Spectral Sets and Factorizations of Finite Abelian Groups

Jeffrey C. Lagarias

AT&T Bell Laboratories
Murray Hill, NJ 07974

Yang Wang[1]

School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332

(revision — March 20, 1996)

Abstract

A *spectral set* is a subset Ω of $\mathbf{R}^n$ with Lebesgue measure $0 < \mu(\Omega) < \infty$ such that there exists a set Λ of exponential functions which form an orthogonal basis of $L^2(\Omega)$. The spectral set conjecture of B. Fuglede states that a set Ω is a spectral set if and only if Ω tiles $\mathbf{R}^n$ by translation. We study sets Ω which tile $\mathbf{R}^n$ using a rational periodic tile set $\mathcal{S} = \mathbf{Z}^n + \mathcal{A}$, where $\mathcal{A} \subseteq \frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}$ is finite. We characterize geometrically sets Ω that tile $\mathbf{R}^n$ with such a tile set. Certain tile sets $\mathcal{S}$ have the property that every bounded measurable set Ω which tiles $\mathbf{R}^n$ with $\mathcal{S}$ is a spectral set, with a fixed spectrum $\Lambda_{\mathcal{S}}$. We call such $\Lambda_{\mathcal{S}}$ a *universal spectrum* for such $\mathcal{S}$. We give a necessary and sufficient condition for a rational periodic set Λ to be a universal spectrum for $\mathcal{S}$, which is expressed in terms of factorizations $A \oplus B = G$ where $G = \mathbf{Z}_{N_1} \times \cdots \times \mathbf{Z}_{N_n}$, and $A := \mathcal{A} \pmod{\mathbf{Z}^n}$. In dimension $n = 1$ we show that $\mathcal{S}$ has a universal spectrum whenever N_1 is the order of a "good" group in the sense of Hajós, and for various other sets $\mathcal{S}$.

[1]Research supported in part by the National Science Foundation, grant DMS-9307601.

Spectral Sets and Factorizations of Finite Abelian Groups

Jeffrey C. Lagarias

AT&T Bell Laboratories
Murray Hill, NJ 07974

Yang Wang

School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332

(revision — March 20, 1996)

1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be a set of finite positive Lebesgue measure $0 < \mu(\Omega) < \infty$. We call Ω a *spectral set* if there exists a subset Λ of $\mathbf{R}^n$ such that the set of functions $\{e_\lambda(x) : \lambda \in \Lambda\}$ forms an *orthogonal basis* for $L^2(\Omega)$, where $e_\lambda(x) := e^{2\pi i \lambda \cdot x}$. In this case (Ω, Λ) is called a *spectral pair* and Λ is called a *spectrum* or *exponent set* for Ω. (A spectral set Ω may have several different spectra.)

The notion of spectral set was introduced by Fuglede [3] in his study of a problem of I. E. Segal: Which open sets $\Omega \subset \mathbf{R}^n$ have the property that the partial differential operators $-i\frac{\partial}{\partial x_1}$, $i\frac{\partial}{\partial x_2}, \ldots, -i\frac{\partial}{\partial x_n}$ acting on the space, $C_c^\infty(\Omega)$ of compactly supported smooth functions on Ω extend to a set of commuting self-adjoint operators $H = (H_1, H_2, \ldots, H_n)$ defined on a dense subspace $dom(H)$ of $L^2(\Omega)$. This property is called the *integrability property* in [17]. Fuglede proved [3, Theorem I(b) and remark (1), p. 108] that any open set Ω which is a spectral set has the integrability property, and that for each spectrum Λ for Ω there is a unique maximal extension H having Λ as its spectrum. He also proved [3, Theorem I(a)] that any connected open set Ω satisfying an extra condition (Nikodym domain) which has the integrability property must conversely be a spectral set. Later Pedersen [16, Theorem 3.2] removed the extra condition from this converse result. Fuglede observed that a converse result does not generally hold for non-connected sets Ω.

This paper studies spectral sets. These sets form a very restrictive class of sets in $\mathbf{R}^n$. For example, neither a circular disk nor a triangle can be a spectral set ([3]). Fuglede proved that if Ω is a fundamental domain of a lattice L, then Ω is a spectral set, with spectrum $\Lambda = L^*$, the dual lattice to L, which is

$$L^* = \{\gamma : \gamma \cdot \lambda \in \mathbf{Z} \text{ for all } \lambda \in L\} . \tag{1.1}$$

However, not all spectral sets are fundamental domains of a lattice. For example, the disconnected set $\Omega = [0,1] \cup [2,3]$ is a spectral set with spectrum $\Lambda = \mathbf{Z} + \{0, \frac{1}{4}\}$.

Fuglede [3, p. 119] conjectured that a set Ω with finite measure $0 < \mu(\Omega) < \infty$ is a spectral set if and only if it is a *direct summand*, which he defined to be a set that tiles $\mathbf{R}^n$ with measure-disjoint translates up to a measure zero set. Let $\Omega_1 \simeq \Omega_2$ mean that Ω_1 and Ω_2 differ in a set of measure zero. We state this conjecture as:

Spectral Set Conjecture. *A set Ω in $\mathbf{R}^n$ having finite Lebesgue measure $0 < \mu(\Omega) < \infty$ is a spectral set if and only if Ω tiles $\mathbf{R}^n$ by translation, i.e. $\mathbf{R}^n$ is the disjoint union (up to sets of measure zero)*

$$\mathbf{R}^n \cong \Omega + \mathcal{S} = \cup_{s \in \mathcal{S}} (\Omega + s) , \tag{1.2}$$

in which $\mathcal{S}$ is a discrete set called a tile set.

Extensive studies have been made of spectral sets,[2] mainly by Fuglede [3], Jorgensen [7], Jorgensen and Pedersen ([9], [10]) and Pedersen ([17], [18]). However the spectral set conjecture has not been resolved in either direction in any dimension, even dimension $n = 1$. In fact it has not even been resolved for all sets $\Omega = [0,1] + \mathcal{B}$ where $\mathcal{B} \subset \mathbf{Z}$ is a finite set.

This paper studies the spectral set property for bounded sets Ω of positive measure which tile $\mathbf{R}^n$ with a rational periodic tile set. A tile set $\mathcal{S}$ is *periodic* if there is a (full-rank) lattice L in $\mathbf{R}^n$ such that

$$\mathcal{S} := L + \{a_1, \ldots, a_m\} ,$$

and it is *rational* if in addition all coset differences $a_i - a_j$ are commensurate with the lattice L, i.e. there exists an integer N such that $N(a_i - a_j) \in L$ for all i, j. By an affine transformation we can always reduce a rational periodic tile set to the case that $L = \mathbf{Z}^n$ and

$$\mathcal{S} := \mathbf{Z}^n + \mathcal{A} , \tag{1.3}$$

[2] Petersen [17, p. 125] uses a more general definition of spectral set that includes certain sets Ω of infinite Lebesgue measure. His definition agrees with the one here finite measure case [17, Corollary 1.11].

with $\mathcal{A} := \{a_1, \ldots, a_m\} \subseteq \frac{1}{N_1}\mathbf{Z} \times \frac{1}{N_2}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}$, for positive integers $N_1, N_2, \ldots, N_n$. We assume that the cosets of $\mathbf{Z}^n$ given by $\mathcal{A}$ are all distinct, that is

$$(\mathcal{A} - \mathcal{A}) \cap \mathbf{Z}^n = \{0\} \ . \tag{1.4}$$

This paper makes the observation that certain tiling sets $\mathcal{S}$ given by (1.3) possess a *universal spectrum* $\Lambda = \Lambda_{\mathcal{S}}$. A *universal spectrum* for $\mathcal{S}$ is a set Λ that is a spectrum simultaneously for *all* bounded sets Ω that tile $\mathbf{R}$ with tiling set $\mathcal{S}$. The result of Fuglede [3, p. 113] stated in (1.1) above is such a result, in that it says that if $\mathcal{S} = L$ is a lattice, then the dual lattice L^* is a universal spectrum for $\mathcal{S}$. We consider as candidates for universal spectra rational periodic sets in $\mathbf{Z}^n$, namely

$$\Lambda = N_1\mathbf{Z} \times N_2\mathbf{Z} \times \cdots \times N_n\mathbf{Z} + \{\gamma_1, \gamma_2, \ldots, \gamma_s\} \ . \tag{1.5}$$

in which $\Gamma = \{\gamma_1, \ldots, \gamma_s\} \subseteq \mathbf{Z}^n$ has $(\Gamma - \Gamma) \cap (N_1\mathbf{Z} \times \cdots \times N_n\mathbf{Z}) = \{0\}$.

We give a necessary and sufficient condition for a given rational periodic set Λ to be a universal spectrum for the tile set $\mathcal{S}$. This condition involves factorizations of the finite abelian group $G = \mathbf{Z}_{N_1} \times \cdots \times \mathbf{Z}_{N_n}$. A *factorization* of a finite abelian group G, written $G = A \oplus B$, is one in which every $g \in G$ has a unique representation

$$g = a + b, \text{ with } a \in A, \quad b \in B \ . \tag{1.6}$$

In what follows, we identify $G = \mathbf{Z}_{N_1} \times \cdots \times \mathbf{Z}_{N_n}$ with the set $\frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z} \pmod{\mathbf{Z}^n}$ viewed as a subgroup of the n-torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$.

Definition 1.1. *For $\mathcal{A} \subseteq \frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}$, define* Comp $(\mathcal{A})$ *to be the collection of all sets*

$$\mathcal{B} = \{b_1, \cdots, b_m\} \subseteq \left(\frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}\right) \cap [0,1]^n \ , \tag{1.7}$$

such that $A := \mathcal{A} \pmod{\mathbf{Z}^n}$ and $B := \mathcal{B} \pmod{\mathbf{Z}^n}$ yield a factorization

$$A \oplus B = \mathbf{Z}_{N_1} \times \cdots \times \mathbf{Z}_{N_n} := \left(\frac{1}{N_1}\mathbf{Z} \times \ldots \times \frac{1}{N_n}\mathbf{Z}\right) / \mathbf{Z}^n \ . \tag{1.8}$$

Such sets $\mathcal{B}$ are called complementing sets for $\mathcal{A}$ in $\mathbf{Z}_{N_1} \times \ldots \times \mathbf{Z}_{N_n}$.

In §3 we prove the following criterion for a universal spectrum.

Theorem 1.1. *Let $\mathcal{S} = \mathbf{Z}^n + \mathcal{A}$ with*

$$\mathcal{A} = \{a_1, \ldots, a_m\} \subseteq \frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z} \ . \tag{1.9}$$

Then a set $\Lambda = (N_1\mathbf{Z} \times \cdots \times N_n\mathbf{Z}) + \Gamma$ with $\Gamma \subseteq \mathbf{Z}^n$ is a spectrum for all bounded sets Ω that tile $\mathbf{R}^n$ with tile set $\mathcal{S}$ if and only if Λ is a spectrum for each of the sets

$$\Omega_{\mathcal{B}} := \left[0, \frac{1}{N_1}\right] \times \cdots \times \left[0, \frac{1}{N_n}\right] + \mathcal{B}, \qquad \mathcal{B} \in \text{ Comp } (\mathcal{A}) \ . \tag{1.10}$$

Since Comp$(\mathcal{A})$ is finite, this result yields an algorithm to test whether a given set Λ as above is a universal spectrum for $\mathcal{S}$, using Theorem 2.3 in §2, which gives a Fourier-analytic criterion to check if $\Omega_{\mathcal{B}}$ has Λ as a spectrum.

Which tiling sets $\mathcal{S}$ have universal spectra? As far as we know at present, the following could be true.

Universal Spectrum Conjecture. *Let $\mathcal{S} := \mathbf{Z}^n + \mathcal{A}$, where $\mathcal{A} \subseteq \frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}$ such that $A := \mathcal{A}$ (mod $\mathbf{Z}^n$) admits some factorization $A \oplus B = \mathbf{Z}_{N_1} \times \cdots \times \mathbf{Z}_{N_n}$. Then $\mathcal{S}$ has a universal spectrum Λ of the form $N_1\mathbf{Z} \times N_2\mathbf{Z} \times \cdots \times N_n\mathbf{Z} + \Gamma$, with $\Gamma \subseteq \mathbf{Z}^n$.*

For any given set $\mathcal{S}$, this conjecture is checkable in a finite number of operations using Theorem 1.1 and Corollary 2.3a.

There is currently little evidence supporting this conjecture in dimensions $n \geq 2$, and it is open even in dimension 1. In this paper we exhibit many one-dimensional tiling sets $\mathcal{S}$ that have universal spectra. We verify the universal spectrum conjecture in many specific cases using Theorem 1.1 together with known results about the structure of factorizations of finite abelian groups G.

Factorizations of abelian groups have been extensively studied, see the discussion in §4 and the paper of Tijdeman [22]. Given a factorization $A \oplus B = G$ of an abelian group G, we call A and B *complementing sets*, and we say that B is an *A-complement* (mod G).

Definition 1.2. *Let $G = \mathbf{Z}_N$ be a finite cyclic group. Call a subset $A \subseteq G$* eligible *if there exists $\mathcal{A} = \{a_1, \ldots, a_m\} \subseteq \mathbf{Z}$ with $0 \in \mathcal{A}$ and* $\gcd(a_1, \ldots, a_m) = 1$*, such that $A := \mathcal{A}$ (mod N). An eligible set A has the* strong Tijdeman property *if there is a proper subgroup $H \subsetneqq G$ such that if B satisfies $A \oplus B = G$ and $0 \in B$, then $B \subseteq H$. Equivalently, there is some prime $p \mid |\mathcal{A}|$ such that if $\mathcal{B} \subseteq \mathbf{Z}$ with $0 \in \mathcal{B}$ and $B := \mathcal{B}$ (mod N), then $p | \mathcal{B}$.*

Definition 1.3. *A group $G = \mathbf{Z}_N$ has the* strong Tijdeman property *if all eligible $A \subseteq G$ have the strong Tijdeman property.*

The strong Tijdeman property is hereditary in the sense that if G has this property, so does any subgroup of G. (Lemma 4.1). We prove:

Theorem 1.2. *If the cyclic group $\mathbf{Z}_N$ has the strong Tijdeman property, then any tile set $\mathcal{S} = \mathbf{Z} + \frac{1}{N}\mathcal{A}$ with $\mathcal{A} \subseteq \mathbf{Z}$ has a universal spectrum $\Lambda = N\mathbf{Z} + \Gamma$ for some $\Gamma \subseteq \mathbf{Z}$.*

We show that all cyclic groups that are *good* in the sense of Hajós [5] have the strong Tijdeman property (Theorem 4.1). The complete list of finite abelian groups that are good was found by Sands [20], cf. Proposition 4.1. The cyclic groups $\mathbf{Z}_N$ which are good are exactly those N that divide one of $pqrs, p^2qr$, p^2q^2 or p^nq, where p, q, r, s are any distinct primes and $n \geq 1$. Fuglede [3] stated without proof that if $\mathcal{B} \subseteq \mathbf{Z}$ is such that $B = \mathcal{B} \pmod N$ is a complementing set for $\mathbf{Z}_N$, and $\mathbf{Z}_N$ is a good group, then $\Omega = [0,1] + \mathcal{B}$ is a spectral set. This result follows from Theorems 1.2 and Theorem 4.1.

As far as we know, the following could be true.

Strong Tijdeman Conjecture. *Every cyclic group $\mathbf{Z}_N$ has the strong Tijdeman property.*

We use results of Sands [19] and Tijdeman [22], which give certain other sets A that have the strong Tijdeman property, to prove:

Theorem 1.3. *Let $\mathcal{S} = \mathbf{Z} + \frac{1}{N}\mathcal{A}$, where $\mathcal{A} \subseteq \mathbf{Z}$ is such that $\mathcal{A} \pmod N$ is a complementing set of $\mathbf{Z}_N$. If either $|\mathcal{A}|$ or $\frac{N}{|\mathcal{A}|}$ is a prime power, then $\mathcal{S}$ has a universal spectrum $\Lambda = N\mathbf{Z} + \Gamma$ with $\Gamma \subseteq \mathbf{Z}$.*

In the one-dimensional case, it was shown in [15] that every compact set T that tiles $\mathbf{R}$ has a rational periodic tiling. We apply this to show the strong Tijdeman conjecture implies that all compact sets that tile $\mathbf{R}$ are spectral sets (Theorem 5.1).

The approach to finding rational periodic sets that are universal spectra via the strong Tijdeman property cannot always work in dimensions $n \geq 3$. In §4 we give an example of an abelian group of rank 3 which contains a set A that does not even have the Tijdeman property, which is a weak version of the strong Tijdeman property.

There is a "dual" approach to the spectral set conjecture, which takes a set Λ and asks: Which sets Ω have Λ as a spectrum? Jorgensen and Pedersen [8] showed that in order for Λ to be a spectrum, it must be uniformly discrete. Fuglede [3, p. 114] showed that if Λ is a lattice L, then it is a spectrum of exactly those sets Ω which are a fundamental domain of its dual lattice L^*. Pedersen [17, Theorem 2] derives necessary and sufficient conditions for a set Ω to be a spectral set with a given rational periodic set $\Lambda = \mathbf{Z}^d + \Gamma$ as spectrum. These conditions do not reveal whether or not Ω tiles $\mathbf{R}^n$ by translation.

We remark that Fuglede [3, p. 120–121] was certainly aware of relations between rational periodic spectra and factorizations of abelian groups. He states several results without proof, which are all proved in this paper, in Theorems 1.2 and 4.1.

The contents of this paper are as follows. In §2 we develop a criterion for a set $\Omega := [0,1]+\mathcal{B}$ with $\mathcal{B} \subseteq \mathbf{Z}$ to have a particular rational periodic set as a spectrum. To do this we use theorems of [8], [18] which we reprove using Fourier-analytic methods. In §3 we give a structure theorem for sets Ω that have a rational periodic tiling $\mathcal{S}$ in terms of complementing sets of finite abelian groups, and prove Theorem 1.1. In §4 we survey results on factorizations of finite abelian groups G and prove that good groups have the strong Tijdeman property. Finally, §5 treats the one-dimensional case, proves Theorems 1.2 and 1.3, and also proves Theorem 5.1 mentioned above.

2. Spectral Set Criterion in $\mathbf{R}^n$

In this section we let $\Omega \subseteq \mathbf{R}^n$ be a Lebesgue measurable set with measure $0 < \mu(\Omega) < \infty$; we do not assume Ω is bounded or that it Ω tiles $\mathbf{R}^n$ by translation. We develop a criterion to decide whether a given set $\Lambda = \mathbf{Z}^n + \Gamma$ with $\Gamma \subseteq \mathbb{Q}$ is a spectrum for Ω.

Let χ_Ω be the characteristic function of Ω and let Z_Ω denote the set of real zeros of its Fourier transform

$$Z_\Omega := \left\{ \lambda \in \mathbf{R}^n : \int_\Omega e^{2\pi i \lambda \cdot x} dx = 0 \right\} . \tag{2.1}$$

In order for a set $\{e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda\}$ to be an orthogonal set in $L^2(\Omega)$ we must have

$$\Lambda - \Lambda \subseteq Z_\Omega \cup \{0\} .$$

Since $\mathbf{Z}^n \subseteq \Lambda - \Lambda$ we certainly need $\mathbf{Z}^n \backslash \{0\} \subseteq \Lambda - \Lambda$. The following result, due to Jorgensen and Pedersen ([8], Theorem 6.2), characterizes sets Ω with $\mathbf{Z}^n \backslash \{0\} \subseteq Z_\Omega \cup \{0\}$. We give a short Fourier-analytic proof, different from that in [8].

Recall that a *fundamental domain* of a lattice L is a set D such that $\cup_{\ell\in L}(D+\ell)$ tiles $\mathbf{R}^n$ almost everywhere, i.e. D is a measurable set such that the mapping $\pi_n : \mathbf{R}^n \to \mathbf{R}^n/L$ restricted to the domain D is one-to-one almost everywhere and is onto except for a measure zero set.

Theorem 2.1. *Let $\Omega \subset \mathbf{R}^n$ be a Lebesgue measurable set with measure $0 < \mu(\Omega) < \infty$, and suppose that $\mathbf{Z}^n\backslash\{0\} \subseteq Z_\Omega$. Then there exist fundamental domains $D_1, \ldots, D_m$ of the lattice $\mathbf{Z}^n$ such that*

$$\Omega = D_1 \cup D_2 \cup \cdots \cup D_m \tag{2.2}$$

where $\mu(D_i \cap D_j) = 0$ for all $i \neq j$. In particular, $\mu(\Omega) = m$ is a positive integer.

Proof. Let $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ be the n-dimensional torus and $\pi_n : \mathbf{R}^n \to \mathbf{T}^n$ be the canonical covering map. We prove that Ω is an m-fold covering of $\mathbf{T}^n$ under π_n (almost everywhere) for some positive integer m. ¿From the assumption,

$$\int_\Omega e^{2\pi i k\cdot x}dx = \int_{\mathbf{T}^n} \sigma(z)e^{2\pi i k\cdot z}dz = 0$$

for all $k \in \mathbf{Z}^n\backslash\{0\}$, where $\sigma(z) := \sigma_{y\in\pi_n^{-1}(z)}\chi_\Omega(y)$. But $\{e_k : k \in \mathbf{Z}^n\}$ is total in $L^2(\mathbf{T}^n)$. Hence $\sigma(z) = \alpha$ almost everywhere for some nonzero constant α. Since $\chi_\Omega(y)$ is integer-valued, we must have $\alpha = m$ for some positive integer m. Therefore Ω is an m-fold covering of $\mathbf{T}^n$ under π_n.

To construct the D_j, put the lexicographic total indexing on $\mathbf{R}^n$, order the preimages of each point $x \in \mathbf{T}^n$ using this ordering, and assign the j-th point in this ordering (when it exists) to D_j. We omit details showing that the D_j are measurable. $\square$

Does Ω tile $\mathbf{R}^n$ by translation? Theorem 2.1 shows that Ω gives at least a multiple tiling of $\mathbf{R}^n$ with multiplicity m, using the tile set $\mathbf{Z}^n$.

The following result was originally obtained by Pedersen [18]; we give an independent proof.

Theorem 2.2. *Let $\Omega \subset \mathbf{R}^n$ be such that $0 < \mu(\Omega) < \infty$. Let $\Gamma \subset \mathbf{R}^n$ be a finite set with $(\Gamma - \Gamma) \cap \mathbf{Z}^n = \{0\}$, and suppose that $\{e_\lambda : \lambda \in \mathbf{Z}^n + \Gamma\}$ is orthogonal in $L^2(\Omega)$. Then $(\Omega, \mathbf{Z}^n + \Gamma)$ is a spectral pair if and only if $|\Gamma| = \mu(\Omega)$.*

Proof. Suppose that $(\Omega, \mathbf{Z}^n + \Gamma)$ is a spectral pair. Then the set $\mathbf{Z}^n + \Gamma$ is both a set of sampling and a set of interpolation for $B(\Omega) := \{\hat{f}(\lambda) : f \in L^2(\Omega)\}$. Theorems 3 and 4 of Landau [16] together imply that $\mu(\Omega) = |\Gamma|$, because $|\Gamma|$ is the asymptotic density of $\mathbf{Z}^n + \Gamma$.

We now prove the converse: if $|\Gamma| = \mu(\Omega)$ then $(\Omega, \mathbf{Z}^n + \Gamma)$ is a spectral pair. We need only show that $\{e_\lambda : \lambda \in \mathbf{Z}^n + \Gamma\}$ is total in $L^2(\Omega)$.

Again we let $\pi_n : \mathbf{R}^n \to \mathbf{T}^n$ denote the canonical covering map from $\mathbf{R}^n$ to the torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$. Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$. Since $\{e_\lambda : \lambda \in \mathbf{Z}^n + \Gamma\}$ is orthogonal in $L^2(\Omega)$, for any $\gamma_\ell \neq \gamma_j$ we have $\hat{\chi}_\Omega(\gamma_j - \gamma_l + k) = 0$ for all $k \in \mathbf{Z}^n$. So

$$\hat{\chi}_\Omega(\gamma_j - \gamma_l + k) = \int_\Omega e^{2\pi i(\gamma_j - \gamma_l + k)\cdot x} dx = \int_{\mathbf{T}^n} \eta(z) e^{2\pi i k\cdot z} dz = 0$$

for all $k \in \mathbf{Z}^n$, where

$$\eta(z) := \sum_{y \in \pi_n^{-1}(z)} \chi_\Omega(y) e^{2\pi i(\gamma_j - \gamma_l)\cdot y} .$$

It follows from the totalness of $\{e_k : k \in \mathbf{Z}^n\}$ in $L^2(\mathbf{T}^n)$ that

$$\eta(z) = \sum_{y \in \pi_n^{-1}(z)} \chi_\Omega(y) e^{2\pi i(\gamma_j - \gamma_l)\cdot y} = 0 . \tag{2.3}$$

Since $\mathbf{Z}^n \backslash \{0\} \subseteq \mathbf{Z}_\Omega$, it follows from Theorem 2.1 that Ω is the disjoint union (up to sets of zero measure) of m fundamental domains of the lattice $\mathbf{Z}^n$, say $\Omega = \cup_{j=1}^m D_j$. The projection map $\pi_n : D_j \to \mathbf{T}^n$ is one-to-one and onto, and we let $\sigma_j : \mathbf{T}^n \to D_j$ denote its inverse, so that $\pi_n \circ \sigma_j =$ identity. Then we may rewrite (2.3) as

$$\sum_{k=1}^m e^{2\pi i(\gamma_j - \gamma_l)\cdot \sigma_k(z)} = 0, \text{ for almost all } z \in \mathbf{T}^n . \tag{2.4}$$

Now, for any $f(x) \in L^2(\Omega)$ we define $v_f(z) \in L^2(\mathbf{T}^n)^m$ by

$$v_f(z) := [f(\sigma_1(z)), \ldots, f(\sigma_m(z))]^T . \tag{2.5}$$

Then $f \mapsto v_f$ defines a linear map from $L^2(\Omega)$ to $L^2(\mathbf{T}^n)^m$ which is one-to-one and onto. Let $H_j(\Omega)$ be the subspace of $L^2(\Omega)$ defined as follows:

$$H_j(\Omega) := \left\{ \sum_{\lambda \in b_j + \mathbf{Z}^n} a_\lambda e_\lambda : a_\lambda \in \mathbf{C} \ , \quad \sum_{\lambda \in b_j + \mathbf{Z}^n} |a_\lambda|^2 < \infty \right\} .$$

For $1 \le j \le m$ denote

$$u_j(z) := [e^{2\pi i \gamma_j \dot{\sigma}_1(z)}, \ldots, e^{2\pi i \gamma_j \cdot \sigma_m(z)}]^T . \tag{2.6}$$

Claim. *Let $g(x) \in L^2(\Omega)$. Then $g(x) \in H_j(\Omega)$ if and only if $v_g(z) = h(z)u_j(z)$ for some $h(z) \in L^2(\mathbf{T}^n)$.*

Proof of Claim. Suppose that $g(x) \in H_j(\Omega)$. Then

$$g(x) = \sum_{k\in\mathbf{Z}^n} a_k e^{2\pi i(\gamma_j+k)\cdot x} = e^{2\pi i\gamma_j\cdot x}\sum_{k\in\mathbf{Z}^n} a_k e^{2\pi i k\cdot x} .$$

Let $h(z) = \sum_{k\in\mathbf{Z}^n} a_k e^{2\pi i k\cdot y}$ where y is any element in $\pi_n^{-1}(z)$. Then for any $z \in \mathbf{T}^n$ and $1 \le l \le m$,

$$g(\sigma_l(z)) = e^{2\pi i\gamma_j\cdot\sigma_l(z)}\sum_{k\in\mathbf{Z}^n} a_k e^{2\pi i k\cdot\sigma_l(z)} = e^{2\pi i\gamma_j\cdot\sigma_l(z)}h(z) .$$

Hence $v_g(z) = h(z)u_j(z)$.

Conversely, suppose that $v_g(z) = h(z)u_j(z)$ for some $h(z) \in L^2(\mathbf{T}^n)$. Then we have $h(z) = \sum_{k\in\mathbf{Z}^n} a_k e_k(z)$. Let $x \in \Omega$. Then $x = \sigma_l(z)$ for some $z \in \mathbf{T}^n$ and some $1 \le l \le m$. So $h(z) = \sum_{k\in\mathbf{Z}^n} a_k e^{2\pi i\cdot x}$. Thus

$$g(x) = g(\sigma_l(z)) = e^{2\pi i\gamma_j\cdot\sigma_l(z)}h(z) = \sum_{k\in\mathbf{Z}^n} a_k e^{2\pi i(\gamma_j+k)\cdot x} .$$

Hence $g(x) \in H_j(\Omega)$. This completes the proof of the claim.

We now prove the totalness of $\{e_\lambda : \lambda \in \Gamma + \mathbf{Z}^n\}$ by establishing that

$$L^2(\Omega) = H_1(\Omega) + \cdots + H_m(\Omega) .$$

Let $A(z)$ be the $m \times m$ matrix with its entries $a_{j,l}(z) = e^{2\pi i\gamma_j\cdot\sigma_l(z)}$. It follows from (2.4) that $AA^* = A^*A = mI$, where $A^* = \bar{A}^T$. Now for any given $f(x) \in L^2(\Omega)$, let

$$v_f(z) = \begin{bmatrix} f_1(z) \\ \vdots \\ f_m(z) \end{bmatrix}, \text{ and } \begin{bmatrix} h_1(z) \\ \vdots \\ h_m(z) \end{bmatrix} = \frac{1}{m}A(z)\begin{bmatrix} f_1(z) \\ \vdots \\ f_m(z) \end{bmatrix} .$$

Then $h_j(z) \in L^2(\mathbf{T}^2)$ for all j and $A^*(z)[h_1(z), \ldots, h_m(z)]^T = [f_1(z), \ldots, f_m(z)]^T$. So

$$\sum_{j=1}^{m} e^{2\pi i\gamma_j\cdot\sigma_l(z)}h_j(z) = f_l(z) . \tag{2.7}$$

Now, for each $1 \le j \le m$ let $g_j(x) \in L^2(\Omega)$ satisfy

$$v_{g_j}(z) = [e^{2\pi i\gamma_j\cdot\sigma_l(z)}h_j(z), \ldots, e^{2\pi i\gamma_j\cdot\sigma_m(z)}h_j(z)] .$$

Then $g_j(x) \in H_j(\Omega)$. Moreover, $v_f = v_{g_1} + \cdots + v_{g_m}$, hence $f(x) = g_1(x) + \cdots + g_m(x)$. This completes the proof. $\Box$

We now study sets $\Omega := [0,1]^n + \mathcal{B}$ with $\mathcal{B} \subseteq \mathbf{Z}^n$. We associate to $\mathcal{B}$ the function

$$f_{\mathcal{B}}(\lambda) := \sum_{b \in \mathcal{B}} e^{2\pi i b\cdot\lambda} = \sum_{b=(b_1,\cdots,b_n)\in\mathcal{B}} e^{2\pi i(\lambda_1 b_1 + \cdots + \lambda_n b_n)} . \tag{2.8}$$

Its *Fourier zero set* is:

$$Z(f_{\mathcal{B}}) := \{\lambda \in \mathbf{R}^n : f_{\mathcal{B}}(\lambda) = 0\} . \tag{2.9}$$

The set $Z(f_{\mathcal{B}})$ is periodic (mod $\mathbf{Z}^n$).

Theorem 2.3. *Let $\Omega = [0,1]^n + \mathcal{B}$, where $\mathcal{B} \subseteq \mathbf{Z}^n$ is a finite set. Suppose that $\Gamma \subseteq \frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}$ is a set of distinct residue classes* (mod $\mathbf{Z}^n$)*, i.e.* $(\Gamma - \Gamma) \cap \mathbf{Z}^n = \{0\}$. *Then $\Lambda = \mathbf{Z}^n + \Gamma$ is a spectrum for Ω if and only if $|\Gamma| = |\mathcal{B}|$ and*

$$\Gamma - \Gamma \subseteq Z(f_{\mathcal{B}}) \cup \{0\} . \tag{2.10}$$

Proof. We have

$$\begin{aligned} \int_\Omega e^{2\pi i\lambda\cdot x} dx &= \sum_{b\in\mathcal{B}} e^{2\pi i\lambda\cdot b} \int_{[0,1]^n} e^{2\pi i\lambda\cdot x} dx \\ &= f_{\mathcal{B}}(\lambda) \int_{[0,1]^n} e^{2\pi i\lambda\cdot x} dx. \end{aligned} \tag{2.11}$$

$\Leftarrow$. Suppose that $|\Gamma| = |\mathcal{B}|$ and (2.10) holds. Given $\lambda_i, \lambda_j \in \Lambda$, set $\lambda_i = m_i + \gamma_i$ with $m_i \in \mathbf{Z}^n$ and $\gamma_i \in \Lambda$, and similarly set $\lambda_j = m_j + \gamma_j$. If $\lambda_i \neq \lambda_j$, (2.11) gives

$$\begin{aligned} \int_\Omega e^{2\pi i(\lambda_i - \lambda_j)\cdot x} &= f_{\mathcal{B}}(\lambda_i - \lambda_j) \int_{[0,1]^n} e^{2\pi i(\lambda_i-\lambda_j)\cdot x} dx \\ &= f_{\mathcal{B}}(\gamma_i - \gamma_j) \int_{[0,1]^n} e^{2\pi i(\gamma_i - \gamma_j + m_i - m_j)\cdot x} dx \\ &= 0, \end{aligned}$$

since either $\gamma_i \neq \gamma_j$ and $f_{\mathcal{B}}(\gamma_i - \gamma_j) = 0$ by (2.10), or else $\gamma_i = \gamma_j$ and $m_i - m_j \in \mathbf{Z}^n - \{0\}$, so the integral over $[0,1]^n$ is 0. Thus $\{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal set. Since $|\Gamma| = |\mathcal{B}| = \mu(\Omega)$, Theorem 2.2 shows that Λ is a spectrum for Ω.

$\Rightarrow$. Suppose that Λ is a spectrum for $\mathcal{B}$. Then $\{e^\lambda : \lambda \in \Lambda\}$ is orthogonal, and Theorem 2.2 shows that $|\Lambda| = \mu(\Omega) = |\mathcal{B}|$. Now orthogonality for $\lambda_i \neq \lambda_j$ and (2.11) together give

$$0 = \int_\Omega e^{2\pi i(\lambda_i - \lambda_j)\cdot x} dx = f_{\mathcal{B}}(\lambda_i - \lambda_j) \int_{[0,1]^n} e^{2\pi i(\lambda_i - \lambda_j)\cdot x} dx .$$

To obtain $\Gamma - \Gamma \subseteq Z(f_{\mathcal{B}}) \cup \{0\}$ it suffices to show that if $\gamma, \gamma' \in \Gamma$ with $\gamma \neq \gamma'$, then

$$\int_{[0,1]^n} e^{2\pi i(\gamma-\gamma')\cdot x} dx \neq 0 \ . \tag{2.12}$$

Now we may choose coset representatives (mod $\mathbf{Z}^n$) so that all $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Lambda$ have $-\frac{1}{2} \leq \gamma < \frac{1}{2}$, hence $\mu := \gamma - \gamma'$ has coordinates $-1 < \mu_j < 1$. Now

$$\int_{[0,1]^n} e^{2\pi i \mu \cdot x} dx = \prod_{j=1}^{n} (\int_0^1 e^{2\pi i \mu_j x_j} dx_j).$$

Each one-dimensional integral

$$\int_0^1 e^{2\pi i \mu_j x_j} dx_j = \begin{cases} 1 & \text{if } \mu_j = 0, \\ \dfrac{1}{2\pi i \mu_j}(e^{2\pi i \mu_j} - 1) & \text{if } \mu_j \neq 0 \ , \end{cases}$$

and since $-1 < \mu_j < 1$ this is never zero, so (2.12) follows. $\square$

For later applications, we rescale this result by sending $x_i \to \frac{1}{N_i} x_i$, to obtain:

Corollary 2.3. *Let $\Omega = [0, \frac{1}{N_1}] \times \cdots \times [0, \frac{1}{N_n}] + \mathcal{B}$, where $\mathcal{B} \subseteq \frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}$ is a finite set. Suppose that $\Gamma \subseteq \mathbf{Z}^n$ is a set of distinct residue classes (*mod $N_1\mathbf{Z} \times \cdots \times N_n\mathbf{Z}$*), i.e. $(\Gamma - \Gamma) \cap (N_1\mathbf{Z} \times \cdots \times N_n\mathbf{Z}) = \{0\}$. Then $\Lambda = (N_1\mathbf{Z} \times \cdots \times N_n\mathbf{Z}) + \Gamma$ is a spectrum for Ω if and only if $|\Gamma| = |\mathcal{B}|$ and*

$$\Gamma - \Gamma \subseteq Z(f_{\mathcal{B}}) \cup \{0\}.$$

Note that in this corollary the function $f_{\mathcal{B}}(\lambda)$ is periodic with period $N_1\mathbf{Z} \times \cdots \times N_n\mathbf{Z}$.

3. Structure Theorem for Tiles

In this section we assume that Ω is a bounded Lebesgue measurable set with measure $0 < \mu(\Omega) < \infty$ that tiles $\mathbf{R}^n$ with a rational periodic tile set $\mathcal{S}$. We characterize those sets Ω that tile with a given $\mathcal{S}$ and apply this to prove Theorem 1.1.

Given any $\tilde{b} \in \mathbf{R}^n$, we write

$$b := \tilde{b} \ (\text{mod } \mathbf{Z}^n)$$

to mean that $b - \tilde{b} \in \mathbf{Z}^n$ and $0 \leq b_i < 1$. If $\tilde{b} \in \frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}$, then so is b, and we view b as an element of the finite abelian group $\mathbf{Z}_{N_1} \times \cdots \times \mathbf{Z}_{N_n}$ in the torus $\mathbf{R}^n/\mathbf{Z}^n$.

The following structure theorem is a weak n-dimensional analogue of Theorem 1.3 in [15].

Theorem 3.1. *Suppose that a bounded measurable set Ω tiles $\mathbf{R}^n$ with the rational periodic tile set $\mathcal{S} = \mathbf{Z}^n + \mathcal{A}$, where $\mathcal{A} \subseteq \frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}$ with integers N_i and $(\mathcal{A} - \mathcal{A}) \cap \mathbf{Z}^n = \{0\}$. Then Ω has a finite partition, up to a set of measure zero, of the form*

$$\Omega = \bigcup_{j=1}^{J} (\tilde{\Omega}_{\mathcal{B}_j} + \mathcal{B}_j) \ , \tag{3.1}$$

satisfying the condition:

(i). *Each $\mathcal{B}_j \subseteq \frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}$ is a set of cardinality $\frac{N_1 N_2 \cdots N_n}{|\mathcal{A}|}$.*

(ii). *The sets $A := \mathcal{A} \pmod{\mathbf{Z}^n}$ and each $B_j := \mathcal{B}_j \pmod{\mathbf{Z}^n}$ yield a group factorization*

$$A \oplus B_j = \mathbf{Z}_{N_1} \times \cdots \times \mathbf{Z}_{N_n} \ . \tag{3.2}$$

(iii). *The sets $\tilde{\Omega}_{\mathcal{B}_j}$ are measurable and satisfy*

$$\bigcup_{j=1}^{J} \tilde{\Omega}_{\mathcal{B}_j} = [0, \frac{1}{N_1}] \times [0, \frac{1}{N_2}] \times \cdots \times [0, \frac{1}{N_n}] \ , \tag{3.3}$$

up to a set of measure zero.

This partition is unique up to measure zero sets.

Proof. For each $x \in [0, \frac{1}{N_1}] \times \cdots \times [0, \frac{1}{N_n}]$, set

$$\mathcal{B}(x) := \Big\{ b \in \frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z} : \ x + b \in \Omega \Big\}. \tag{3.4}$$

Because Ω is bounded, there are only finitely many possible such sets $\mathcal{B}$ that satisfy $\mathcal{B} = \mathcal{B}(x)$ for some x. Define

$$\tilde{\Omega}_{\mathcal{B}} := \Big\{ x \in [0, \frac{1}{N_1}] \times \cdots \times [0, \frac{1}{N_n}] : \ \mathcal{B}(x) = \mathcal{B} \Big\}. \tag{3.5}$$

Each $\tilde{\Omega}_{\mathcal{B}}$ is measurable, since

$$\bigcup_{\mathcal{B} \subseteq \mathcal{B}'} \tilde{\Omega}_{\mathcal{B}'} = ([0, \frac{1}{N_1}] \times \cdots \times [0, \frac{1}{N_n}]) \cup (\bigcup_{b \in \mathcal{B}} (\Omega - b)) \ ,$$

from which $\tilde{\Omega}_{\mathcal{B}}$ is extracted by inclusion-exclusion.

We discard all sets $\tilde{\Omega}_{\mathcal{B}}$ of measure zero, and retain the rest, which by construction gives the partition (3.1). Property (iii) holds by definition.

The tiling of $\mathbf{R}^n$ by Ω yields a measure-disjoint partition

$$\begin{aligned}\mathbf{R}^n &= \Omega + \mathcal{A} + \mathbf{Z}^n \\ &= \bigcup_{j=1}^{J} (\tilde{\Omega}_{\mathcal{B}_j} + \mathcal{B}_j + \mathcal{A} + \mathbf{Z}^n) \ .\end{aligned}$$

Since

$$\mathcal{B}_j + \mathcal{A} + \mathbf{Z}^n \subseteq \frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z},$$

we have

$$\begin{aligned}\bigcup_{j=1}^{J} (\tilde{\Omega}_{\mathcal{B}_j} + \mathcal{B}_j + \mathcal{A} + \mathbf{Z}^n) &\subseteq \bigcup_{j=1}^{J} \tilde{\Omega}_{\mathcal{B}_j} + (\frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}) \\ &= ([0, \frac{1}{N_1}] \times \cdots \times [0, \frac{1}{N_n}]) + \left(\frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}\right) \\ &= \mathbf{R}^n \ , \end{aligned} \tag{3.6}$$

Both sides of (3.6) are measure-disjoint partitions of $\mathbf{R}^n$, hence $\mathcal{B}_j + \mathcal{A} + \mathbf{Z}^n = \frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z}$, and it follows that

$$A \oplus B_j = \mathbf{Z}_{N_1} \times \cdots \times \mathbf{Z}_{N_n} \ ,$$

which is (ii). Finally $|\mathcal{B}_j| = |B_j| = \frac{N_1 \cdots N_n}{|A|} = \frac{N_1 \cdots N_n}{|\mathcal{A}|}$, proving (i).

The steps of this construction are reversible, to prove that there is a unique partition (3.1) satisfying (i)–(iii), up to sets of measure zero. $\square$

Various examples of sets Ω having rational periodic tilings arise from self-affine constructions, see [4], [14] and [22].

We now combine Theorem 3.1 with Theorem 2.3 to prove Theorem 1.1.

Proof of Theorem 1.1. $\Rightarrow$ Suppose that Λ is a universal spectrum for $\mathcal{S}$. Each set $\Omega_{\mathcal{B}}$ given by (1.10) tiles $\mathbf{R}^n$ using tile set $\mathcal{S}$. For (1.7) yields the partition

$$\bigcup_{s \in \mathcal{S}} (\Omega_{\mathcal{B}} + s) = ([0, \frac{1}{N_1}] \times \cdots \times [0, \frac{1}{N_n}]) + (\mathcal{A} + \mathcal{B} + \mathbf{Z}^n) = \mathbf{R}^n,$$

up to measure zero sets. Thus Λ is a spectrum for $\Omega_{\mathcal{B}}$.

$\Leftarrow$ We are given a set $\Lambda = N_1\mathbf{Z} \times \cdots \times N_n\mathbf{Z} + \Gamma$ with $\Gamma = \{\gamma_1, \ldots, \gamma_s\} \subseteq \mathbf{Z}^n$, which is a spectrum for all sets

$$\Omega_{\mathcal{B}} = [0, \frac{1}{N_1}] \times \cdots \times [0, \frac{1}{N_n}] + \mathcal{B}$$

where $\mathcal{B}$ satisfies (1.7) and (1.8). Now let Ω tile $\mathbf{R}^n$ with the tile set $\mathcal{S} = \mathbf{Z}^n + \mathcal{A}$. It suffices to prove that $\{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal set, for Theorem 2.2 then applies to show that Λ is a spectrum for Ω.

Now the partition of Ω given by Theorem 3.1 gives, for $\lambda, \lambda' \in \Lambda$,

$$\begin{aligned} \int_\Omega e^{2\pi i(\lambda-\lambda')\cdot x} dx &= \sum_{j=1}^{J} \int_{\tilde{\Omega}_j + \mathcal{B}_j} e^{2\pi i(\lambda-\lambda')\cdot x} dx \\ &= \sum_{j=1}^{J} f_{\mathcal{B}_j}(\lambda - \lambda') \int_{\tilde{\Omega}_j} e^{2\pi i(\lambda-\lambda')\cdot x} dx \ . \end{aligned} \tag{3.7}$$

Define $\mathcal{B}_j^* \subseteq [0,1]^n \cap (\frac{1}{N_1}\mathbf{Z} \times \cdots \times \frac{1}{N_n}\mathbf{Z})$ by $\mathcal{B}_j^* := \mathcal{B}_j \pmod{\mathbf{Z}^n}$. Then $\mathcal{B}_j^*$ satisfies (1.7) and (1.8), so Λ is a spectrum of $\Omega_{\mathcal{B}_j^*}$ by hypothesis. Write $\lambda = m + \gamma$ and $\lambda' = m' + \gamma'$ with $m', m \in N_1\mathbf{Z} \times \cdots \times N_n\mathbf{Z}$ and $\gamma, \gamma' \in \Gamma$. Since $\lambda - \lambda' \in \mathbf{Z}^n$ we have

$$f_{\mathcal{B}_j}(\lambda - \lambda') = f_{\mathcal{B}_j^*}(\lambda - \lambda') = f_{\mathcal{B}_j^*}(\gamma - \gamma') \ .$$

If $\gamma \neq \gamma'$, Theorem 2.3 gives $f_{\mathcal{B}_j^*}(\gamma - \gamma') = 0$, hence all $f_{\mathcal{B}_j}(\lambda - \lambda') = 0$ in (3.7), so that

$$\int_\Omega e^{2\pi i(\lambda-\lambda')\cdot x} dx = 0 \ .$$

If $\gamma = \gamma'$, then $f_{\mathcal{B}_j^*}(\gamma - \gamma') = |\mathcal{B}_j^*| = \frac{N}{|\mathcal{A}|}$. In this case, substituting (3.3) in (3.7) yields

$$\int_\Omega e^{2\pi i(\lambda-\lambda')\cdot x} dx = \frac{N}{|\mathcal{A}|} \int_{[0,\frac{1}{N_1}]\times\cdots\times[0,\frac{1}{N_n}]} e^{2\pi i(m-m')\cdot x} dx = 0 \ .$$

since $m - m' \in (N_1\mathbf{Z} \times \cdots \times N_n\mathbf{Z}) \backslash \{0\}$. This establishes the orthogonality of $\{e_\lambda : \lambda \in \Lambda\}$. $\square$

4. Factorizations of Finite Abelian Groups

To apply Theorem 1.1 to obtain universal spectra we need to know properties of factorizations of finite abelian groups.

Let G be an abelian group and let $A, B \subseteq G$. We call the sum $A + B$ a *direct sum* if all $a + b$ with $a \in A$ and $b \in B$ are distinct. We denote a direct sum by $A \oplus B$. We call any $A \oplus B = G$ a *factorization* of G, and we say that A (or B) is a *complementing set* (mod G) and that B is an *A-complement (*mod G*)*.

The *rank* of a finite abelian group G is the minimal $n \geq 1$ such that G is a quotient group of $\mathbf{Z}^n$. The complexity of the set of factorizations of G grows with the number of prime factors in $|G|$, and with its rank.

The structure of complementing sets is well-understood for a special class of groups, called *good groups* by Hajós [5]. A subset A of G is *periodic* if there exists $g \in G$ with $g \neq 0$ such that $g + A = A$. A group G is *good* if for every factorization $A \oplus B = G$, at least one of A and B is periodic. Subgroups of good groups are good, and a complete structure theory for factorizations of good groups is obtained by induction on the order of G. de Bruijn [1] asked if all groups are good, and Hajós [5] found a counterexample. Later de Bruijn [2] showed that the group $\mathbf{Z}_{72}$ is not good, using the factorization[3]

$$A = \{0, 8, 16, 18, 26, 34\}, \ B = \{0, 5, 6, 9, 12, 29, 33, 36, 42, 48, 53, 57\} .$$

Sands [19], [20] determined the complete set of finite abelian groups which are good.

Proposition 4.1. (Sands) *A finite abelian group is good if and only if it is contained in a group of one of the following types:* $(p, 2, 2, 2, 2)$, (p, q, r, s), $(p, q, 2, 2)$, $(p^2, 2, 2, 2)$, (p^2, q, r), $(p^3, 2, 2)$, $(p, 2^2, 2)$, $(p, 3, 3)$, (p^n, q), (p^2, q^2), $(3^2, 3)$, $(2^n, 2)$, $(2^2, 2^2)$ *and* (p, p), *where* p, q, r, s *are distinct primes and* p *may equal 2.*

In particular, the good groups that are cyclic are $\mathbf{Z}_N$ where N divides one of $pqrs, p^2qr, p^2q^2$ or p^nq, where p, q, r, s are any distinct primes. The cyclic group of smallest order which is not good is $\mathbf{Z}_{72}$.

A number of weaker structural properties have been studied for factorizations of cyclic groups. Call a factorization $A \oplus B = G$ *quasiperiodic* if either A or B, say B, can be partitioned into disjoint subsets $B_1, \ldots, B_m$ with $m > 1$ such that there is a subgroup $H = \{h_1, \ldots, h_m\}$ of G with

$$A + B_i = A + B_1 + h_i .$$

The example of de Bruijn above is quasiperiodic, with $H = \{0, 36\}$ where B is partitioned as:

$$B_1 = \{0, 6, 9, 12, 33, 57\}, \quad B_2 = \{5, 29, 36, 42, 48, 53\} .$$

Call a group G *quasiperiodic* if all factorizations of G are quasiperiodic.

Quasiperiodicity Conjecture. (Hajós [5]) *All finite abelian groups G are quasiperiodic.*

For cyclic groups de Bruijn [2] obtained some partial results on this question.

[3] We have applied a translation to de Bruijn's second set to make $0 \in B$.

Tijdeman [22] has studied a somewhat stronger property that a factor A of a cyclic group $G = \mathbf{Z}_N$ may have. Call a subset A of a cyclic group $\mathbf{Z}_N$ *primitive* if $0 \in A$ and there is $\mathcal{A} \subseteq \mathbf{Z}$ with $\mathcal{A} = \{a_1, \ldots, a_n\}$ having gcd $(a_i) = 1$ and $A := \mathcal{A} \pmod{N}$.

Definition 4.1. *A primitive complementing set $A \subseteq \mathbf{Z}_N$ has the* Tijdeman property *if for each $B \subseteq \mathbf{Z}_N$ such that $A \oplus B = \mathbf{Z}_N$ and $0 \in B$, there is a subgroup $H \subsetneqq G$ such that $B \subseteq H$. Equivalently, if $B = \{0, b_1, \ldots, b_m\}$, there is a prime $p|\ |A|$ such that $p|b_i$ for all i, and $B \subseteq \mathbf{Z}_{\frac{N}{p}}$. We say that A has the* strong Tijdeman property *if the subgroup H can be chosen to depend only on A, i.e. it can be taken the same for all complements B with $0 \in B$.*

Tijdeman [22] gives examples showing that the subgroup H of $\mathbf{Z}_N$ cannot be chosen independent of A.

Definition 4.2. *A finite cyclic group $G = \mathbf{Z}_N$ has the* Tijdeman property *(resp.* strong Tijdeman property*) if every eligible complementing set A has the Tijdeman property (resp. strong Tijdeman property).*

Tijdeman [22] shows that if a group G and all its subgroups have the Tijdeman property, then G is quasiperiodic.

Conjecture. (Tijdeman) *Every finite cyclic group G has the Tijdeman property.*

In order to construct universal spectra we need to use the uniformity condition embodied in the strong Tijdeman property. We proceed to show that all good groups have the strong Tijdeman property.

Lemma 4.1. *If a finite cyclic group G has the strong Tijdeman property, then every subgroup of G has the strong Tijdeman property.*

Proof. We argue by contradiction. Let $G = \mathbf{Z}_N$, and suppose there exists $d|N$ such that $\mathbf{Z}_d$ does not have the strong Tijdeman property. Then there exists a set $\mathcal{A} \subseteq \mathbf{Z}$ with $0 \in \mathcal{A}$ and gcd $\{a \in \mathcal{A}\} = 1$, such that $A := \mathcal{A} \pmod{d}$ has $|A| = |\mathcal{A}|$, and there are complementing sets $A \oplus B_j = \mathbf{Z}_d$, with $\mathcal{B}_j \subseteq \mathbf{Z}$ such that $0 \in \mathcal{B}_j$ and $B_j := \mathcal{B}_j \pmod{N}$ for $1 \le j \le k$, such that no prime factor p of $|A|$ divides all elements of B_j. Let $N = md$ and define

$$\mathcal{C}_j := \mathcal{B}_j \oplus \{0, d, \ldots, (m-1)d\} \subseteq \mathbf{Z},\ 1 \le j \le k\ .$$

Then $(\mathcal{A}, \mathcal{C}_j)$ is a complementing pair $\pmod{N}$ for $1 \le j \le k$, and no prime factor $p|\ |A|$ divides all elements of all $\mathcal{C}_j$. This contradicts $\mathbf{Z}_N$ having the strong Tijdeman property. $\square$

Theorem 4.1. *Let G be a finite cyclic group which is a good group. Then G has the strong Tijdeman property.*

Proof. Let $G = \mathbf{Z}_N$. We prove the result by induction on $d(N)$, the number of divisors of N. The base case is $d(N) = 2$, where $N = p$ is prime, and the hypothesis is trivially true.

Assume that the induction hypothesis is true if $d(N) < k$. Now let $d(N) = k$ and assume that $\mathbf{Z}_N$ is good. Let $\mathcal{A}$ be a complementing set $(\mathrm{mod}\ N)$ such that $0 \in \mathcal{A}$ and $\gcd(a : a \in \mathcal{A}) = 1$. We prove that there exists a prime factor p of $|\mathcal{A}|$ such that $p|\mathcal{B}$ for all $\mathcal{A}$-complementing sets $\mathcal{B}$ with $0 \in \mathcal{B}$.

Suppose first that $\mathcal{A}$ is periodic, i.e. $\mathcal{A} + g \equiv \mathcal{A} \pmod{N}$, with $g \not\equiv 0 \pmod{N}$. Since any multiple mg of g also makes $\mathcal{A}$ periodic, i.e. $\mathcal{A} + mg = \mathcal{A}$, we may without loss of generality take g to be a (proper) factor of N. Then there exists $\mathcal{A}_1 \subseteq \mathbf{Z}$ with $0 \in \mathcal{A}_1$ and

$$\mathcal{A} \equiv \{0, g, 2g, \ldots, N - g\} \oplus \mathcal{A}_1 \pmod{N} . \tag{4.1}$$

For any $\mathcal{A}$-complementing set $\mathcal{B}$ $(\mathrm{mod}\ N)$ with $0 \in \mathcal{B}$, the set $(\mathcal{A}_1, \mathcal{B})$ $(\mathrm{mod}\ g)$ is a complementing pair for $\mathbf{Z}_g$. Set

$$f_1 = \gcd(a : a \in \mathcal{A}_1) .$$

Then $(f_1, g) = 1$, otherwise we contradict $\gcd(a : a \in \mathcal{A}) = 1$. Thus $(\frac{1}{f_1}\mathcal{A}_1, \mathcal{B})$ $(\mathrm{mod}\ g)$ is also a complementing pair for $\mathbf{Z}_g$, hence $\frac{1}{f_1}\mathcal{A}_1$ $(\mathrm{mod}\ g)$ is a complementing set for $\mathbf{Z}_g$. Since $d(g) < k$, the induction hypothesis gives a prime factor p of $|\mathcal{A}_1|$, and hence of $|\mathcal{A}|$, such that $p|\mathcal{B}$ for every $\mathcal{A}$-complementing set $\mathcal{B}$ $(\mathrm{mod}\ g)$ with $0 \in \mathcal{B}$.

Suppose next that $\mathcal{A}$ is not periodic, and that $|\mathcal{A}|$ is a prime power. Then the strong Tijdeman property for $\mathcal{A}$ follows from a result of Tijdeman [22, Theorem 3].

Suppose now that $\mathcal{A}$ is not periodic, and that $|\mathcal{A}|$ has at least two distinct prime factors. Since $\mathbf{Z}_N$ is a good group, if $\mathcal{B}$ is an $\mathcal{A}$-complementing set $(\mathrm{mod}\ N)$ with $0 \in \mathcal{B}$ then $\mathcal{B}$ is periodic. Since $|\mathcal{A}|\ |N$, N has at least two prime factors. We have the following cases.

Case 1. $N = p^k q$ *with* $k \geq 2$, *and* $|\mathcal{A}| = p^e q$ *with* $1 \leq e < k$.

Write $\mathcal{B} + g \equiv \mathcal{B} \pmod{N}$, with $g|N$ and $g \neq N$. Then g is a multiple of $|\mathcal{A}|$, hence $g = p^\ell q$

with $e \le \ell < k$. Set $g^* = p^{k-1}q$, and then for all $\mathcal{B}$, we have $\mathcal{B} + g^* \equiv \mathcal{B} \pmod{N}$. As above,

$$\mathcal{B} \equiv \{0, g^*, 2g^*, \ldots, N - g^*\} \oplus \mathcal{B}_1 \pmod{N}$$

for some set $\mathcal{B}_1 \subseteq \mathbf{Z}$ with $0 \in \mathcal{B}_1$, such that $(\mathcal{A}, \mathcal{B}_1)$ is a complementing pair (mod g^*). Since $d(g^*) < k$, the induction hypothesis says there exists a prime factor t of $|\mathcal{A}|$ independent of $\mathcal{B}_1$, that divides $\mathcal{B}_1$. Now t is either p or q, and since $pq | g^*$, we obtain $t | \mathcal{B}$. Thus the induction step holds.

Case 2. $N = p^2q^2$ *and* $|\mathcal{A}| = pqd$ *with* $d = 1, p$ *or* q.

Write $\mathcal{B} + g \equiv \mathcal{B} \pmod{N}$ with $g | N$ and $g \neq N$, and note that g is a multiple of $|\mathcal{A}|$. If $d = p$ or q, then $g = |\mathcal{A}|$, hence $\mathcal{B} \pmod{N}$ can only be the subgroup of $\mathbf{Z}_N$ of order q or p, respectively. So in this case, $pq | \mathcal{B}$.

The hard case is $d = 1$ where $|\mathcal{A}| = pq$. All complementing sets $\mathcal{B}$ have $|\mathcal{B}| = pq$ and fall into three classes, according to whether their minimal period g dividing p^2q^2 is $g = pq$, p^2q or pq^2. (The periods of $\mathcal{B}$ form a subgroup of $\mathbf{Z}_N$, so if p^2q and pq^2 are both periods, then so is pq). If $g = pq$ then $\mathcal{B} + pq = \mathcal{B} \pmod{N}$ implies that $\mathcal{B}$ is the subgroup of $\mathbf{Z}_N$ of order pq, hence $pq | \mathcal{B}$. In the remainder of the proof we show that for at least one of the classes $g = pq^2$ and $g = pq^2$ all $\mathcal{B}$ have $pq | \mathcal{B}$, while all $\mathcal{B}$ in the other class have $t | \mathcal{B}$ for $t = p$ or q. This then implies that $t | \mathcal{B}$ for all complementing sets $\mathcal{B}$.

Consider first the case $g = p^2q$. Then there exists a set $\mathcal{B}_1 \subseteq \mathbf{Z}$ with $0 \in \mathcal{B}_1$ such that

$$\mathcal{B} \equiv \{0, p^2q, \ldots, (q-1)p^2q\} \oplus \mathcal{B}_1 \pmod{N} , \tag{4.2}$$

and $(\mathcal{A}, \mathcal{B}_1)$ is a complementing pair (mod p^2q) with $|\mathcal{B}_1| = p$. If $\mathcal{A}$ is not periodic (mod p^2q) then the argument of Case 1 applies to $(\mathcal{A}, \mathcal{B}_1)$ to show that

$$\mathcal{B}_1 \equiv \{0, pq, 2pq, \ldots, (p-1)pq\} \pmod{p^2q} .$$

Thus $pq | \mathcal{B}_1$, so that $pq | \mathcal{B}$, for all $\mathcal{B}$ having $g = p^2q$. If $\mathcal{A}$ is periodic (mod p^2q) then, since $d(p^2q) < k$, we apply the induction hypothesis to conclude there is a prime $t_1 | \mathcal{B}_1$, independent of the complementing set $\mathcal{B}_1$ (mod p^2q) to $\mathcal{A}$, with $t_1 = p$ or q. Now (4.2) gives $t_1 | \mathcal{B}$ in this case, for all $\mathcal{B}$ having $g = p^2q$. If $g = pq^2$ an identical argument says that if $\mathcal{A}$ is not periodic (mod pq^2) then $pq|\mathcal{B}$, while if $\mathcal{A}$ is periodic (mod pq^2) then there exists a $t_2 = p$ or q such that $t_2|\mathcal{B}$ for all $\mathcal{B}$ having $g = pq^2$.

There remains an exceptional case, in which $\mathcal{A}$ is periodic both (mod p^2q) and (mod pq^2). We show that this case never occurs. We argue by contradiction. Suppose $\mathcal{A}$ were periodic (mod p^2q). Then $\mathcal{A} + g' \equiv \mathcal{A}$ (mod p^2q), with $g'|p^2q$ and $|\mathcal{A}| = pq$ divides g', so $g' = pq$. Then (4.1) gives

$$\mathcal{A} \equiv \{0, pq, \ldots, (p-1)pq\} \oplus \mathcal{A}' \pmod{p^2q} .$$

It follows that the number of elements $a \in \mathcal{A}$ with $a \equiv 0 \pmod{pq}$ is a multiple of p. If $\mathcal{A}$ were also periodic (mod pq^2) a similar argument shows that the number of elements of $\mathcal{A}$ with $a \equiv 0 \pmod{pq}$ is a multiple of q. This number is thus a multiple of pq, and since $0 \in \mathcal{A}$ and $|\mathcal{A}| = pq$, we conclude all elements of $\mathcal{A}$ have $a \equiv 0 \pmod{pq}$. This forces

$$\mathcal{A} \equiv \{0, pq, 2pq, \ldots, (pq-1)pq\} \pmod{p^2q^2}$$

hence $\mathcal{A} + pq \equiv \mathcal{A} \pmod{N}$, contradicting the hypothesis that $\mathcal{A}$ is not periodic (mod N).

Case 3. $N = pqr, p^2qr$ or $pqrs$.

The arguments are similar to case 2. As one example: $N = p^2qr$ and $|\mathcal{A}| = qr$. If $\mathcal{B} + g \equiv \mathcal{B}$ (mod N) with $g = qr$ then $\mathcal{B}$ is necessarily the subgroup of N of order p^2, hence $qr|\mathcal{B}$; if $g = pqr$ then

$$\mathcal{B} \equiv \{0, pqr, \ldots, (p-1)pqr\} \oplus \mathcal{B}_1 \pmod{N}$$

hence any prime $t|\mathcal{B}_1$ has $t|\mathcal{B}$, and the induction hypothesis gives $t|\mathcal{B}_1$ for all $\mathcal{B}_1$, for some $t = q$ or r.

These cases are exhaustive, and the induction step follows. $\square$

For finite abelian groups G of rank n, with n large, there are many "exotic" factorizations. In particular, a natural analogue of the Tijdeman property is not valid for rank $n \geq 3$. For example, consider the factorization $A \oplus B = (\mathbf{Z}_4)^3$ with $A = \{0,1\}^3 = \{(0,0,0), \ldots, (1,1,1)\}$ and

$$B = \{(0,0,0),\ (2,0,1),\ (1,2,0),\ (0,1,2),\ (2,0,3),\ (3,2,0),\ (0,3,2),\ 2,2,2)\} ,$$

which appears in Table I of [13]. In this example both A and B generate $(\mathbf{Z}_4)^3$, so neither A nor B are contained in a proper subgroup H of $(\mathbf{Z}_4)^3$.

5. One-Dimensional Case

We establish the existence of universal spectra for certain one-dimensional tile sets $\mathcal{S} = \mathbf{Z} + \frac{1}{N}\mathcal{A}$ with $\mathcal{A} \subseteq \mathbf{Z}$, by establishing the strong Tijdeman property.

Proof of Theorem 1.2. Since translating $\mathcal{A} \subseteq \mathbf{Z}$ does not affect the result, we assume without loss of generality that $0 \in \mathcal{A}$. Set $A := \mathcal{A} \pmod N$, and by hypothesis $|A| = |\mathcal{A}|$.

Theorem 1.1 says that $\Lambda = N\mathbf{Z} + \Gamma$ with $|\Gamma| = \frac{N}{|\mathcal{A}|}$ is a universal spectrum for $\mathcal{S} = \mathbf{Z} + \frac{1}{N}\mathcal{A}$ if and only if Λ is a spectrum for all sets $\Omega = [0, \frac{1}{N}] + \frac{1}{N}\mathcal{B}$ with $\mathcal{B} \subseteq 0, 1, 2, \ldots, N-1$, such that $0 \in \mathcal{B}$ and $A \oplus B = \mathbf{Z}_N$, where $B := \mathcal{B} \pmod N$. In particular a necessary and sufficient condition for the existence of a set Ω that tiles $\mathbf{R}$ with tile set $\mathcal{S}$ is that B be a complementing set for $\mathbf{Z}_N$. Now Corollary 2.3 states that Λ is a spectrum for a particular Ω if and only if

$$\Gamma - \Gamma \subseteq Z(f_{\mathcal{B}}) \cup \{0\} , \tag{5.1}$$

where $Z(f_{\mathcal{B}})$ is the real zero set of

$$f_{\mathcal{B}}(\lambda) = \sum_{b \in \mathcal{B}} e^{2\pi i \lambda b}, \ \lambda \in \mathbf{R} . \tag{5.2}$$

We prove the theorem by induction on $d(N)$, the number of divisors of N (counting 1 and N as divisors). The base case is $d(N) = 2$, where $N = p$ is prime, and the only possibilities are $A = \{0\}$, $B = \mathbf{Z}_p$, in which case $\Gamma = \{0, 1, \ldots, p-1\}$ and $\Lambda = \mathbf{Z}$, and $A = \mathbf{Z}_p$, $B = \{0\}$, in which case $\Gamma = \{0\}$ and $\Lambda = p\mathbf{Z}$.

Now suppose that the theorem is true for all N with $d(N) < k$ that have the strong Tijdeman property. We treat two cases, depending on the value of $f := \gcd\,(a : a \in \mathcal{A})$.

Case 1. f and N are relatively prime.

In this case, let $\mathcal{A}' := \{\frac{a}{f} : a \in \mathcal{A}\}$. If $(\mathcal{A}, \mathcal{B})$ is a complementing pair (mod N), then so is $(\mathcal{A}', \mathcal{B})$, by [21, Theorem 1]. Thus without loss of generality we may suppose that $f = 1$. Thus $\mathcal{A}$ is eligible, so the strong Tijdeman property applies to give a prime $p|\ |\mathcal{A}|$ such that $p|\mathcal{B}$ for every $\mathcal{A}$-complementing set $\mathcal{B}$ (mod N) with $0 \in \mathcal{B}$. Set $\mathcal{C} = \{a \in \mathcal{A} : a \equiv 0 \pmod p\}$ and $\mathcal{C}' = \{\frac{c}{p} : c \in \mathcal{C}\}$. Then for each $\mathcal{A}$-complementing set $\mathcal{B}$ (mod N) with $0 \in \mathcal{B}$, let $\mathcal{B}' = \frac{1}{p}\mathcal{B}$ and observe that $(\mathcal{C}', \mathcal{B}')$ is a complementing pair (mod $\frac{N}{p}$). Now let $N' = \frac{N}{p}$, and observe that $\mathbf{Z}_{N'}$ has the strong Tijdeman property by Lemma 4.1. Since $d(\frac{N}{p}) < k$, there exists a universal spectrum $\Lambda' = \frac{N}{p}\mathbf{Z} + \Gamma'$ for the tile set $\mathbf{Z} + \frac{1}{N'}\mathcal{C}'$, which has $|\Gamma'| = \frac{N'}{|\mathcal{C}'|}$ and $(\Gamma' - \Gamma') \cap \frac{N}{p}\mathbf{Z} = \{0\}$. Observe that $\frac{1}{N'}\mathcal{B}' = \frac{1}{N}\mathcal{B}$, hence Corollary 2.3 applied to $\Omega' = [0, \frac{1}{N'}] + \frac{1}{N'}\mathcal{B}'$ gives

$$\Gamma' - \Gamma' \subseteq Z(f_{\frac{1}{N'}\mathcal{B}'}) \cup \{0\} = Z(f_{\frac{1}{N}\mathcal{B}}) \cup \{0\} \ , \tag{5.3}$$

and this holds for all $\mathcal{A}$-complementing sets $\mathcal{B}$ (mod N) with $0 \in \mathcal{B}$. We claim that

$$\Lambda := N\mathbf{Z} + \Gamma'$$

is a spectrum for all $\mathcal{B}$ above. This follows immediately from (5.3), by Corollary 2.3 applied to $\Omega = [0, \frac{1}{N}] + \frac{1}{N}\mathcal{B}$. Thus Λ is a universal spectrum for the tile set $\mathbf{Z} + \frac{1}{N}\mathcal{A}$, completing the induction step.

Case 2. f and N are not relatively prime.

In this case, let p be a prime dividing both f and N. Suppose that $\mathcal{B}$ is an $\mathcal{A}$-complementing set (mod N). Then $\mathcal{B}$ (mod p) contains the same number of elements in each residue class (mod p). For $0 \le j \le p-1$ set

$$\mathcal{B}_j = \{b \in \mathcal{B} : b \equiv j \pmod{p}, \qquad \mathcal{C}_j = \{\frac{b-j}{p} : b \in \mathcal{B}_j\} \ .$$

Then $(\frac{1}{p}\mathcal{A}, \mathcal{C}_j)$ is a complementing pair (mod N') where $N' = \frac{N}{p}$ for $0 \le j \le p-1$. In particular $\frac{1}{p}\mathcal{A}$ is a complementing set (mod N'), and $d(N') < k$. The induction hypothesis applies to give a universal spectrum for the tile set $\mathbf{Z} + \frac{1}{N'}(\frac{1}{p}\mathcal{A}) = \mathbf{Z} + \frac{1}{N}\mathcal{A}$ of the form $\Lambda = N'\mathbf{Z} + \Gamma'$ with $\Gamma' \subset \mathbf{Z}$, having $|\Gamma'| = \frac{N}{p|\mathcal{A}|}$ and $(\Gamma' - \Gamma') \cap \frac{N}{p}\mathbf{Z} = \{0\}$. But $\Lambda = \frac{N}{p}\mathbf{Z} + \Gamma' = N\mathbf{Z} + \Gamma$ where

$$\Gamma = \Gamma' \oplus \{0, \frac{N}{p}, 2\frac{N}{p}, \ldots, (p-1)\frac{N}{p}\} \ .$$

This proves the hypothesis for $d(N) = k$, completing the induction step. $\square$

Proof of Theorem 1.3. We treat the cases where $|\mathcal{A}|$ and $\frac{N}{|\mathcal{A}|}$ are prime powers separately. By Theorem 1.1 and Corollary 2.3 it suffices to show that there exists $\Lambda = N\mathbf{Z} + \Gamma$ with $|\Gamma| = N/|\mathcal{A}|$ and $(\Gamma - \Gamma) \cap N\mathbf{Z} = \{0\}$ such that

$$\Gamma - \Gamma \subseteq Z(f_{\frac{1}{N}\mathcal{B}}) \cup \{0\} \tag{5.4}$$

for all $\mathcal{A}$-complementing sets $\mathcal{B}$ (mod N) with $0 \in \mathcal{B}$.

(i). Suppose $|\mathcal{A}| = q^e$, where q is prime. We proceed by induction on the number of divisors $d(N)$ of N. If $d(N) = 2$ then N is a prime, and the result is true since $\mathbf{Z}_N$ is a good group (Theorems 1.2 and 4.1).

For the induction step, suppose that a universal spectrum as above exists whenever we have $d(N) < k$, and let $d(N) = k$. We use an argument similar to that of Theorem 1.2, treating two cases according to the value of $f = \gcd(a : a \in \mathcal{A})$.

Case 1. f and N are relatively prime.

We reduce to the case $f = 1$ by noting that $(\mathcal{A}, \mathcal{B})$ is a complementing set $(\mathrm{mod}\ N)$ if and only if $(\frac{1}{f}\mathcal{A}, \mathcal{B})$ is a complementing set $(\mathrm{mod}\ N)$, according to Tijdeman [22, Theorem 1]. Now [22, Theorem 3] states that if $|\mathcal{A}| = q^e$ and then $q|\mathcal{B}$ for all $\mathcal{A}$-complementing sets $\mathcal{B}$ $(\mathrm{mod}\ N)$ with $0 \in \mathcal{B}$, which is the strong Tijdeman property. This case is now handled exactly as in Case 1 of the proof of Theorem 1.2. There is a complementing pair $(\mathcal{C}', \frac{1}{q}\mathcal{B})$ $(\mathrm{mod}\ \frac{N}{q})$ with $\mathcal{C}' = \{\frac{1}{q} : a \in \mathcal{A} \text{ with } a \equiv 0 \pmod{q}\}$, and any universal spectrum $\Lambda' = \frac{N}{q}\mathbf{Z} + \Gamma'$ for the tile set $\mathbf{Z} + \frac{q}{N}\mathcal{C}$ lifts to a universal spectrum $\Gamma = N\mathbf{Z} + \Gamma'$ for $\mathcal{A}$.

Case 2. f and N are not relatively prime.

Let p be a prime factor of both f and N. Then any $\mathcal{A}$-complementing set $\mathcal{B}$ $(\mathrm{mod}\ p)$ contains equal numbers of elements in each residue class $(\mathrm{mod}\ p)$. Set $\mathcal{B}_j = \{b \in \mathcal{B} : b \equiv j \pmod{p}\}$ and $\mathcal{C}_j' = \{\frac{b-j}{p} : b \in \mathcal{B}_j\}$. Then $(\frac{1}{p}\mathcal{A}, \mathcal{C}_j')$ is a complementing pair $(\mathrm{mod}\ \frac{N}{p})$ for $0 \le j \le p-1$, and $|\frac{1}{p}\mathcal{A}| = q^e$, and $d(\frac{N}{p}) < k$, so the induction hypothesis now applies to $\frac{1}{p}\mathcal{A}$. This case is now completed identically to Case 2 in the proof of Theorem 1.2.

These cases are exhaustive, so the induction is complete.

(ii). Suppose that $\frac{N}{|\mathcal{A}|} = q^e$ for a prime q. We prove the result by induction on the number of divisors $d(N)$, with the base case $d(N) = 2$ being immediate.

Assume the universal spectrum exists when $d(N) < k$, and suppose that $d(N) = k$. We again treat two cases.

Case 1. $\mathcal{A}$ is periodic $(\mathrm{mod}\ N)$.

There exists a proper divisor g of N such that $\mathcal{A} + g = \mathcal{A}$ $(\mathrm{mod}\ N)$, therefore there exists $\mathcal{A}_1 \subseteq \mathbf{Z}$ such that

$$\mathcal{A} \equiv \{0, g, 2g, \ldots, N-g\} \oplus \mathcal{A}_1 \pmod{N} .$$

Let $\mathcal{B}$ be any $\mathcal{A}$-complementing set (mod N). Then $\mathcal{B}$ is also an $\mathcal{A}_1$-complementing set (mod g). Since $\frac{N}{g}|\mathcal{A}_1| = |\mathcal{A}|$, we have $\frac{g}{|\mathcal{A}_1|} = \frac{N}{|\mathcal{A}|} = q^e$, and $d(g) < k$, so the induction hypothesis applies to give a universal spectrum $\Lambda' = g\mathbf{Z} + \Gamma'$ for the tile set $\mathbf{Z} + \frac{1}{g}\mathcal{A}_1$ with $|\Gamma'| = \frac{g}{|\mathcal{A}_1|} = q^e$ and

$$\Gamma' - \Gamma' \subseteq Z(f_{\frac{1}{g}\mathcal{B}}) \cup \{0\} \ .$$

We take $\Lambda = N\mathbf{Z} + \Gamma$ with $\Gamma = \Gamma'$, and observe that (5.4) holds, hence Λ is a universal spectrum for the tile set $\mathbf{Z} + \frac{1}{N}\mathcal{A}$.

*Case 2. $\mathcal{A}$ is not periodic (*mod N*).*

If $\mathcal{B}$ is any $\mathcal{A}$-complementing set (mod N) then Sands [18, Theorem 2] proved that $\mathcal{B}$ must be periodic. Thus there exists a proper factor g of N such that $\mathcal{B} + g \equiv \mathcal{B}$ (mod N). Now any multiple g^* of g is a period, and there exists $g^* = \frac{N}{p}$ for some prime p. The elements of $\mathcal{B}$ are then grouped into cycles $\{b, b+g^*, b+2g^*, \ldots, b+(p-1)g^*\}$ of length p, hence $p|\ |\mathcal{B}| = \frac{N}{|\mathcal{A}|} = q^e$. Thus $p = q$, hence $g^* = \frac{N}{q}$ is a period for *all* $\mathcal{A}$-complementing sets $\mathcal{B}$ (mod N). Therefore for any such $\mathcal{B}$ there exists $\mathcal{B}_1 \subseteq \mathbf{Z}$ such that

$$\mathcal{B} = \{0, g^*, 2g^*, \ldots, (q-1)g^*\} \oplus \mathcal{B}_1 \pmod{N}, \tag{5.5}$$

hence $(\mathcal{A}, \mathcal{B}_1)$ is a complementing pair (mod g^*).

We now know that $\mathcal{A}$ is a complementing set (mod g^*). Furthermore $\frac{g^*}{|\mathcal{A}|} = q^{e-1}$ and $d(g^*) < k$, so the induction hypothesis applies to give a universal spectrum $\Lambda' = g^*\mathbf{Z} + \Gamma'$ for the tile set $\mathbf{Z} + \frac{1}{g^*}\mathcal{A}$, where $\Gamma' \subseteq \mathbf{Z}$ is such that $|\Gamma'| = q^{e-1}$ and $(\Gamma' - \Gamma') \cap g^*\mathbf{Z} = \{0\}$. Thus

$$\Gamma' - \Gamma' \subseteq Z(f_{\frac{1}{g^*}\mathcal{B}_1}) \cup \{0\} \tag{5.6}$$

for all $\mathcal{A}$-complementing sets $\mathcal{B}_1$ (mod g^*) containing 0. We claim that $\Lambda = N\mathbf{Z} + \Gamma$, where

$$\Gamma = q\Gamma' + \{0, 1, \ldots, q-1\} \ ,$$

is a universal spectrum for $\mathbf{Z} + \frac{1}{N}\mathcal{A}$.

Clearly $|\Gamma| = q|\Gamma'| = q^e = \frac{N}{|\mathcal{A}|}$, and $(\Gamma - \Gamma) \cap N\mathbf{Z} = \{0\}$. It remains to show that (5.4) holds for all $\mathcal{A}$-complementing sets $\mathcal{B}$. Given $\gamma_1, \gamma_2 \in \Gamma$, write

$$\gamma_1 - \gamma_2 = q(\gamma_1' - \gamma_2') + (m_1 - m_2)$$

with $\gamma_i' \in \Gamma'$ and $m_i \in \{0, 1, \ldots, q-1\}$, so that $-q < m_1 - m_2 < q$. Now (5.5) and $g^* = \frac{N}{q}$ yield

$$f_{\frac{1}{N}\mathcal{B}}(\lambda) = \Big(\sum_{j=0}^{q-1} e^{2\pi i j\lambda/q}\Big) f_{\frac{1}{N}\mathcal{B}_1}(\lambda) + (e^{2\pi i\lambda} - 1)g(\lambda) \ .$$

for some exponential polynomial $g(\lambda)$. For $\lambda = \gamma_1 - \gamma_2 \in \mathbf{Z}$ the last term vanishes, and using $f_{\frac{1}{N}\mathcal{B}_1}(q\lambda) = f_{\frac{q}{N}\mathcal{B}_1}(\lambda)$, we have

$$f_{\frac{1}{N}\mathcal{B}}(\gamma_1 - \gamma) = \Big(\sum_{j=0}^{q-1} e^{2\pi i j(\gamma_1-\gamma_2)/q}\Big) f_{\frac{q}{N}\mathcal{B}_1}(\gamma_1' - \gamma_2') \ ,$$

because $f_{\frac{q}{N}\mathcal{B}_1}$ is periodic with period $\frac{N}{q}\mathbf{Z}$. If $\gamma_1' \neq \gamma_2'$, then $f_{\frac{q}{N}\mathcal{B}_1}(\gamma_1' - \gamma_2') = 0$, while if $\gamma_1' = \gamma_2'$ and $m_1 \neq m_2$, then

$$\sum_{j=0}^{q-1} e^{2\pi i j(\gamma_1-\gamma_2)/q} = \sum_{j=0}^{q-1} e^{2\pi i j(m_1-m_2)/q} = 0 \ .$$

Thus (5.4) holds, and the induction step is complete. $\square$

Theorem 5.1. *If the strong Tijdeman conjecture is true, then all compact sets Ω that tile $\mathbf{R}$ by translation are spectral sets.*

Proof. Every such set Ω has a rational periodic tiling of $\mathbf{R}$ by [15, Theorem 2]. The result now follows from Theorem 1.2 and the strong Tijdeman conjecture. $\square$

All bounded measurable sets Ω that tile $\mathbf{R}$ have a periodic tiling by [12, Theorem 6.1].

Acknowledgment. We are indebted to P. Jorgensen for introducing us to spectral sets and to the work of Fuglede [3], and to P. Jorgensen and S. Pedersen for discussion concerning this work. Although we gave an independent proof of Theorem 2.2, we did so only after S. Pedersen informed us that he had a proof of the result after we posed the question to him. His proof is given in [18].

References

[1] de Bruijn, N. G., On the factorisation of finite abelian groups, Indag. Math. **15** (1953), 258–246.

[2] de Bruijn, N. G., On the factorisation of cyclic groups, Indag. Math. **17** (1955), 370–377.

[3] Fuglede, B., Commuting self-adjoint partial differential operators and a group theoretic problem, J. Funct. Anal. **16** (1974), 101–121.

[4] Gröchenig, K.-H. and A. Haas, Self-Similar Lattice Tilings, J. Fourier Anal. Appl. **1** (1994) 131–170.

[5] Hajós, G., Sur la factorisation des groupes abéliens, Časopis Pěst Mat. Fys. **74** (1950), 157–162.

[6] Hajós, G., Sur la problème de factorisation des groupes cycliques, Acta. Math. Acad. Sci. Hungar. **1** (1950), 189–195.

[7] Jorgensen, P., Spectral theory of finite volume domains in $\mathbf{R}^n$, Adv. Math. **44** (1982), 105–120.

[8] Jorgensen, P., and Pedersen, S., Spectral theory for Borel sets in $\mathbf{R}^n$ of finite measure, J. Funct. Anal. **107** (1992), 72–104.

[9] Jorgensen, P., and Pedersen, S., Group theoretic and geometric properties of multivariable Fourier series, Expo. Math. **11** (1993), 309–329.

[10] Jorgensen, P., and Pedersen, S., Harmonic analysis of fractal measures, Const. Approx., to appear.

[11] Jorgensen, P., and Pedersen, S., Estimates on the spectrum of fractals arising from affine iterations, preprint.

[12] Kolountzakis, M. and J. C. Lagarias, Structure of tilings of the line by a function, Duke Math. J., to appear, 1996.

[13] Lagarias, J. C. and Shor, P., Keller's cube-tiling conjecture is false in high dimensions, Bull. Amer. Math. Soc. **27** (1992), 279–283.

[14] Lagarias, J. C., and Wang, Y., Integral self-affine tiles in $\mathbf{R}^n$ I. Standard and nonstandard digit sets, J. London Math. Soc., to appear.

[15] Lagarias, J. C., and Wang, Y., Tiling the line by translates of one tile, Inventiones Math., **124** (1996), 341–365.

[16] Landau, H., Necessary density conditions for sampling and interpolation of certain entire functions, Acta Math. **117** (1967), 37–52.

[17] Pedersen, S., Spectral theory of commuting self-adjoint partial differential operators, J. Funct. Anal. **73** (1987), 122–134.

[18] Pedersen, S., Spectral sets whose spectrum is a lattice with a base, preprint, 1995.

[19] Sands, A. D., On the factorisation of finite abelian groups, Acta. Math. Acad. Sci. Hungar. **8** (1957), 65–86.

[20] Sands, A. D., On the factorization of finite abelian groups. II., Acta. Math. Acad. Sci. Hungar. **13** (1962) 153–159.

[21] Strichartz, R. S., Wavelets and self-affine tilings, Constr. Approx. **9** (1993), 327–346.

[22] Tijdeman, R., Decomposition of the integers as a direct sum of two subsets, in: *Number Theory Seminar Paris 1992–1993* (ed. S. David), Cambridge University Press: Cambridge 1995, pp. 261–276.

email: `jcl@research.att.com`
`wang@math.gatech.edu`