An Elementary Problem Equivalent to the Riemann Hypothesis

Jeffrey C. Lagarias

(July 29, 2001 version)

1. INTRODUCTION.

We consider the following problem.

Problem E. *Let $H_n = \sum_{j=1}^{n} 1/j$. Show that, for each $n \geq 1$,*

$$\sum_{d|n} d \leq H_n + \exp(H_n)\log(H_n), \tag{1.1}$$

with equality only for $n = 1$.

The function $\sigma(n) = \sum_{d|n} d$ is the *sum-of-divisors function*, so for example $\sigma(6) = 12$. The number H_n is called the *nth harmonic number* by Knuth, Graham, and Patashnik [12, sect. 6.3], who detail various properties of harmonic numbers.

The 'E' in Problem E might stand for either 'easy' or 'elementary'. Perhaps 'H' for 'hard' would be a better letter to use, since our object is to show the following equivalence.

Theorem 1.1 *Problem E is equivalent to the Riemann hypothesis.*

The Riemann hypothesis, stated by Riemann [21] in 1859, concerns the complex zeros of the Riemann zeta function. The Riemann zeta function $\zeta(s)$ is defined in the half-plane $\Re(s) > 1$ by the convergent Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

and is extended by analytic continuation to the complex plane, where it has one singularity; namely, a simple pole with residue 1 at $s = 1$. The Riemann hypothesis states that the nonreal zeros of the Riemann zeta function $\zeta(s)$ all lie on the line $\Re(s) = 1/2$. One reason for the great interest in the Riemann hypothesis, currently regarded as the outstanding unsolved problem in pure mathematics, is its connection with the distribution of prime numbers, as described in what follows. Its importance in pure mathematics arises as part of a vast edifice of generalizations of the zeta function (L-functions) and their interrelations with central problems in number theory, algebraic geometry, topology, and representation theory (see Berry and Keating [3], Gelbart [8], Katz and Sarnak [11], and Murty[15]). There are even connections with physics, in zeta-function regularization in quantum field theory, and possibly in string theory (see Elizalde [6], Hawking [10], and Miller and Moore [14]).

The connection of the Riemann hypothesis with prime numbers was the original question studied by Riemann. Let $\pi(x)$ count the number of primes p with $1 < p \le x$. C. F. Gauss noted empirically that $\pi(x)$ is well approximated by the logarithmic integral

$$Li(x) = \int_2^x \frac{dt}{\log t},$$

which itself satisfies

$$Li(x) = \frac{x}{\log x} + O(\frac{x}{(\log x)^2}).$$

The Riemann hypothesis is equivalent to the assertion that for each $\epsilon > 0$ there is a positive constant C_ϵ such that

$$|\pi(x) - Li(x)| \le C_\epsilon x^{1/2+\epsilon}$$

for all $x \ge 2$ (see Edwards [5, p. 90]). The force of the Riemann hypothesis lies in the small size of the error term. The strongest form known of the *Prime Number Theorem* with error term asserts that

$$|\pi(x) - Li(x)| \le C_1 x \ \exp(-C_2(\log x)^{3/5-\epsilon}),$$

for any positive ϵ and certain positive constants C_1 and C_2 depending on ϵ. This result is due to Vinogradov and Korobov in 1958.

Problem E is derived from a criterion of Guy Robin [22] for the Riemann hypothesis. Robin's criterion states that the Riemann hypothesis is true if and only if

$$\sigma(n) < e^\gamma n \log\log n \ \text{ for all } \ n \ge 5041, \tag{1.2}$$

where $\gamma \approx 0.57721$ is Euler's constant. This criterion is related to the density of primes, as explained in Section 2. Our aim here is to obtain a problem statement as elementary as possible, containing no undefined constants such as Euler's constant. However, the hard work underlying the equivalence resides in the results of Robin stated in Section 3. There we deduce Theorem 1.1 by appealing to Robin's results.

In the next section we describe how the Riemann hypothesis is related to the sum-of-divisors function. The connection traces back to Ramanujan's work on highly composite numbers and involves several results of Erdős with various coauthors.

There are many other equivalent forms for the Riemann hypothesis, described in Edwards [5, chap. 12] and Titchmarsh [26, chap. 14]. Some known equivalences might be called "elementary," for example, one given by Davis [4, p. 335].

2. COLOSSALLY ABUNDANT NUMBERS.

The Riemann hypothesis is encoded in the criterion of Theorem 1.1 in terms of the very thin set of values of $\sigma(n)$ that are "large." The sum-of-divisors function can be represented as follows:

$$\sigma(n) = \prod_{p^a||n} (1 + p + p^2 + ... + p^a) = n \prod_{p^a||n} (1 + \frac{1}{p} + ... + \frac{1}{p^a}), \tag{2.1}$$

where the product is taken over all primes p dividing n and the notation $p^a||n$ signifies that p^a divides n, but p^{a+1} does not divide n. The average size of $\sigma(n)$ is on the order of Cn.

A quantitative form of this is provided by the following result of Bachmann (see Hardy and Wright [9, Theorem 324]).

Theorem 2.1 (Bachmann). *The sum-of-divisors function $\sigma(n)$ satisfies*

$$\frac{1}{n}\sum_{j=1}^{n}\sigma(j) = \frac{\pi^2}{12}n + O(\log n)$$

as $n \to \infty$.

This result implies that the average order of $\sigma(n)$ is given by $g(n) = \pi^2 n/6$, where we say that an arithmetical function $f(n)$ has *average order* $g(n)$ provided that

$$f(1) + f(2) + \cdots + f(n) \sim g(1) + g(2) + \cdots + g(n)$$

as $n \to \infty$, and $g(n)$ is a smooth function of n (see Hardy and Wright [9, sect. 18.2]).

The maximal size of $\sigma(n)$ can be somewhat larger than n and was determined by Gronwall in 1913 (see Hardy and Wright [9, Theorem 323]).

Theorem 2.2 (Gronwall). *The asymptotic maximal size of $\sigma(n)$ satisfies*

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma},$$

where γ is Euler's constant.

This result can be deduced from Mertens's theorem, which asserts that

$$\prod_{p\le x}(1 - \frac{1}{p}) \sim \frac{e^{\gamma}}{\log x}$$

as $x \to \infty$ (see Hardy and Wright [9, Theorem 429])). A much more refined version of the asymptotic upper bound, due to Robin [22, Theorem 2], asserts that for all $n \ge 3$,

$$\sigma(n) < e^{\gamma} n \log\log n + 0.6482\frac{n}{\log\log n}. \tag{2.2}$$

In Section 3 we will show that for all $n \ge 3$,

$$H_n + \exp(H_n)\log(H_n) \le e^{\gamma} n \log\log n + \frac{4n}{\log n},$$

which is only slightly stronger than (2.2). Now (2.2) shows that the inequality (1.1), if ever false, cannot be false by very much.

The study of extremal values of functions of the divisors of n is a branch of number theory with a long history. Let $d(n)$ count the number of divisors of n (including 1 and n itself). *Highly composite numbers* are those positive integers n such that

$$d(n) > d(k) \qquad \text{for } 1 \le k \le n-1.$$

n	Factorization of n	$\frac{\sigma(n)}{n}$
2	2	1.500
6	$2 \cdot 3$	2.000
12	$2^2 \cdot 3$	2.333
60	$2^2 \cdot 3 \cdot 5$	2.800
120	$2^3 \cdot 3 \cdot 5$	3.000
360	$2^3 \cdot 3^2 \cdot 5$	3.250
2520	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	3.714
5040	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	3.838
55440	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	4.187
720720	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	4.509
1441440	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	4.581
4324320	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	4.699
21621600	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	4.855
367567200	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$	5.141
6983776800	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	5.412
160626866400	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdots 23$	5.647
321253732800	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdots 23$	5.692
9316358251200	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdots 29$	5.888
288807105787200	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdots 31$	6.078
2021649740510400	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 31$	6.187
6064949221531200	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 31$	6.238
224403121196654400	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 37$	6.407

Table 1: Colossally abundant numbers up to 10^{18}.

Superior highly composite numbers are those positive integers n for which there is a positive exponent ϵ such that

$$\frac{d(n)}{n^\epsilon} \geq \frac{d(k)}{k^\epsilon} \qquad \text{for all } k > 1,$$

so that they maximize $d(n)/n^\epsilon$ over all n. These form a subset of the highly composite numbers. The study of such numbers was initiated by Ramanujan. One can formulate similar extrema for the sum-of-divisors function. *Superabundant numbers* are those positive integers n such that

$$\frac{\sigma(n)}{n} > \frac{\sigma(k)}{k} \qquad \text{for } 1 \leq k \leq n-1.$$

Colossally abundant numbers are those numbers n for which there is a positive exponent ϵ such that

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(k)}{k^{1+\epsilon}} \qquad \text{for all } k > 1,$$

so that n attains the maximum value of $\sigma(k)/k^{1+\epsilon}$ over all k. The set of colossally abundant numbers is infinite. They form a subset of the superabundant numbers, a fact that can be deduced from the definition. Table 1 displays the colossally abundant numbers up to 10^{18}, as in [1]. Robin showed that, if the Riemann hypothesis is false, then there will necessarily exist a counterexample to the inequality (1.2) that is a colossally abundant number (see [22, sect. 3, Proposition 1]); the same property can be established for counterexamples to (1.1). (There could potentially exist other counterexamples as well.)

Superabundant and colossally abundant numbers were studied[1] in detail by Alaoglu and Erdős [1] in 1944. As evidenced in the table, colossally abundant numbers consist of a product of all the small primes up to some bound, with exponents that are nonincreasing as the prime increases. The values of these exponents have a characteristic smooth shape that can be almost completely described (see formula (2.3)). In fact, these classes of numbers had been studied earlier, by Ramanujan, in his 1915 work on highly composite numbers [18]. The notes in Ramanjuan's Collected Papers [18, p. 339] report: "The London Mathematical Society was in some financial difficulty at the time, and Ramanujan suppressed part of what he had written in order to save expense." Only the first fifty-two of seventy-five sections were printed. The manuscript of the unpublished part was rediscovered in a box among the possessions of G. N. Watson after his death, along with a collection of loose pages now called "Ramanujan's Lost Notebook" (see Andrews [2] and Ramanujan [19, pp. 280–308]). The remaining twenty-three sections of the paper were finally published in 1997, in [20]. Superabundant and colossally abundant numbers are considered in Section 59, as special cases of the concepts of "generalised highly composite numbers" and "superior generalised highly composite numbers," for the parameter value $s = 1$. Ramanujan derived upper and lower bounds for the maximal order of generalised highly composite numbers in Section 71, assuming the Riemann hypothesis. (The Riemann hypothesis was assumed from Section 40 onward in his paper.) His bounds imply, assuming the Riemann hypothesis, that (1.2) holds for all sufficiently large n. The detailed estimates of Robin then establish (1.2) for all $n \geq 5041$.

The results of Alaoglu and Erdős in their 1944 paper are unconditional and are mainly concerned with the exponents of primes occurring in highly composite and superabundant numbers. In considering the exponents of primes appearing in colossally abundant numbers, they raised the following question, which is still unsolved.

Question (Alaoglu and Erdős). *If p and q are both primes, is it true that p^x and q^x are both rational only if x is an integer?*

This question is expected to have a positive answer. This would follow as a consequence[2] of the *four exponentials conjecture* in transcendental number theory, which asserts that if a_1, a_2 form a pair of complex numbers, linearly independent over the rationals $\mathbb{Q}$, and b_1, b_2 is another such pair, also linearly independent over the rationals, then at least one of the four exponentials $\{e^{a_i b_j} : i, j = 1 \text{ or } 2\}$ is transcendental (see Lang [13, pp. 8-11]). Alaoglu and Erdős [1, Theorem 10] showed that, for "generic" values of ϵ, there is exactly one colossally abundant number n maximizing $\sigma(k)/k^{1+\epsilon}$, and the exponent $a_p(\epsilon)$ of each prime p in it is

$$a_p(\epsilon) = \lfloor \frac{\log(p^{1+\epsilon} - 1) - \log(p^{\epsilon} - 1)}{\log p} \rfloor - 1. \tag{2.3}$$

Furthermore, for all $\epsilon > 0$ the value of n defined by this formula is a colossally abundant number. For a discrete set of ϵ there will be more than one maximizing integer. Erdős and

[1] Alaoglu and Erdős [1] use a slightly stronger definition of colossally abundant number; namely, they impose the additional requirement that $\sigma(n)/n^{1+\epsilon} > \sigma(k)/k^{1+\epsilon}$ hold for $1 \leq k < n$. With their definition, the colossally abundant numbers are exactly those given by (2.3).

[2] For irrational x consider $a_1 = \log p$, $a_2 = \log q$, $b_1 = 1$, and $b_2 = x$ in the four exponentials conjecture. For noninteger rational x a direct argument can be given.

Nicolas [7, p. 70] later showed that, for a given value of ϵ, there will be exactly one, two, or four integers n that attain the maximum value of $\sigma(k)/k^{1+\epsilon}$. If the question of Alaoglu and Erdős has a positive answer, then no values of ϵ will have four extremal integers. This would imply that the ratio of two consecutive colossally abundant numbers, the larger divided by the smaller, will always be a prime, and that every colossally abundant number has a factorization (2.3) for some nonempty open interval of "generic" values of ϵ.

Now we can explain the relation of the Riemann hypothesis to the behavior of extremal values of $\sigma(n)/n$. Colossally abundant numbers are products of all the small primes raised to powers that are a smoothly decreasing function of their size. Fluctuations in the distribution of primes will be reflected in fluctuations in the growth rate of $\sigma(n)/n$ taken over the set of colossally abundant numbers. Recall that the Riemann hypothesis is equivalent to the assertion that, for each fixed $\epsilon > 0$, the estimate

$$|\pi(x) - Li(x)| < x^{1/2+\epsilon}$$

holds for all sufficiently large x. It is also known that, if the Riemann hypothesis is false, then there will exist a specific positive constant δ such that the one-sided inequality

$$\pi(x) > Li(x) + x^{1/2+\delta}$$

is true for an infinite set of values x with $x \to \infty$. Heuristically, if we choose a value of x where such an excess of primes over $Li(x)$ occurs and then take a product of appropriate powers of primes up to this bound, we may hope to construct numbers n with $\sigma(n)$ exceeding $e^{\gamma} n \log\log n$ by a small amount. If the Riemann hypothesis holds, there is a smaller upper bound for excess of the number of primes above $Li(x)$, and from this one can deduce a slighly better upper bound for $\sigma(n)$. The analysis of Robin [22] gives a quantitative version of this, as formulated in Theorems 3.1 and 3.2 below. Note that the Riemann hypothesis does not influence the main term in the extremal asymptotic behavior of $\sigma(n)/n$ only the size of lower order terms in the asymptotics.

One can prove unconditionally that inequality (1.1) holds for nearly all integers. Even if the Riemann hypothesis is false, the set of exceptions to (1.1) will form a very sparse set. Furthermore, if there exists any counterexample to (1.1), the value of n will be very large.

3. PROOFS.

Theorem 1.1 will be deduced from the following two results of Robin [22].

Theorem 3.1 *(*Robin*). If the Riemann hypothesis is true, then for each $n \geq 5041$,*

$$\sum_{d|n} d \leq e^{\gamma} n \log\log n \ , \tag{3.1}$$

where γ is Euler's constant.

Proof. This is Theorem 1 of Robin [22]. Its main advantage over earlier results is that it establishes the explicit bound 5041 beyond which (3.1) holds. The proof contains very careful inequality estimates using "explicit formulas" for prime-counting functions in terms of zeros of the zeta function. It also makes use of explicit error estimates for prime-counting functions due to Rosser and Schoenfeld [23], [24], [25]. ∎

Theorem 3.2 *(*Robin*). If the Riemann hypothesis is false, then there exist constants $0 < \beta < 1/2$ and $C > 0$ such that*

$$\sum_{d|n} d \geq e^{\gamma} n \log\log n + \frac{Cn \log\log n}{(\log n)^{\beta}} \tag{3.2}$$

holds for infinitely many n.

Proof. This follows from Proposition 1 of Section 4 of Robin [22]. The constant β can be chosen to have any value satisfying $1 - b < \beta < 1/2$, where $b = \Re(\rho)$ for some zero ρ of $\zeta(s)$ with $\Re(\rho) > 1/2$, and $C > 0$ must be chosen sufficiently small, depending on ρ. The proof uses ideas from a result of Nicolas [16], [17, Proposition 3], which itself uses a method of Landau. ∎

The reduction of Theorem 1.1 to these results follows from the elementary inequalities given in the following two lemmas.

Lemma 3.1 . *For $n \geq 3$,*

$$\exp(H_n)\log(H_n) \geq e^{\gamma} n \log\log n \ . \tag{3.3}$$

Proof. Letting $\lfloor t \rfloor$ denote the integer part of t and $\{t\}$ the fractional part of t, we have

$$\int_1^n \frac{\lfloor t \rfloor}{t^2} dt = \int_1^n \frac{1}{t^2}(\sum_{1 \leq r \leq t} 1) dt = \sum_{1 \leq r \leq n} \int_r^n \frac{1}{t^2} dt = \sum_{r=1}^{n} (\frac{1}{r} - \frac{1}{n}) = H_n - 1.$$

Thus

$$H_n = 1 + \int_1^n \frac{t - \{t\}}{t^2} dt = \log n + 1 - \int_1^n \frac{\{t\}}{t^2} dt. \tag{3.4}$$

From this we obtain

$$H_n = \log n + \gamma + \int_n^{\infty} \frac{\{t\}}{t^2} dt, \tag{3.5}$$

where we have set

$$\gamma = 1 - \int_1^{\infty} \frac{\{t\}}{t^2} dt.$$

This is Euler's constant $\gamma = 0.57721\ldots$, since letting $n \to \infty$ yields

$$\gamma = \lim_{n \to \infty} (H_n - \log n),$$

which is its usual definition. Now (3.5) gives

$$H_n > \log n + \gamma,$$

which on exponentiating yields

$$\exp(H_n) \geq e^{\gamma} n. \tag{3.6}$$

Finally $H_n \geq \log n$, so $\log(H_n) \geq \log\log n > 0$ for $n \geq 3$. Combining this with (3.6) yields (3.3). ∎

Lemma 3.2 . *For $n \geq 3$,*

$$H_n + \exp(H_n)\log(H_n) \leq e^{\gamma} n \log\log n + \frac{4n}{\log n} \ . \tag{3.7}$$

Proof. For $n \geq 1$ define R_n by

$$R_n = H_n - \log(n+1) = \int_1^{n+1} (\frac{1}{\lfloor t \rfloor} - \frac{1}{t}) dt.$$

The expression on the far right reveals that the quantities R_n are nonnegative and monotonically increasing with n. Since $\lim_{n\to\infty}(H_n - \log(n+1)) = \gamma$, we obtain

$$H_n - \log(n+1) \leq \gamma. \tag{3.8}$$

Exponentiating this inequality gives

$$exp(H_n) \leq e^{\gamma}(n+1). \tag{3.9}$$

Formula (3.4) implies that, for $n \geq 3$,

$$\log H_n \leq \log(\log n + 1) = \log(\log n(1 + \frac{1}{\log n})) \leq \log\log n + \frac{1}{\log n} \ , \tag{3.10}$$

where we have used the elementary fact that $\log(1+x) \leq x$ for $x \geq 0$. Multiplying (3.9) and (3.10) yields, for $n \geq 3$,

$$\exp(H_n)\log(H_n) \leq e^{\gamma} n \log\log n + \frac{e^{\gamma} n}{\log n} + e^{\gamma}(\log\log n + \frac{1}{\log n}) \tag{3.11}$$

We next observe that, for $n \geq 3$,

$$\log\log n + \frac{1}{\log n} \leq \frac{n}{2\log n}.$$

Substituting this bound in (3.11) yields

$$\exp(H_n)\log(H_n) \leq e^{\gamma} n \log\log n + \frac{3e^{\gamma} n}{2\log n}.$$

Now (3.4) gives, for $n \geq 3$,

$$H_n \leq \log n + 1 \leq \frac{n}{\log n}.$$

Adding this inequality to the preceding one yields, for $n \geq 3$,

$$H_n + \exp(H_n)\log(H_n) \leq e^{\gamma} n \log\log n + \frac{4n}{\log n},$$

since $1 + 3e^{\gamma}/2 < 4$. ∎

Proof of Theorem 1.1. ($\Leftarrow$) Suppose the Riemann hypothesis is true. Then Theorem 3.1 and Lemma 3.1 together give, for $n \geq 5041$,

$$\sum_{d|n} d \leq e^{\gamma} n \log\log n < H_n + \exp(H_n)\log H_n \ .$$

For $1 \leq n \leq 5040$ one verifies (1.1) directly by computer, the only case of equality being $n = 1$.

($\Rightarrow$) Suppose (1.1) holds for all n. We argue by contradiction, assuming that the Riemann hypothesis is false. Then Theorem 3.2 applies. However its lower bound (which is valid for infinitely many n) contradicts the upper bound of Lemma 3.2 (which holds for all sufficiently large n). We conclude that the Riemann hypothesis must be true. ■

M. Kaneko notes that these results together with a small computation can be used to show that the Riemann hypothesis is equivalent to the assertion that $\sigma(n) < exp(H_n)\log(H_n)$ for $n > 60$.

ACKNOWLEDGMENTS. I thank Michel Balazard for bringing Robin's criterion to my attention and J.-L. Nicolas for references, particularly concerning Ramanujan. Eric Rains checked (1.1) for $1 \leq n \leq 5040$ on the computer, and Jim Reeds found errata. I thank Masanobu Kaneko for a sharpened statement and proof of Lemma 3.2, and the reviewers for an improved proof of Lemma 3.1.

References

[1] L. Alaoglu and P. Erdős, On Highly Composite and Similar Numbers, Trans. Amer. Math. Soc. **56** (1944) 448–469.

[2] G. E. Andrews, An Introduction to Ramanujan's Lost Notebook, Amer. Math. Monthly **86** (1979) 89–108.

[3] M. V. Berry and J. P. Keating, The Riemann zeros and eigenvalue asymptotics, SIAM Review **41** (1999) 236–266.

[4] M. Davis, Hilbert's Tenth Problem: Aspects of a negative solution, in: *Mathematical Developments arising from Hilbert Problems*, Proc. Symp. Pure Math. No. XXVIII, American Math. Society: Provindence 1976, pp. 323–378.

[5] H. E. Edwards, *Riemann's Zeta Function*, Academic Press: New York 1974.

[6] E. Elizalde, *Ten physical applications of spectral zeta functions,* Lecture Notes in Physics, Monograph 35, Springer-Verlag: Berlin 1995.

[7] P. Erdős and J.-L. Nicolas, Répartition des nombres superabondants, Bull. Math. Soc. France **103** (1975) 65–90.

[8] S. Gelbart, An elementary introduction to the Langlands program, Bull. Amer. Math. Soc. **10** (1984) 177–219.

[9] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers. Fifth Edition.* Oxford Univ. Press: Oxford 1979.

[10] S. W. Hawking, Zeta function regularization of path integrals in curved spacetime, Comm. Math. Phys. **55** (1977) 133–148.

[11] N. Katz and P. Sarnak, Zeros of zeta functions and symmetry, Bull. Amer. Math. Soc. *36* (1999) 1–26.

[12] D. Knuth, R. L. Graham and O. Patashnik, *Concrete Mathematics* (Second Edition), Addison-Wesley: Reading, MA 1994.

[13] S. Lang, *Transcendental Number Theory*, Addison-Wesley: Reading, Mass. 1966.

[14] S. D. Miller and G. Moore, Landau-Siegel zeroes and black hole entropy, Asian J. Math. **4** (2000) 183–211.

[15] M. R. Murty, A motivated introduction to the Langlands program, in: *Advances in Number Theory*, Oxford U. Press: New York 1993, pp. 37–66.

[16] J. L. Nicolas, Petites valeurs de la fonction d'Euler et hypothèse de Riemann, *Seminar on number theory, Paris 1981-82 (Paris 1981/1982)*, Birkhäuser, Boston 1983, pp. 207–218.

[17] J. L. Nicolas, Petites valeurs de la fonction d'Euler, J. Number Theory **17** (1983) 375–388.

[18] S. Ramanujan, Highly composite numbers, Proc. London Math. Soc. **14** (1915), 347–407. (Also, S. Ramanujan, *Collected Papers*, Chelsea 1962, pp. 78–129, notes pp. 338-339.)

[19] S. Ramanujan, *The Lost notebook and other unpublished papers.* With an introduction by George E. Andrews, Narosa: New Delhi and Springer-Verlag: Berlin-New York 1988.

[20] S. Ramanujan, Highly composite numbers. Annotated and with a foreword by J.-L. Nicholas and G. Robin, Ramanujan J. **1** (1997) 119–153.

[21] G. F. B. Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, Monatsber. Akad. Berlin (1859), 671–680. (Also in: G. F. B. Riemann, *Gesammelte Mathematische Werke* (Ed. H. Weber), Dover reprint 1953.) [English translation in: Edwards (1974), pp. 299-305.]

[22] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, J. Math. Pures Appl. **63** (1984) 187–213.

[23] J. B. Rosser and L. Schoenfeld, Approximate Formulas for some Functions of Prime Numbers, Illinois J. Math. **6** (1962) 64–94.

[24] J. B. Rosser and L. Schoenfeld, Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $psi(x)$, Math. Comp. **29** (1975) 243–269.

[25] J. B. Rosser and L. Schoenfeld, Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $psi(x)$ II, Math. Comp. **30** (1976) 337–360.

[26] E. C. Titchmarsh, revised by D. R. Heath-Brown, *The Theory of the Riemann Zeta Function, Second Edition*, Clarendon Press: Oxford 1986.

AT&T Labs–Research, Florham Park, NJ 07932-0971, USA
email address: `jcl@research.att.com`