.2 there exists a chain (2.4) at points of X connecting $\mathbf{x}$ to $\mathbf{x}'$ that satisfies $||\mathbf{x}_{i+1} - \mathbf{x}_i|| \leq 2R$ and the number of points fit at most $\frac{4R}{r^2}||\mathbf{x}_{i+1} - \mathbf{x}_i||$. Using the linearity of ϕ on $[X]$, we have

$$\begin{aligned} ||\phi(\mathbf{x}) - \phi(\mathbf{x}')|| &\leq \sum_{i=1}^{m} ||\phi(\mathbf{x}_{i+1}) - \phi(\mathbf{x}_i)|| \\ &= \sum_{i=1}^{m} ||\phi(\mathbf{x}_{i+1} - \mathbf{x}_i)|| \\ &\leq C_2 m \leq \frac{4C_2R}{r^2}||\mathbf{x} - \mathbf{x}'|| \ , \end{aligned}$$

which proves (iv) with $C_0 = \frac{4C_2R}{r^2}$.

(v) $\Rightarrow$ (i). If $\mathbf{x}, \mathbf{x}' \in X$ satisfy $||\mathbf{x}' - \mathbf{x}|| \leq 2R$, then by hypothesis

$$||\phi(\mathbf{x} - \mathbf{x}')|| = ||\phi(\mathbf{x}) - \phi(\mathbf{x}')|| \leq 2CR \ .$$

Addresses all lie in $\mathbb{Z}^s$, hence there are only finitely many choices for $\phi(\mathbf{x} - \mathbf{x}')$, hence for $\mathbf{x} - \mathbf{x}'$, since $\phi : X \to \mathbb{Z}^s$ is one-to-one. Thus $(X - X) \cap B(\mathbf{0}; 2R)$ is finite, so X is of finite type by Theorem 2.1. $\square$

Theorem 2.2 yields an upper bound on the growth of the distance counting function $D_T(X)$.

Corollary 2.2 *Let X be a Delone set of finite type, of rank s. There exists a positive constant C_1 depending on X such that*

$$D_X(T) \leq C_1 T^s \ , \quad \textit{all} \ \ T \geq 1 \ . \tag{2.12}$$

Proof. Suppose that $\mathbf{x}, \mathbf{x}' \in X$ with $||\mathbf{x} - \mathbf{x}'|| \leq T$. The point $\mathbf{x} - \mathbf{x}'$ is determined by its address $\phi(\mathbf{x} - \mathbf{x}')$, and by (iv) of Theorem 2.2, there is a constant C_0 such that

$$||\phi(\mathbf{x} - \mathbf{x}')|| \leq C_0||\mathbf{x} - \mathbf{x}'|| = C_0 T \ .$$

The set of elements in $\mathbb{Z}^s$ in a ball of radius T is at most CT^s for all $T \geq 1$, and (2.12) follows. $\square$

We next obtain an upper bound on the atlas counting function $N_X^*(T)$ of a Delone set of finite type.

Definition 2.8 The *configurational entropy* $H_c(X)$ of a Delone set X in $\mathbb{R}^n$ is given by

$$H_c(X) := \limsup_{T \to \infty} \frac{\log N_X^*(T)}{T^n} .$$

We prove:

Theorem 2.3 *If X is a Delone set of finite type in $\mathbb{R}^n$, there is a constant C_2 depending on X such that*

$$N_X^*(T) \leq \exp(C_2 T^n) , \quad \textit{all} \quad T \geq 1 . \tag{2.13}$$

That is, X has finite configurational entropy.

Remark. The factor T^n in the bound (2.13) cannot be improved, see Example 3.1 in §3.

Proof. We may build up all possible patches of radius T out of *cubical patches* of the form $X \cap C(\mathbf{x}; 8R)$ for $\mathbf{x} \in X$, where $C(\mathbf{x}; \rho)$ denotes a cube of side 2ρ centered at $\mathbf{x}$. Since $X \cap C(\mathbf{x}; 8R) \subseteq X \cap B(\mathbf{x}; 8\sqrt{n}R)$, the number C' of translation-inequivalent cubical patches $X \cap C(\mathbf{x}; 4R)$ is at most $D_X(8\sqrt{n}R)$.

We bound the number of different cubical patches of side $2T$ centered at $\mathbf{0}$, all of those subpatches of side $2R$ match those in the list $\mathcal{L}$. To begin, there are C' choices for the patch centered at $\mathbf{0}$. Now we proceed to add cubical patches in concentric layers. The centers of the new patches are chosen at already constructed points, within distance R of the boundary, so the new cubical patch extends at least $3R$ outside. At the k-th stage we have covered a cube of side $8 + 6kR$. We can arrange that all cube centers chosen are separated by at least $2R$, hence there are at most $O((\frac{T}{R})^n)$ cubical patches to choose in the process. Thus there are at most $(C')^{C''T^n}$ possible choices of cubical patches of side $2T$, which proves (2.13). $\square$

We conclude this section with an example showing that the requirement that $X - X$ be closed is necessary in the definition of a Delone set of finite type. The Delone set

$$X = \{n + \frac{1}{3n^2 + 3} : n \in \mathbb{Z}\}$$

in $\mathbb{R}$ has the property that $X - X$ is discrete. However $X - X$ has 1 as a finite limit point, so X is not a Delone set of finite type; it is not even finitely generated.

3. Meyer Sets

Meyer [54] recently defined a mathematical notion which he called a " quasicrystal", which is based on concepts in harmonic analysis that he introduced around 1970, see Meyer [52, 53]. These sets are termed *Meyer sets* by Moody [55, 56], who unifies and extends the known characterizations of Meyer sets. Moody [56] takes a harmonic analysis criterion as the basic definition of Meyer set (relatively dense harmonious set), while Meyer [54] defines them to be those Delone sets X such that $X - X \subseteq X + F$ for some finite set F. These properties are equivalent to $X - X$ being a Delone set, see [39].

It is known that Meyer sets can also be characterized in terms of cut-and-project sets.

Definition 3.1 Let Λ be a full rank lattice in $\mathbb{R}^d = \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$, and let $\pi^{||}$ and $\pi^{\perp}$ be orthogonal projections onto the factors $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A *window* Ω is a bounded open subset of $\mathbb{R}^m$, and the *strip* $S(\Omega)$ in $\mathbb{R}^d$ associated to the window Ω is

$$S(\Omega) := \mathbb{R}^n \times \Omega = \{\mathbf{w} \in \mathbb{R}^d : \pi^{\perp}(\mathbf{w}) \in \Omega\} .$$

The *cut-and-project set* $X(\Lambda, \Omega)$ associated to the data (Λ, Ω) is

$$X(\Lambda, \Omega) = \pi^{||}(\Lambda \cap S(\Omega)) . \tag{3.1}$$

We call d the *dimension* of the data (Λ, Ω). A given cut-and-project set X may be constructed in many ways, using different pairs (Λ, Ω) and (Λ', Ω') of different dimensions. Some authors allow the window Ω to be a closed set.

An alternate definition of cut-and-project sets, which is used in [25] and [60], appears slightly different but gives the same sets. It takes a fixed lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$, and projects it onto an arbitrary pair $(E^{||}, E^{\perp})$ of orthogonal subspaces of dimensions n and m, respectively, using orthogonal projections $\pi^{||}$ and $\pi^{\perp}$, respectively. It identifies $E^{||}$ with $\mathbb{R}^n$ by a linear map, and takes a window set Ω in $E^{\perp}$.

Definition 3.2 A cut-and-project set is *nondegenerate* if $\pi^{||} : \mathbb{R}^d \rightarrow \mathbb{R}^n$ is one-to-one on Λ, i.e. if

$$\Lambda \cap (\{\mathbf{0}\} \times \mathbb{R}^m) = \{(\mathbf{0}, \mathbf{0})\} .$$

It is *irreducible* if it is nondegenerate and $\pi^{\perp}(\Lambda)$ is dense in $\mathbb{R}^m$.

An irreducible cut-and-project set coincides with the concept of *model set* of Meyer [53]. Meyer proved that a Delone set X is a Meyer set if and only if there is an irreducible cut-and-project set X' and a finite set F such that

$$X \subseteq X' + F' , \tag{3.2}$$

see [24,25,27].

Meyer sets are Delone sets of finite type, hence are finitely generated. We characterize which finitely generated Delone sets are Meyer sets in terms of their address maps, using the following concept due to Meyer.

Definition 3.3 A map $\psi : [X] \to \mathbb{R}^d$ is an *almost linear mapping* on X if there exists a linear map $L_\psi : \mathbb{R}^n \to \mathbb{R}^d$ such that

$$||\psi(\mathbf{x}) - L_\psi(\mathbf{x})|| < C , \quad \text{all } \mathbf{x} \in X . \tag{3.3}$$

This condition (3.3) applies only to $\mathbf{x} \in X$ and not to general elements of $[X]$.

The following result gives several characterizations of Meyer sets. These characterizations are mainly due to Meyer, (see also Moody [56]); the equivalence of (iv) and (v) seems to be new.

Theorem 3.1 *The following properties of a discrete set X in $\mathbb{R}^n$ are equivalent.*

(i) X is a Meyer set. That is, $X - X$ is a Delone set.

(ii) X is a Delone set and there is a finite set F such that

$$X - X \subseteq X + F .$$

(iii) X is a finitely generated Delone set and every homomorphism $\psi : [X] \to \mathbb{R}^d$ for some $d \geq 1$ is an almost linear mapping on X.

(iv) X is a finitely generated Delone set and the address map $\phi : [X] \to \mathbb{Z}^s$ is an almost linear mapping on X, i.e. there is a linear map $\tilde{L} : \mathbb{R}^n \to \mathbb{R}^s$ and a constant C such that

$$||\phi(\mathbf{x}) - \tilde{L}(\mathbf{x})|| \leq C , \quad \textit{all } \mathbf{x} \in X . \tag{3.4}$$

(v) X is a finitely generated Delone set and there exists a nondegenerate cut-and-project set X' of dimension at most $rank(X)$ such that

$$X \subseteq X' . \tag{3.5}$$

Remark. The cut-and-project set X' containing X that appears in (v) is not necessarily irreducible.

Proof. (i) $\Leftrightarrow$ (ii). This is proved in Lagarias [39].

(ii) $\Rightarrow$ (iii). This result is due to Meyer and a proof can be found in §8 of Moody [56]. We mention some important points. The map $\tilde{L} : \mathbb{R}^n \to \mathbb{R}^s$ can be constructed as an "ideal address map," as follows. For each $\mathbf{y} \in \mathbb{R}^n$ define

$$\tilde{L}(\mathbf{y}) := \lim_{k\to\infty} \frac{\phi(\mathbf{x}_k)}{2^k} \tag{3.6}$$

where $\mathbf{x}_k \in X$ satisfies $||\mathbf{x}_k - 2^k\mathbf{y}|| \le R$. Using the Meyer set property one proves that this limit exists and is unique, and satisfies

$$\tilde{L}(c\mathbf{y} + \mathbf{y}') = c\tilde{L}(\mathbf{y}) + \tilde{L}(\mathbf{y}') , \quad \mathbf{y}, \mathbf{y}' \in \mathbb{R}^n ,$$

hence is linear. We call $\tilde{L}$ an *ideal address map* because it satisfies the identity

$$\mathbf{y} = \sum_{j=1}^{s} \hat{L}(\mathbf{y})_j \mathbf{v}_j , \quad \text{all } \mathbf{y} \in \mathbb{R}^n . \tag{3.7}$$

This identity is derived from (3.6) using the address identity

$$\mathbf{x} = \sum_{j=1}^{s} \phi(\mathbf{x})_j \mathbf{v}_j , \quad \text{all } \mathbf{x} \in [X] .$$

(iii) $\Rightarrow$ (iv). Immediate.

(iv) $\Rightarrow$ (ii). Given $\mathbf{x}, \mathbf{x}' \in X$, the hypothesis (iv) gives

$$||\tilde{L}(\mathbf{x} - \mathbf{x}') - \phi(\mathbf{x} - \mathbf{x}')|| \le C \tag{3.8}$$

Since X is a Delone set, there is some point $\mathbf{x}'' \in X$ with

$$||\mathbf{x}'' - (\mathbf{x} + \mathbf{x}')|| \le R ,$$

and by hypothesis

$$||\tilde{L}(\mathbf{x}'') - \phi(\mathbf{x}'')|| \le C . \tag{3.9}$$

Now

$$||\tilde{L}(\mathbf{x}'' - \mathbf{x} + \mathbf{x}')|| \le C'||\mathbf{x}'' - (\mathbf{x} + \mathbf{x}')|| \le C'R , \tag{3.10}$$

where $C' = ||\tilde{L}||$. The triangle inequality applied to (3.8)–(3.10) gives

$$||\phi(\mathbf{x}'' - (\mathbf{x} - \mathbf{x}'))|| = ||\phi(\mathbf{x}'') - \phi(\mathbf{x} - \mathbf{x}')|| \le 2C + C'R . \tag{3.11}$$

hence there are a finite number of choices for addresses in $\mathbb{Z}^s$ satisfying (3.11). Since the address map is one-to-one on $[X]$, there are a finite number of $\mathbf{y} = \mathbf{x}'' - (\mathbf{x} - \mathbf{x}') \in [X]$ satisfying (3.11), and we call this set F. We have shown that

$$X - X \subseteq X + F ,$$

which proves (ii).

(iv) $\Rightarrow$ (v). We use the almost linear mapping $\tilde{L} : \mathbb{R}^n \to \mathbb{R}^s$ associated to the address map by (3.7) to construct a suitable nondegenerate cut-and-project set. Set $s = n + m$, so $m \geq 0$, and let the vector space

$$V := \{\hat{L}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$$

by the image of the map $\tilde{L}$. We first show that $dim(V) = n$. Define $\psi : \mathbb{R}^s \to \mathbb{R}^n$ by

$$\psi(u_1, \ldots, u_s) = \sum_{j=1}^{s} u_j \mathbf{v}_j$$

and observe that (3.7) gives

$$\psi \circ \hat{L}(\mathbf{y}) = \mathbf{y} \quad \text{all } \mathbf{y} \in \mathbb{R}^n .$$

Thus $\psi(V) = \mathbb{R}^n$ hence $dim(V) = n$. Now set

$$W := ker(\psi) = \{(\ell_1, \ldots, \ell_s) : \sum_{j=1}^{s} \ell_j \mathbf{v}_j = 0\} .$$

Now $dim(W) = s - n = m$. Choose an orthonormal basis of W,

$$\boldsymbol{\ell}^{(i)} = (\ell_{i1}, \ldots, \ell_{is}) , \quad 1 \leq i \leq m ,$$

and set

$$\mathbf{w}_j := \tilde{L}(\mathbf{v}_j) - \mathbf{e}_j , \quad 1 \leq j \leq s ,$$

where $\mathbf{e}_j = (0, , \ldots, 1, \ldots, 0)$ is the j-th coordinate vector. Now (3.7) implies that $\mathbf{w}_j \in W$, hence we may define coefficients $\{\alpha_{ji}\}$ by

$$\mathbf{w}_j = \sum_{i=1}^{m} \alpha_{ji} \boldsymbol{\ell}^{(i)} , \quad 1 \leq j \leq s . \tag{3.12}$$

Let $\Lambda = \mathbb{Z}[\mathbf{v}_1^*, \ldots, \mathbf{v}_s^*]$ be the additive subgroup of $\mathbb{R}^s$ spanned by

$$\mathbf{v}_j^* := (\mathbf{v}_j, \alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jm}) , \quad 1 \leq j \leq s .$$

We claim that Λ is a full rank lattice in $\mathbb{R}^d$. To show this, it suffices to prove that the vectors $\mathbf{v}_j^*$ are $\mathbb{R}$-linearly independent. So suppose that

$$\sum_{j=1}^{s} u_j \mathbf{v}_j^* = \mathbf{0} \ .$$

The first n coordinates give

$$\sum_{j=1}^{s} u_j \mathbf{v}_j = \mathbf{0} \ ,$$

hence $\mathbf{u} = (u_1, \ldots, u_d) \in W$. Now the last m coordinates give

$$\sum_{j=1}^{s} u_j \alpha_{ji} = 0 \ , \quad 1 \le i \le m \ .$$

Thus

$$\sum_{j=1}^{s} u_j \mathbf{w}_j = \sum_{j=1}^{s} u_j \left(\sum_{i=1}^{m} \alpha_{ji} \boldsymbol{\ell}^{(i)} \right) = \sum_{i=1}^{m} \boldsymbol{\ell}^{(i)} \left(\sum_{j=1}^{s} u_j \alpha_{ji} \right) = \mathbf{0} \ ,$$

and the definition of $\mathbf{w}_j$ yields

$$\sum_{j=1}^{s} u_j \tilde{L}(\mathbf{v}_j) - (u_1, u_2, \ldots, u_s) = \mathbf{0} \ . \tag{3.13}$$

Since each coordinate is linear,

$$(u_1, \ldots, u_s) = \tilde{L}(\sum_{j=1}^{s} u_j \mathbf{v}_j) = \tilde{L}(\mathbf{0}) = \mathbf{0} \ .$$

Thus $[\mathbf{v}_1^*, \ldots, \mathbf{v}_s^*]$ are linearly independent, and Λ is a full rank lattice in $\mathbb{R}^s$.

The projection $\pi^{||} : \mathbb{R}^s \to \mathbb{R}^n$ sends

$$\pi^{||}(\mathbf{w}_j) = \mathbf{v}_j \ , \quad 1 \le j \le s \ ,$$

hence it is one-to-one on Λ since $[X] = \mathbb{Z}[\mathbf{v}_1, \ldots, \mathbf{v}_s]$ is a free module.

Now suppose that $\mathbf{x} \in [X]$ and its address is

$$\mathbf{x} = \sum_{j=1}^{s} \phi(\mathbf{x})_j \mathbf{v}_j \ , \quad \text{with } \phi(\mathbf{x})_j \in \mathbb{Z} \ .$$

Then

$$\mathbf{x}^* := \sum_{j=1}^{s} \phi(\mathbf{x})_j \mathbf{w}_j$$

has $\mathbf{x}^* \in \Lambda$, and $\pi^{||}(\mathbf{x}^*) = \mathbf{x}$. Now set

$$\pi^{\perp}(\mathbf{x}^*) = \sum_{j=1}^{s} \phi(\mathbf{x})_j (\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jm}) = (z_1, \ldots, z_m) \ .$$

Since the vectors $\boldsymbol{\ell}^{(i)}$ are orthonormal, we have

$$\begin{aligned}
||\pi^{\perp}(\mathbf{x}^*)|| &= \left(\sum_{j=1}^{m} z_i^2\right)^{1/2} = ||\sum_{i=1}^{m} z_i \boldsymbol{\ell}^{(i)}|| \\
&= ||\sum_{i=1}^{m} \sum_{j=1}^{s} \phi(\mathbf{x})_j \alpha_{ji} \boldsymbol{\ell}^{(i)}|| \\
&= ||\sum_{j=1}^{s} \phi(\mathbf{x})_j \mathbf{w}_j|| \\
&= ||\phi(\mathbf{x}) - \tilde{L}(\mathbf{x})|| \ ,
\end{aligned}$$

where (3.12) and (3.13) were used at the last step. The almost linear property (3.4) then yields

$$||\pi^{\perp}(\mathbf{x}^*)|| \leq C \ .$$

It therefore suffices to take the window set

$$\Omega = \{\mathbf{z} \in \mathbb{R}^m : ||\mathbf{z}|| < C + 1\} \ .$$

to obtain a nondegenerate cut-and-project set $X(\Lambda, \Omega)$ of dimension s with

$$X \subseteq X(\Lambda, \Omega) \ ,$$

which is (v).

(v) $\Rightarrow$ (iv). This is a result of Meyer, for which see Moody [56, Theorem 9.1]. Alternatively one can directly reconstruct the map $\tilde{L}(\mathbf{x})$ from X', using

$$\tilde{L}(\mathbf{x}) := (\pi^{||})^{-1}(\mathbf{x}) \cap \Lambda \quad \text{for } \mathbf{x} \in X \ .$$

This is well-defined since X' is nondegenerate. Now $\tilde{L}$ extends to a linear map on $[X]$, and (3.4) holds by the cut-and-project property of X'. $\square$

The following result of Meyer follows directly from property (v) above.

Corollary 3.1a *If X is a Meyer set and Y is a Delone set with $Y \subseteq X$, then Y is a Meyer set.*

We obtain a similar result for sets locally derivable from Meyer sets.

Corollary 3.1b *If X is a Meyer set and Y is a Delone set locally derivable from X, then Y is a Meyer set.*

Proof. We claim that $Y \subseteq X + F'$ for some finite set F'. Indeed there are only finitely many different R'-patches of X, and the elements of Y can be viewed as obtained by individually shifting the center of each R'-patch of X, in a finite number of different ways, so the claim follows. Since X is a Meyer set, the set $X' = X + F'$ is a Meyer set because it satisfies property (ii) above:

$$X' - X' \quad \subseteq X' + (F + F' - F'). \tag{3.14}$$

Now Y is a Meyer set by Corollary 3.1a. $\square$

We also apply Corollary 3.1a in the following example.

Example 3.1. There exists a Meyer set $X \subseteq \mathbb{Z}^n$ in $\mathbb{R}^n$ such that

$$N_X^*(T) \geq \exp(C_3 T^n) \ , \quad \textit{all} \ \ T \geq 1 \ , \tag{3.15}$$

with $C_3 = \frac{1}{2}\log 2$.

We will arrange that $2\mathbb{Z}^n \subseteq X \subseteq \mathbb{Z}^n$, which guarantees that X is a Meyer set by Corollary 3.1a. We use a simple random construction: pick each point in $\mathbb{Z}^n \backslash 2\mathbb{Z}^n$ independently to be in X with probability 1/2. Now consider a box of side $2^n T$ centered at a point in $2\mathbb{Z}^n$. Every possible assignment of cells in the box not in the lattice $2\mathbb{Z}^n$ occurs with positive probability, hence for any fixed assignment the possibility that no such configuration occurs in X is zero. The union over all such exceptional events, for $T = 1, 2, 3, \ldots$, then has probability zero, hence with probability one every possible assignment occurs, for every $T \geq 1$. Thus, with probability one,

$$N_X^*(2^n T) \geq \exp((\log 2)(2^n - 1)T^n) \ , \quad \text{for all} \quad T \geq 1 \ ,$$

as required.

There are much stronger examples known than the one above. A Delone set X of finite type is *minimal* if it has the *repetitivity property*: For each $T > 0$ there is a finite value $M_X(T)$ such that every ball of radius $M_X(T)$ contains a copy of each patch of X of radius T. Furstenberg [16, Theorem III.2] constructs a symbolic sequence giving a minimal set with positive entropy. By using intervals of different integer lengths to represent the symbols, we obtain a one-dimensional Meyer set which is minimal and satisfies (3.15) for some C_3. Pleasants

[62] constructs such examples in arbitrary dimensions. Minimal sets are important because they are an analogue of "ground state" configurations, cf. Radin [65], and the introduction to part II.

4. Inflation Symmetries

We recall the basic definition.

Definition 4.1 A Delone set X in $\mathbb{R}^n$ has an *inflation symmetry* by the real number $\eta > 1$ if $\eta X \subseteq X$.

If $\mathbf{0} \notin X$ then $X' = X \cup \{\mathbf{0}\}$ is also a Delone set with an inflation symmetry by η, so without loss of generality we may assume that $\mathbf{0} \in X$. An inflation symmetry is not a complete symmetry of a Delone set X, because the set $X \backslash \eta X$ contains infinitely many points. Inflation similarities occur in Delone sets constructed as control points of self-similar tilings, see [33], [75].

A (complex) number η is an *algebraic integer* if $f(\eta) = 0$ for some nonzero monic polynomial $f(u) \in \mathbb{Z}[u]$, i.e.

$$f(u) = u^n + \sum_{j=0}^{n-1} a_j u^j \ .$$

The *degree* of η is the minimal degree of any nonzero polynomial $f(u) \in \mathbb{Z}[u]$ with $f(\eta) = 0$. The *algebraic conjugates* of η are the other roots of the minimal degree monic polynomial in $\mathbb{Z}[u]$ which η satisfies.

Several classes of real algebraic integers appear in studying inflation symmetries.

Definition 4.2 Let η be a real algebraic integer with $\eta > 1$.

(i) η is a *Pisot number* or *Pisot-Vijayaraghavan number* if all algebraic conjugates $|\eta'| < 1$.

(ii) η is a *Salem number* if all algebraic conjugates $|\eta'| \le 1$ and at least one $|\eta'| = 1$.

(iii) η is a *Perron number* if all algebraic conjugates $|\eta'| < \eta$.

(iv) η is a *Lind number* if all algebraic conjugates $|\eta'| \le \eta$ and at least one $|\eta'| = \eta$.

Pisot and Salem numbers have been extensively studied, see Bertin et al. [4]. Perron numbers were introduced and studied by Lind [47], [48], [52] and Lind numbers are introduced here.

Theorem 4.1 *Let X be a Delone set in $\mathbb{R}^n$ such that $\eta X \subseteq X$ for a real number $\eta > 1$.*

(i) If X is finitely generated, then η is an algebraic integer. If X has rank s, then the degree of η divides s.

(ii) If X is a Delone set of finite type, then η is an algebraic integer with all algebraic conjugates $|\eta'| \leq \eta$. That is, η is a Perron number or a Lind number.

(iii) If X is a Meyer set, then η is an algebraic integer with all algebraic conjugates $|\eta'| \leq 1$. That is, η is a Pisot number or a Salem number.

Proof. (i). Let $\mathbf{0} \in X$ and suppose that X has *rank* s, so that

$$[X] = \mathbb{Z}[X] = \mathbb{Z}[\mathbf{v}_1, \ldots, \mathbf{v}_s] \tag{4.1}$$

for suitable vectors $\{\mathbf{v}_i : 1 \leq i \leq s\}$ independent over $\mathbb{Q}$. Now $\eta X \subseteq X$ implies that $\eta[X] \subseteq [X]$, hence

$$\eta \mathbf{V} = \mathbf{M}\mathbf{V} \ , \quad \text{with} \ \ \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_s \end{bmatrix} \ , \tag{4.2}$$

in which $\mathbf{M}$ is an $s \times s$ integral matrix and $\mathbf{V}$ is an $s \times n$ real matrix. Now

$$f(x) = \det(\lambda \mathbf{I} - \mathbf{M}) \in \mathbb{Z}[\lambda]$$

is a monic polynomial, and $f(\eta) = 0$ since $\eta \mathbf{I} - \mathbf{M}$ annihilates an n-dimensional subspace of $\mathbb{R}^s$. Thus η is an algebraic integer, of degree at most s.

To show that $d = degree(\eta)$ divides s, we use the rational canonical form of $\mathbf{M}$. Let the $f(\lambda) \in \mathbb{Z}[\lambda]$ be the minimal degree monic polynomial that η satisfies, and write

$$f(\lambda) = \lambda^d + \sum_{j=1}^{d} a_j \lambda^{d-j} \ .$$

Now (6.2) gives $\eta^j \mathbf{V} = \mathbf{M}^j \mathbf{V}$ for $1 \leq j \leq d$, hence

$$f(\mathbf{M})\mathbf{V} = (\mathbf{M}^d + \sum_{j=1}^{d} a_j \mathbf{M}^{d-j})\mathbf{V} = \mathbf{0} \ . \tag{4.3}$$

Each row of $f(\mathbf{M})$ is an integer vector, so (4.2) gives

$$\sum_{j=1}^{s} f(\mathbf{M})_{ij} \mathbf{v}_j = \mathbf{0}, \quad 1 \leq i \leq s \ . \tag{4.4}$$

The vectors $\mathbf{v}_j$ are independent over $\mathbb{Q}$, so it follows that all $f(\mathbf{M})_{ij} = 0$, hence $f(\mathbf{M}) = \mathbf{0}$. Thus the minimal polynomial of $\mathbf{M}$ divides $f(\lambda)$, and since $f(\lambda)$ is irreducible over $\mathbb{Q}$, it must equal $f(\lambda)$. It follows that the rational canonical form of $\mathbf{M}$ (over $\mathbb{Q}$) can only be a block diagonal matrix with all blocks equal to the $d \times d$ companion matrix $\mathbf{C}_f$ of f, see [2, p. 199]. Thus d divides s.

(ii). Since X is a Delone set of finite type, by Theorem 2.1 it is finitely generated, of rank s, say. Now η is a real algebraic integer by (i). Let η be of degree d and set $\mathbb{Z}[\eta] := \mathbb{Z}[1, \eta, \ldots \eta^{d-1}]$, which is a ring.

We now argue by contradiction, and suppose that η' is an algebraic conjugate of η with $|\eta'| > \eta > 1$. We first observe that the $\mathbb{Z}$-module $[X]$ is actually a $\mathbb{Z}[\eta]$-module, because if $\mathbf{y} \in [X]$, then so is $\eta\mathbf{y}$, since we can express $\mathbf{y}$ as a finite integral combination of elements of X, and if $\mathbf{x} \in X$ then $\eta\mathbf{x} \in X$. Now $[X]$ is not necessarily a free $\mathbb{Z}[\eta]$-module, but there exists a free $\mathbb{Z}[\eta]$-module $[W]$ in $\mathbb{R}^n$ containing $[X]$ as a submodule of finite index. It can be obtained from the matrix converting $\mathbf{M}$ to canonical form in (i) above. Now d divides s and we have

$$[X] \subseteq [W] = \mathbb{Z}[\eta][\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{s/d}] \ ,$$

i.e. there is some $\mathbb{Z}$-basis $[\mathbf{v}_1, \ldots, \mathbf{v}_s]$ of $[W]$ with

$$\mathbf{v}_{i+(j-1)s/d} = \eta^{i-1}\mathbf{w}_j \ , \quad 1 \le i \le d \ , \quad 1 \le j \le s/d \ . \tag{4.5}$$

We now define the $\mathbb{Z}[\eta']$-module

$$[W'] := \mathbb{Z}[\eta'][\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{s/d}] \ ,$$

which we regard as a set of points in $\mathbb{C}^n$, viewing $\eta' \in \mathbb{C}$ even if η' is real. We call this space $\mathbb{C}^n$ the *shadow space*, and we measure distances on it, using the Hermitian norm

$$||\mathbf{z} - \mathbf{z}'||_H^2 := \sum_{i=1}^{n} |z_i - z_i'|^2 \ , \quad \text{where } \ \mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \ .$$

We define a *shadow map* $\sigma : [W] \to [W']$ by $\sigma(\mathbf{w}_j) = \mathbf{w}_j$ for $1 \le j \le s/d$ and extend it using the Galois map from $\mathbb{Z}[\eta]$ to $\mathbb{Z}[\eta']$, so

$$\begin{aligned} \sigma(\mathbf{y}_1 + \mathbf{y}_2) &= \sigma(\mathbf{y}_1) + \sigma(\mathbf{y}_2) \\ \sigma(\eta\mathbf{y}_1) &= \eta'\sigma(\mathbf{y}_1) \end{aligned}$$

for $\mathbf{y}_1, \mathbf{y}_2 \in [X]$. Let $[X']$ denote the image of $[X]$ under the shadow map, and choose a point $\mathbf{x} \in X$ whose image $\sigma(\mathbf{x}) \neq \mathbf{0}$. We obtain a contradiction by measuring the length of the shadow image of the vectors $\mathbf{y}_k = \eta^k\mathbf{x} - \mathbf{x}$ in two different ways, as $k\to\infty$. First, $\sigma(\mathbf{y}_k) = (\eta')^k\sigma(\mathbf{x}) - \sigma(\mathbf{x})$ hence

$$\begin{aligned} ||\sigma(\mathbf{y}_k)|| &= |(\eta')^k - 1| \; ||\sigma(\mathbf{x})||_H \\ &\geq c_0|\eta'|^k \quad \text{as } k\to\infty \; . \end{aligned} \tag{4.6}$$

On the other hand, both $\mathbf{x}$, $\eta^k\mathbf{x} \in X$ and using the Delone property of X we can find a path of elements $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m \in X$ from $\mathbf{x}_0 = \mathbf{x}$ to $\mathbf{x}_m = T^k\mathbf{x}$, where

$$||\mathbf{x}_k - \mathbf{x}_{k-1}|| \leq 3R \; , \quad 1 \leq k \leq m \; , \tag{4.7}$$

and $m = \frac{1}{R}||(\eta^k - 1)\mathbf{x}||$. Indeed, draw a straight line connecting $\mathbf{x}$ and $\eta^k\mathbf{x}$ in $\mathbb{R}^n$, and by the Delone property there is a vector $\mathbf{x}_k \in X$ within distance R of the point $(1 + k\frac{||\mathbf{x}||}{R})\mathbf{x}$, and the bound (4.6) follows from the triangle inequality. Since X is of finite type, there are only finitely many vectors in

$$(X - X) \cap B(\mathbf{0}, 3R) \; ,$$

hence

$$C := \max\{||\sigma(\mathbf{y})||_H : \mathbf{y} \in (X - X) \cap B(0; 3R)\} \; . \tag{4.8}$$

is finite. It follows from (4.7) that

$$||\sigma(\mathbf{x}_k - \mathbf{x}_{k-1})||_H \leq C \; . \tag{4.9}$$

Now the linearity of σ on $[X]$ gives

$$\sigma(\mathbf{y}_k) = \sum_{i=1}^{m} \sigma(\mathbf{x}_k) - \sigma(\mathbf{x}_{k-1}) \; ,$$

hence (4.9) gives

$$\begin{aligned} ||\sigma(\mathbf{y}_k)||_H &\leq \sum_{i=1}^{m} ||\sigma(\mathbf{x}_k) - \sigma(\mathbf{x}_{k-1})||_H \\ &\leq mC = \frac{1}{R}(\eta^k - 1)||\mathbf{x}||C \\ &\leq C_1\eta^k \quad \text{as } k\to\infty \; . \end{aligned}$$

If $|\eta'| > \eta$, this contradicts (4.6) for k large enough, so (ii) follows.

(iii). This is a result of Meyer [17, Theorem 6]. Here we present an alternative proof. Since X is a Meyer set, the address map $\phi : [X]\to\mathbb{R}^s$ is an almost linear map. Let $\tilde{L} : \mathbb{R}^n\to\mathbb{R}^s$ be the (unique) linear map such that

$$|\phi(\mathbf{x}) - \tilde{L}(\mathbf{x})| \leq C \; , \quad \text{all } \mathbf{x} \in X \; . \tag{4.10}$$

Part (i) of this proof showed that there is an integer matrix $\mathbf{M}$ such that

$$\phi(\eta \mathbf{x}) = \mathbf{M}\phi(\mathbf{x}) \ , \quad \text{all } \mathbf{x} \in [X] \ , \tag{4.11}$$

and that the minimal polynomial of $\mathbf{M}$ is the monic irreducible polynomial $f(\lambda) \in \mathbb{Z}[\lambda]$ that η satisfies. Write

$$f(\lambda) = \prod_{i=1}^{d} (\lambda - \eta_i) = \lambda^d + \sum_{j=1}^{d} a_j \lambda^{d-j} \ ,$$

with $\eta = \eta_1$. Part (i) also showed that $\mathbf{M}$ is diagonalizable over $\mathbb{C}$, and has a direct sum decomposition into eigenspaces

$$\mathbb{C}^n = V_1 + V_2 + \ldots + V_d \ , \tag{4.12}$$

in which V_j is the eigenspace for the eigenvalue η_j. We also have

$$\mathbf{M}\mathbf{v} = \eta_j \mathbf{v} \quad \text{for all } \mathbf{v} \in V_j, \quad 1 \le j \le d \ ,$$

and $\dim(V_j) = \frac{s}{d}$. The image $\phi([X])$ spans $\mathbb{R}^s$, so there exists $\mathbf{x} \in X$ with $\phi(\mathbf{x}) \neq \mathbf{0}$. Now $\phi(\mathbf{x})$ uniquely decomposes into eigenvectors using (4.12), i.e.

$$\phi(\mathbf{x}) = \sum_{j=1}^{d} \mathbf{v}_j \ , \quad \text{with each } \mathbf{v}_j \in V_j \ .$$

Since $\phi(\mathbf{x}) \in \mathbb{Z}^s \backslash \{\mathbf{0}\}$, every $\mathbf{v}_j \neq \mathbf{0}$, because the Galois group of $f(\lambda)$ permutes all the $\mathbf{v}_j$ transitively.

Iterating (4.11) gives

$$\mathbf{M}^k \phi(\mathbf{x}) = \sum_{j=1}^{d} (\eta_j)^k \mathbf{v}_j \ , \quad \text{all } k \ge 1 \ . \tag{4.13}$$

On the other hand (4.10) gives

$$|\tilde{L}(\eta^k \mathbf{x}) - \phi(\eta^k \mathbf{x})| = |\eta^k \tilde{L}(\mathbf{x}) - \mathbf{M}^k \phi(\mathbf{x})| \le C \ .$$

Combining this bound with (4.13) yields

$$||\eta^k (L(\mathbf{x}) - \mathbf{v}_1) - \sum_{j=2}^{d} (\eta_j)^k \mathbf{v}_j|| \le C \ , \quad \text{all } k \ge 0 \ . \tag{4.14}$$

We claim that this implies that $L(\mathbf{x}) = \mathbf{v}_1$ and that $|\eta_j| \le 1$ for all $j \ge 2$. To show this, consider the i^{th} coordinate vector. Then (4.14) gives $|h_k^{(i)}| \le C$, where

$$h_k^{(i)} = \sum_{j=1}^{d} v_{ji} (\eta_j)^k \ , \tag{4.15}$$

where v_{ji} is the i-th entry of $L(\mathbf{x}) - \mathbf{v}_1$ for $i = 1$ and of $-\mathbf{v}_i$ for $i \geq 2$. This relation yields the identity

$$\sum_{k=0}^{\infty} h_k^{(i)} z^k = \sum_{j=1}^{d} \frac{v_{ji}}{1 - \eta_j z} , \quad 1 \leq i \leq s ,$$

where both sides converge as functions of the complex variable z for sufficiently small $|z|$. The relations $|h_k^{(i)}| \leq C$ imply that the power series on the left converges in the unit disk $|z| < 1$. Since all η_j are distinct we must have $v_{ji} = 0$ for any $|\eta_j| > 1$. Thus $v_{j1} = 0$ for $1 \leq i \leq s$, hence $L(\mathbf{x}) = \mathbf{v}_1$. Since each $\mathbf{v}_j \neq \mathbf{0}$ for $j \geq 2$, for each such j there exists some $v_{ji} \neq 0$ hence we infer that $|\eta_j| \leq 1$ for $j \geq 2$. $\square$

Remarks. Each of (i)–(iii) is apparently sharp, in the sense that given any algebraic integer η one can construct a one-dimensional Delone set $X \subseteq \mathbb{R}$ of the appropriate type having $\eta X \subseteq X$. For (iii) this was shown by Meyer. For (i) it is quite easy to give a recursive construction of $X \subseteq \mathbb{R}$, with $[X] = \mathbb{Z}[\eta]$, such that $0 \in X$, $X = -X$ and $\eta X \subseteq X$. To do this, construct points in $[1, \eta)$, then recursively construct points in $[\eta^k, \eta^{k+1})$ by first including all points $\eta\mathbf{y}$ for $\mathbf{y} \in [\eta^{k-1}, \eta^k)$, and adding some new points as necessary to preserve the Delone property. For (ii) this construction can apparently be carried out but requires ingenuity in choosing the correct new points to add; I have verified it for $\eta = \sqrt{2}$, which is a Lind number. The proof of Theorem 4.1 above works more generally for real numbers η such that $|\eta| > 1$, with the condition in (ii) modified to require that all algebraic conjugates $|\eta'| \leq |\eta|$. R. Kenyon observes that the Delone set of finite type $X := \{-n\sqrt{2} : n = 1, 2, ...\} \cup \{n : n = 0, 1, 2, ...\}$ has an inflation by $-\sqrt{2}$.

There are results in the literature for self-similar tilings which are analogous to Theorem 4.1. Kenyon [34] proves a result analogous to (i) for tilings. The results for self-similar tilings (see Thurston [79] and Kenyon [33], [35]) analogous to cases (ii) and (iii) differ slightly in that they rule out inflation symmetry factors η that have an algebraic conjugate η' which satisfies the equality $|\eta'| = \eta$ in (ii) or $|\eta'| = 1$ in (iii). That is, they rule out Lind numbers and Salem numbers in (ii) and (iii), respectively. R. Kenyon (private communication) observes that if to Theorem 4.1 one adds the extra hypothesis that the Delone set X is repetitive (i.e. minimal), then Lind numbers can be ruled out in (ii), and Salem numbers can be ruled out in (iii). The repetitivity property holds for self-affine tilings with the unique composition property, see Solomyak [76].

5. One-Dimensional Delone Sets of Finite Type and Symbolic Dynamics

In this section we contrast properties of general Delone sets of finite type and Meyer sets by studying various one-dimensional examples. In particular, we consider diffraction properties of such sets.

One-dimensional Delone sets of finite type X are in one-to-one correspondence with tilings of the line by intervals of finitely many different lengths. If $X = \{x_n : -\infty < n < \infty\}$ with the points of X numbered in increasing order, then we may regard X as tiling the line with intervals $I_n = [x_n, x_{n+1}]$. Theorem 2.1 implies that these intervals have finitely many different lengths, call them $\alpha_0, \alpha_1, \ldots, \alpha_m$. It also implies the converse assertion that any such tiling gives a Delone set X of finite type. Such a tiling is determined by the location of the point x_0 together with the symbolic expression

$$S_X = (\ldots, s_{-1}, s_0, s_1, \ldots) \in \{0, 1, \ldots, m\}^{\mathbb{Z}} , \tag{5.1}$$

in which s_i is a symbol indicating the *type* of the i^{th} interval, and type j denotes an interval of length α_j. Since we only consider sets X up to translation, we may suppose that $x_0 = 0$, in which case the symbol sequence S_X and the assignment of lengths α_j to symbol type j completely describes X.

We now specialize to the case in which X has intervals of only two lengths, 1 and α, in which case a symbol sequence $S \in \{0,1\}^{\mathbb{Z}}$ and the length α completely specifies a Delone set $X = X_\alpha(S)$ with $x_0 = 0$. We will show that conditions for the sets $X_\alpha(S)$ to have a unique autocorrelation measure γ and a unique diffraction measure $\hat{\gamma}$ depend on the symbolic address S but not on α, while in general the property of X being a Meyer set depends on *both* S and α.

Definition 5.1 A symbol sequence $S \in \{0,1\}^{\mathbb{Z}}$ is *almost linear* if there is a real number β such that the partial sums $S_{n+1} := s_0 + s_1 + \ldots + s_n$ and $S_{-n} := s_{-1} + s_{-2} + \ldots + s_{-n}$, for $n \geq 0$, satisfy

$$|S_n - n\beta| \leq C , \quad \text{all } n \in \mathbb{Z} , \tag{5.2}$$

for some constant C.

The following result characterizes when $X_\alpha(S)$ is a Meyer set.

Theorem 5.1 *Let $S \in \{0,1\}^{\mathbb{Z}}$ be a symbol sequence. If S is almost linear then $X_\alpha(S)$ is a Meyer set for all positive α. If S is not almost linear then $X_\alpha = X_\alpha(S)$ is a Delone set of finite type and satisfies:*

(i) If α is rational then $rank(X_\alpha) = 1$ and X_α is a Meyer set.

(ii) If α is irrational then $rank(X_\alpha) = 2$ and X_α is not a Meyer set.

Proof. If $\alpha \in \mathbb{Q}$ then $X_\alpha(S)$ has rank 1 for any $S \subseteq \{0,1\}^{\mathbb{Z}}$. If α has denominator m then $X_\alpha(S) \subseteq \frac{1}{m}\mathbb{Z}$ hence $S_\alpha(S)$ is a Meyer set by Corollary 3.1.

If $\alpha \notin \mathbb{Q}$, then if $S \in \{0,1\}^{\mathbb{Z}}$ is not 0^∞ or 1^∞, then $X_\alpha(S)$ has rank 2. Now $x_n = (n - S_n) + S_n\alpha$, so the address map $\phi : [X] \to \mathbb{Z}^2$ obtained from the basis $v_1 = 1, v_2 = \alpha$ is

$$\phi(x_n) = (n - S_n, \;\; S_n) \;, \quad \text{all } n \in \mathbb{Z} \;.$$

By Theorem 3.1 X_α is a Meyer set if and only if there exists a linear function $\tilde{L} : \mathbb{R} \to \mathbb{R}^2$, say $\tilde{L}(x) = (\beta_1 x, \beta_2 x)$ such that

$$||\phi(x_n) - \tilde{L}(x_n)|| \le C \quad \text{all } n \in \mathbb{Z} \;,$$

for some constant C. This gives

$$||((1-\beta_1)n + (\beta_1\alpha - \beta_1 - 1)S_n \;, \quad \beta_2 n + (\beta_2\alpha - \beta_2 - 1)S_n)|| \le C \;, \quad \text{all } n \in \mathbb{Z} \;, \tag{5.3}$$

Each coordinate separately is bounded, and at least one of them implies (5.2), so that for X_α to be a Meyer set, S must be almost linear. Conversely, if S is almost linear, then (5.6) holds for the choice

$$\beta_1 = \frac{1-\beta}{1-\beta+\alpha\beta} \quad \text{and} \quad \beta_2 = \frac{\beta}{1-\beta+\alpha\,\beta} \;,$$

so $X_\alpha(S)$ is a Meyer set. This includes the cases when S is 0^∞ and 1^∞. $\square$

Results analogous to Theorem 5.1 hold for sets X derived from symbol sequences drawn from an alphabet with m letters, and tiling $\mathbb{R}$ with intervals of length $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}$.

For results concerning autocorrelation measures and diffraction intensities for one-dimensional sets X, we rely on results of Hof [25] [27] and Solomyak [74]–[77]. Recall from [25] that an *autocorrelation measure* γ of a Delone set X in $\mathbb{R}^n$ is any limit point in the vague topology of the sequence of measures

$$\gamma^L := (2L)^{-n} \sum_{\mathbf{x},\mathbf{x}' \in X \cap [-L,L]^d} \delta_{\mathbf{x}-\mathbf{x}'} \;, \qquad \text{for } L = 1, 2, 3, \ldots \,.$$

We are primarily interested in sets X that have a unique autocorrelation measure γ. In order for a unique autocorrelation measure γ for $X_\alpha(S)$ to exist, the symbol sequence S must have limiting frequencies for the tiles of lengths 1 and α, i.e. there must exist a value $\bar{d}$ such that

$$\lim_{n\to\infty} \frac{1}{n} S_n = \lim_{n\to\infty} \frac{1}{n} S_{-n} = \bar{d} \ . \tag{5.4}$$

We can easily construct Meyer sets X for which such limits do not exist. For example take $X = X_2(S)$ with $s_n = s_{-n}$ and with the s_n for $n \geq 0$ being given by

$$S^+ = \prod_{k=1}^{\infty} (00)^{2^k} (01)^{2^k} (11)^{2^k} \ . \tag{5.5}$$

Here $X_2(S) \subseteq \mathbb{Z}$ is a Delone set with $R = 2$, hence is a Meyer set, and (5.4) fails to hold.

Definition 5.2 (i) A Delone set X of finite type is said to have *uniform patch frequencies* if for each patch $\mathcal{P} = (X - x) \cap B(0;T)$ with $x \in X$, and each $\epsilon > 0$ there is a value $L = L(\mathcal{P}, \epsilon)$ such that for any interval of length L on the line X contains copies of the patch $\mathcal{P}$ having a density between $d(\mathcal{P}) - \epsilon$ and $d(\mathcal{P}) + \epsilon$. That is,

$$d(\mathcal{P}) - \epsilon \leq \frac{\#\{y : y + \mathcal{P} \subseteq X \text{ and } u \leq y \leq u + L\}}{\#\{x : x \in X \text{ and } u \leq x \leq u + L\}} \leq d(\mathcal{P}) + \epsilon \ . \tag{5.6}$$

We say that a symbol sequence $S \subseteq \{0,1\}^{\mathbb{Z}}$ has *uniform patch frequencies* if this holds for the Delone set $X_\alpha(S)$ obtained from S by taking $\alpha = 2$. It is easy to see that the symbol sequence S has uniform patch frequencies if and only if the same is true for all $X_\alpha(S)$, $\alpha > 0$. Solomyak [77] observes that this property implies that the tiling dynamical system attached to each such $X_\alpha(S)$ is uniquely ergodic, and that they each have a well-defined autocorrelation measure γ in the sense of Hof [25], [27]. Furthermore γ is a pure discrete measure supported on $X - X$, with

$$\gamma = \sum_{y \in X - X} f(y) \delta_y \ , \tag{5.7}$$

in which $f(y)$ is the uniform limiting frequency of occurrence of the distance y in X. The Fourier transform $\hat{\gamma}$ of the autocorrelation measure is the diffraction measure of $X_\alpha(S)$, and is a positive measure.

Solomyak [75, Theorem 5.1] gives sufficient conditions under which X has uniform limiting patch frequencies. In particular, these conditions are satisfied for S arising as fix-points of irreducible substitutions, and $X = X_\alpha(S)$ have the properties above for all $\alpha > 0$. Such $X_\alpha(S)$ are non-Meyer sets if α is irrational and the largest eigenvalue of the substitution system S is not a Pisot number.

Under the assumption that S has uniform patch frequencies, each set $X_\alpha(S)$ has a well-defined *diffraction measure* $\hat{\gamma}_\alpha$. The occurrence of a nonzero discrete component in $\hat{\gamma}_\alpha$ depends on both S and α, as is shown by examples in Kolàr et al. [37]. Their examples suggest that a two-letter irreducible substitution system can only have a nonzero discrete component when $X_\alpha(S)$ is a Meyer set. The associated expansion constant of the substitution S is then necessarily a Pisot number. Solomyak [74] has shown there exist multi-letter substitution rules with largest eigenvalue not a Pisot number, which have a discrete component in the spectrum of the associated substitution dynamical system. (See also [13].) At present it is not known whether there exist non-Meyer sets $X_\alpha(S)$ with S having uniform limiting patch frequencies, such that the diffraction measure $\hat{\gamma}$ contains some nontrivial discrete spectrum.

Several authors have considered analogous problems for self-similar tilings and tiling dynamical systems. Gähler and Klitzing [19, Theorem 4.1 ff] state that the set of control points of a self-similar tiling with inflation factor η can have a nontrivial discrete component in the diffraction measure only if η is a Pisot number. Solomyak ([75, Theorem 8.1], and [76, Theorem 1.4]) gives sufficient conditions for a self-similar tiling dynamical system to have pure discrete spectrum. When these conditions apply, the diffraction measure $\hat{\gamma}$ of any Delone set X constructed from a tiling in the dynamical system by marking tiles is a pure discrete measure. It seems likely that such Delone sets X must always be Meyer sets.

Acknowledgment. I am indebted to Michael Baake, Ludwig Danzer, Franz Gähler, Richard Kenyon, Doug Lind, Peter Pleasants, Jim Reeds, Marjorie Senechal, Boris Solomyak, Paul Wright and the referees for helpful comments. Martin Schlottman supplied a correction concerning Voronoi tesselations on an earlier version of the paper, and suggested including Delaunay tesselations in Theorem 2.2.

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