Beyond the Descartes Circle Theorem

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1. Introduction

A *Descartes configuration* is a configuration of four mutually tangent circles in the plane, in which no three circles have a common tangent. The possible arrangements of such configurations appear in Figure 1; we allow certain degenerate arrangements where some of the circles are straight lines. Suppose the radii of the circles are r_1, r_2, r_3, r_4. The reciprocals of these are the curvatures (or "bends") $b_j = 1/r_j$. A straight line is assigned infinite radius, so its "bend" is zero.

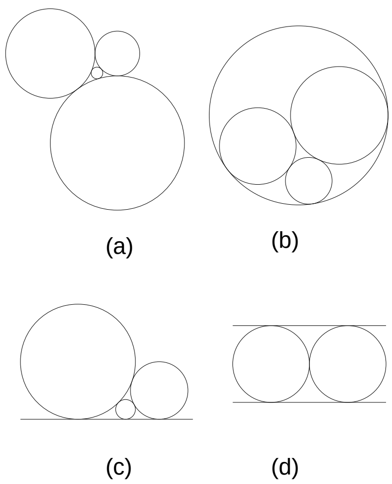

Figure 1: Descartes configurations

It is an old problem to determine relations between the circles in Descartes configurations. Relations among the radii for some particular Descartes configurations of type (b) appear in Greek mathematics, concerning the geometrical figure called an *arbelos*, or shoemaker's knife, in a proposition attributed to Archimedes [19, Prop. 6, p. 102].

In a 1643 letter to Princess Elizabeth of Bohemia, Rene Descartes stated a relation connecting the four radii [12, pp. 45–50]. This relation can be written as a quadratic equation connecting the four curvatures:

Theorem 1.1 (Descartes Circle Theorem) *In a Descartes configuration of four mutually tangent circles, the curvatures satisfy*

$$\sum_{j=1}^{4} {b_j}^2 = \frac{1}{2}(\sum_{j=1}^{4} b_j)^2. \tag{1.1}$$

Descartes considered only the configuration (a) in Figure 1. He did not state the result in this form, but gave a more complicated relation that is algebraically equivalent to (1.1), and his sketched proof is incomplete. In 1826 Jakob Steiner [30, pp. 61–63] independently found the result and gave a complete proof. Another independent rediscovery with a complete proof was given in 1842 by H. Beecroft [4], and is described in Coxeter [8]. Many other proofs have been discovered (and rediscovered), some of which are given in Pedoe [24].

The Descartes Circle Theorem applies to all Descartes configurations of types (a)-(d), provided we define the curvatures to have appropriate signs, as follows. An *oriented circle* is a circle together with an assigned direction of unit normal vector, which can point inward or outward. If it has radius r then its *oriented radius* is r for an inward pointing normal and $-r$ for an outward pointing normal. Its *oriented curvature* (or "signed curvature") is $1/r$ for an inward pointing normal and $-1/r$ for an outward pointing normal. By convention, the *interior* of an oriented circle is its interior for an inward pointing normal and its exterior for an outward pointing normal. An *oriented Descartes configuration* is a Descartes configuration in which the orientations of the circles are compatible in the following sense: either (i) the interiors of all four oriented circles are disjoint, or (ii) the interiors are disjoint when all orientations are reversed. Each Descartes configuration has exactly two compatible orientations in this sense, one obtained from the other by reversing all orientations. The *inward pointing orientation* of a Descartes configuration is the one in which the sum of the signed curvatures is positive, while the *outward pointing orientation* is the one in which the sum of the curvatures is negative. (It is an interesting exercise to prove that the sum of the signed curvatues cannot be zero.) With these definitions, the Descartes Circle Theorem remains valid for all oriented Descartes configurations, using oriented curvatures.

In 1936 Frederick Soddy (who earned a 1921 Nobel prize for discovering isotopes) published in *Nature* [28] a poem entitled "The Kiss Precise" in which he reported the result we have just described and a generalization to three dimensions. The following year Thorold Gossett [14] contributed another stanza that gave the general n-dimensional result. To state it, we define an *n-dimensional Descartes configuration* to consist of $n+2$ mutually tangent $(n-1)$-spheres in $\mathbb{R}^n$ in which all pairs of tangent $(n-1)$-spheres have distinct points of tangency, and orientation is done as in the 2-dimensional case.

Theorem 1.2 (Soddy-Gossett Theorem) *Given an oriented Descartes configuration in $\mathbb{R}^n$, if we let $b_j = 1/r_j$ be the oriented curvatures of the $n+2$ mutually tangent spheres, then*

$$\sum_{j=1}^{n+2} {b_j}^2 = \frac{1}{n}(\sum_{j=1}^{n+2} b_j)^2. \tag{1.2}$$

The case $n = 3$ of this result appears in an 1886 paper of Lachlan [22, p. 498] and his proof is given in the 1916 book of Coolidge [7, p. 258]. Thus in calling this result the "Soddy-Gossett Theorem" we are continuing the tradition that theorems are often not named for their first discoverers; see [32]. Proofs of the n-dimensional theorem appear in Pedoe [24] and Coxeter [9]. Pedoe observes that this result is actually a theorem of real algebraic geometry, rather than of complex algebraic geometry, in dimensions 3 and above. That is, the theorem depends on the fact that the number of real spheres, simultaneously tangent to each of $n+1$ mutually tangent real spheres with distinct tangents, is exactly two. However the total number of complex spheres with this tangency property is two in dimension $n = 2$ but typically exceeds two when $n \geq 3$.

In this paper we present some very simple and elegant extensions of these results, which involve the centers of the circles. We show that there are relations, similar to (1.2), that involve the centers, and the curvatures in the combination curvature$\times$center. Furthermore, all these relations generalize to arrangements of $n+2$ mutually tangent $(n-1)$-spheres in n-dimensional Euclidean, spherical and hyperbolic spaces, and have a matrix formulation, provided we use an appropriate notion of "bend" in spherical and hyperbolic space. In the process we recover spherical and hyperbolic analogues of the Soddy-Gossett Theorem; these were first obtained by Mauldon [23] in 1962. In §2 and §3 we state successively more general theorems, without giving any proofs, finally arriving at our most general result, the augmented Euclidean Descartes Theorem 3.3. Then in §4 we state and prove an analogue in spherical geometry, the Spherical

Generalized Descartes Theorem, which has a remarkably simple proof. In §5 we deduce from it the Augmented Euclidean Descartes Theorem by stereographic projection. In §6 we treat the hyperbolic geometry analogue.

The vast literature on this subject spans two centuries, but (so far) we have not found our matrix formulations in it. In spirit many of the ideas trace back at least to Wilker; [35, p. 390], see the remark at the end of §4.

2. The Complex Descartes Theorem

Given any three mutually tangent circles with curvatures b_1, b_2, b_3, there are exactly two other circles that are tangent to each of these; each gives a four-circle Descartes configuration. See Figure 2 for the possible arrangements of the resulting five circles; the three initial circles are given by dotted lines.

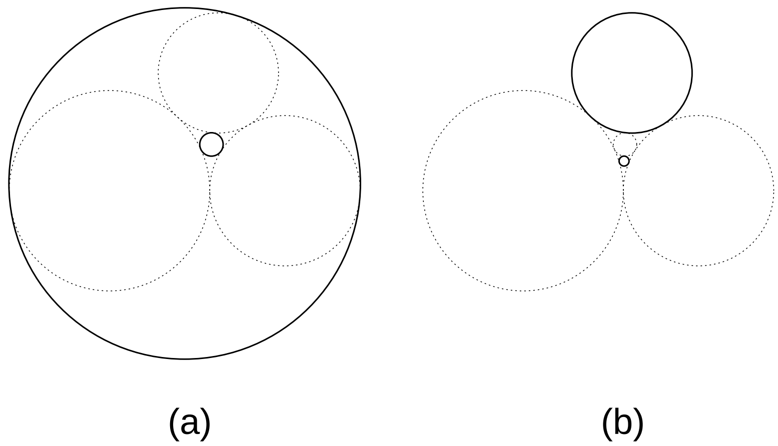

Figure 2: Circles Tangent to Three Tangent Circles

The curvatures of these two new circles are the roots of the quadratic equation (1.1), treating b_4 as the variable. Suppose these roots are b_4 and b_4'. Both can be positive, as in Figure 2(b), or one may be negative, as in Figure 2(a). From (1.1) we have

$$b_4 + b_4' = 2(b_1 + b_2 + b_3). \tag{2.1}$$

Thus, starting from a Descartes configuration, we can select any one of the four circles and replace it by the other circle that is tangent to the remaining three; this gives a new Descartes configuration, which may be called a *Descartes reflection* of the original configuration. (The new configuration is obtained from the old by a Möbius transformation that is an inversion with respect to the circle through the three tangent points of the other three circles.) The curvature of the new circle can be obtained from the original four by using (2.1). This operation can be repeated indefinitely; doing it in all possible ways gives a packing of circles that fills either (i) a single circle, as for example in Figure 3, or (ii) a strip between two parallel lines, or (iii) a half-plane, or (iv) the whole plane. Such a figure is called an *Apollonian packing,* in honor of Apollonius of Perga, who considered (about 200 BC) the eight circles that are tangent to each of three given circles in general position [21]. An Apollonian packing is completely specified by any three mutually tangent circles in it.

In constructing the Apollonian packing pictured in Figure 3, we started with four circles with oriented curvatures -1, 2, 2, and 3. Each circle has been labelled with its curvature; all are integers. It is clear from (2.1) that whenever we start with a Descartes configuration with all curvatures integral, then in this construction all the curvatures in the packing are integers.

In 1998, while computing Figure 3 with the center of the outer circle located at the origin, we noticed that the centers of all the circles are rational; in fact in this figure, if a circle has curvature b and center (x, y) then (it appeared) bx and by are always integers. Following this clue, we were led to the following generalization of (1.1), in which the centers are taken to be the complex numbers $z_j = x_j + iy_j$.

Theorem 2.1 (Complex Descartes Theorem) *Any Descartes configuration of four mutually tangent circles, with curvatures b_j and centers $z_j = x_j + iy_j$ satisfies*

$$\sum_{j=1}^{4}(b_j z_j)^2 = \frac{1}{2}(\sum_{j=1}^{4} b_j z_j)^2. \tag{2.2}$$

The relation (2.2) has the same form as the original Descartes relation (1.1). The Complex Descartes Theorem implies both the Descartes Circle Theorem (1.1) and a third relation

$$\sum_{j=1}^{4} b_j(b_j z_j) = \frac{1}{2}(\sum_{j=1}^{4} b_j)(\sum_{j=1}^{4} b_j z_j), \tag{2.3}$$

These results are obtained by replacing z_j by $z_j + w$ in (2.2), where w is an arbitrary complex number, and identifying coefficients of powers of w.

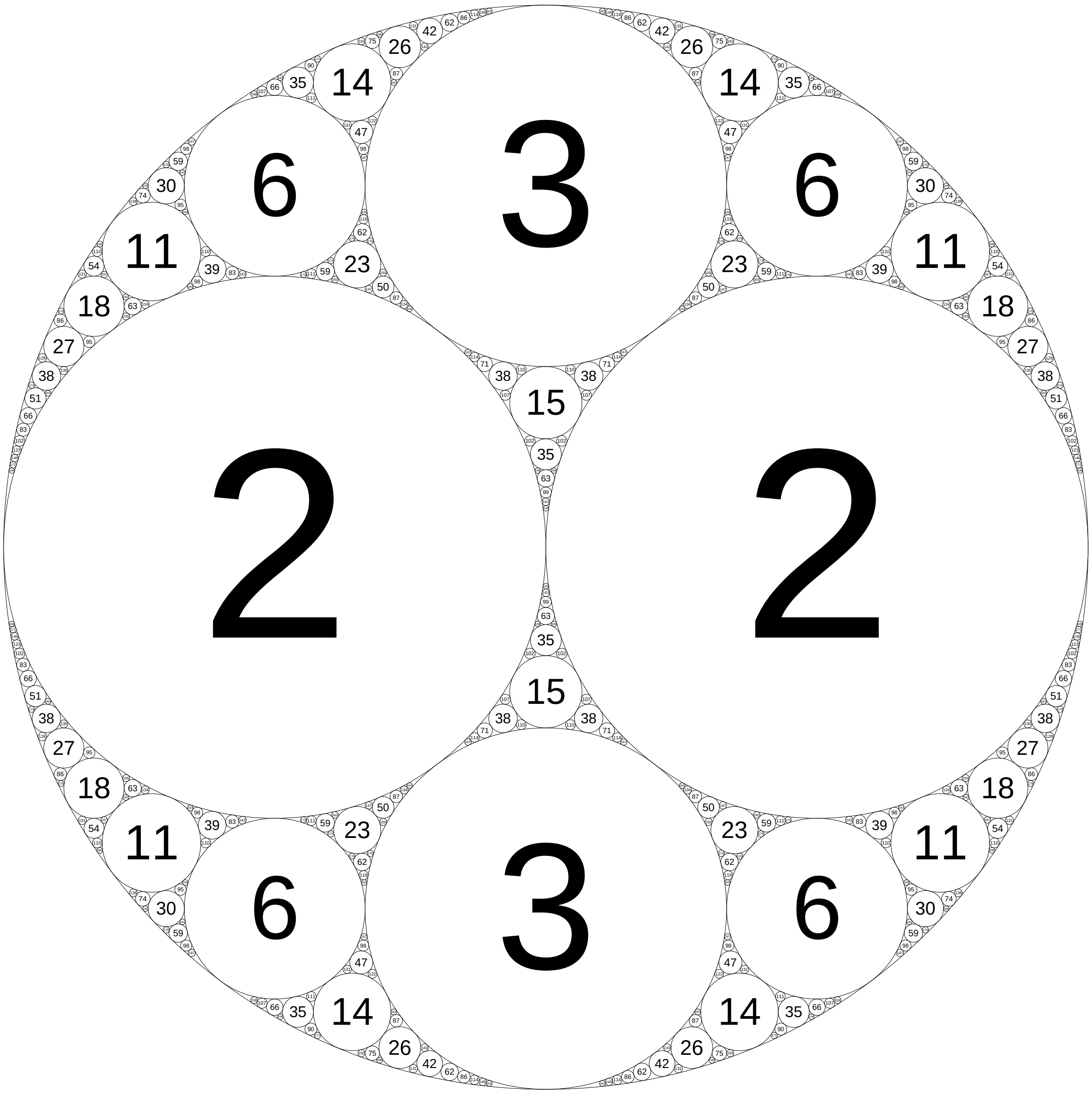

Figure 3: An Apollonian packing

The Complex Descartes Theorem also implies a relation similar to (2.1) that connects the centers of two circles, each of which is tangent to each of three given mutually tangent circles:

$$b_4 z_4 + b'_4 z'_4 = 2(b_1 z_1 + b_2 z_2 + b_3 z_3). \tag{2.4}$$

Thus in the iterative construction of an Apollonian packing, both the curvatures and the centers of the new circles can be obtained by simple linear operations (followed by divisions). This makes it very easy to draw figures such as Figure 3 using a computer.

The relations in the Complex Descartes Theorem can be expressed in an elegant form using the matrix

$$\mathbf{Q}_2 := I_4 - \frac{1}{2}\mathbf{1}_4\mathbf{1}_4^T = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \tag{2.5}$$

in which $\mathbf{1}_n$ denotes a column of n 1's, and $\mathbf{Q}_2$ is the coefficient matrix of the *Descartes quadratic form*

$$Q_2(x_1, x_2, x_3, x_4) := \mathbf{x}^T\mathbf{Q}_2\mathbf{x} = (x_1^2 + x_2^2 + x_3^2 + x_4^2) - \frac{1}{2}(x_1 + x_2 + x_3 + x_4)^2.$$

The subscript 2 in $\mathbf{Q}_2$ refers to the dimension of the space we are considering,

If $\mathbf{b} = (b_1, b_2, b_3, b_4)^T$ denotes the column vector of curvatures and $\mathbf{c} = (b_1z_1, b_2z_2, b_3z_3, b_4z_4)^T$, then the Descartes Circle Theorem asserts that

$$\mathbf{b}^T\mathbf{Q}_2\mathbf{b} = 0, \tag{2.6}$$

and the Complex Descartes Theorem asserts that

$$\mathbf{c}^T\mathbf{Q}_2\mathbf{c} = 0. \tag{2.7}$$

The Complex Descartes Theorem does not completely characterize Descartes configurations in the Euclidean plane, but a slightly stronger result does:

Theorem 2.2 (Extended Descartes Theorem) *Given a configuration of four oriented circles with non-zero curvatures* $(b_1,\ b_2,\ b_3,\ b_4)$ *and centers* $\{(x_i, y_i)\ : 1 \le i \le 4\}$, *let* $\mathbf{M}$ *be the* 4×3 *matrix*

$$\mathbf{M} := \begin{bmatrix} b_1 & b_1x_1 & b_1y_1 \\ b_2 & b_2x_2 & b_2y_2 \\ b_3 & b_3x_3 & b_3y_3 \\ b_4 & b_4x_4 & b_4y_4 \end{bmatrix}. \tag{2.8}$$

Then this configuration is an oriented Descartes configuration if and only if

$$\mathbf{M}^T\mathbf{Q}_2\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{2.9}$$

If one or two curvatures b_i *are zero, and the corresponding centers are infinite, then* $\mathbf{M}$ *can be defined in such a way that (2.9) remains true.*

The Complex Descartes Theorem follows from this result by writing the vector $\mathbf{c} = \mathbf{x} + i\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are the second and third columns of $\mathbf{M}$. The extended Descartes Theorem generalizes gracefully to n-dimensions, to which we turn next.

3. Descartes Configurations in n-Dimensional Euclidean Space

An *n-dimensional oriented Descartes configuration* consists of $n+2$ mutually tangent oriented $(n-1)$-spheres S_i in n-dimensional space $\mathbb{R}^n$, having distinct tangencies, whose orientations are compatible in the sense that all interiors are disjoint, either with the given orientation or with the reversal of all orientation vectors. Here we suppose that $n \ge 2$; the one-dimensional case is treated in §8. We often regard a hyperplane as a limiting case of a sphere, having zero curvature, with orientation given by a unit normal vector. In what follows an "oriented sphere" includes the hyperplane case unless otherwise stated.

The Soddy-Gossett Theorem (1.2) relates the curvatures of such a configuration of mutually tangent n-spheres, and can be written

$$Q_n(\mathbf{b}) := \mathbf{b}^T\mathbf{Q}_n\mathbf{b} = 0,$$

where $\mathbf{b} = (b_1, \ldots, b_{n+2})^T$ and $Q_n(\mathbf{x}) = \mathbf{x}^T\mathbf{Q}_n\mathbf{x}$ is the *n-dimensional Descartes quadratic form* whose associated symmetric $(n+2) \times (n+2)$ matrix $\mathbf{Q}_n$ is

$$\mathbf{Q}_n := I_{n+2} - \frac{1}{n}\mathbf{1}_{n+2}\mathbf{1}_{n+2}^T. \tag{3.1}$$

The Soddy-Gossett Theorem has a converse.

Theorem 3.1 (Converse to Soddy-Gosset Theorem) *If* $\mathbf{b} = (b_1, \ldots, b_{n+2})^T$ *is a nonzero real column vector that satisfies*

$$\mathbf{b}^T \mathbf{Q}_n \mathbf{b} = 0, \tag{3.2}$$

then there exists an oriented Descartes configuration whose oriented curvature vector is $\mathbf{b}$*. Furthermore any two oriented Descartes configurations with the same oriented curvature vector are congruent; that is, there is a Euclidean motion taking one to the other.*

A Euclidean motion is one that preserves angles and distances; it includes reflections. We do not know an easy proof of Theorem 3.1; a proof appears in [17].

The geometry of Descartes configurations is encoded in the curvature vector $\mathbf{b}$. If $\sum_{j=1}^{n+2} b_j > 0$, then one of the following holds: (i) all of $b_1,\ b_2, \ldots,\ b_{n+2}$ are positive; (ii) $n+1$ are positive and one is negative; (iii) $n+1$ are positive and one is zero; or (iv) n are positive and equal and the other two are zero. These four cases correspond respectively to the following configurations of mutually tangent spheres: (i) $n+1$ spheres, with another in the curvilinear simplex that they enclose; (ii) $n+1$ spheres inscribed inside another larger sphere; (iii) $n+1$ spheres with one hyperplane (the $(n+2)$-nd "sphere"), tangent to each of them; (iv) n equal spheres with two common parallel tangent planes.

Definition 3.1. Given an oriented sphere S in $\mathbb{R}^n$, its *curvature-center coordinates* consist of the $(n+1)$-vector

$$\mathbf{m}(S) = (b, bx_1, \ldots, bx_n) \tag{3.3}$$

in which b is the signed curvature of S (assumed nonzero) and $\mathbf{x}(S) = \mathbf{x} = (x_1, x_2, \ldots, x_n)$ is its center. For the degenerate case of an oriented hyperplane H, its *curvature-center coordinates* $\mathbf{m}(H)$ are defined to be

$$\mathbf{m}(S) = (0, \mathbf{h}), \tag{3.4}$$

where $\mathbf{h} := (h_1, h_2, \ldots, h_n)$ is the unit normal vector that gives the orientation of the hyperplane.

To see the origin of this definition in the degenerate case, let the point of H closest to the origin be $\mathbf{z} = a\mathbf{h}$ for some real value a. For $t > |a|$, let S_t be the oriented sphere of radius t centered at $(t+a)\mathbf{h}$, which has center in direction $\mathbf{h}$ from the origin and contains $\mathbf{z}$. As $t \to \infty$ the oriented spheres S_t clearly converge geometrically to the oriented hyperplane H, and $\mathbf{m}(S_t) = (\frac{1}{t}, (1+\frac{a}{t})\mathbf{h}) \to \mathbf{m}(H) = (0, \mathbf{h})$.

Curvature-center coordinates are not quite a global coordinate system, because they do not always uniquely specify an oriented sphere. Given $\mathbf{m} \in \mathbb{R}^{n+1}$, if its first coordinate a is nonzero then there exists a unique sphere having $\mathbf{m} = \mathbf{m}(S)$. But if $a = 0$, the hyperplane case, there is a hyperplane if and only if $\sum h_i^2 = 1$, and in that case there is a pencil of hyperplanes that have the given value $\mathbf{m}$, which differ from each other by a translation.

Theorem 3.2 (Euclidean Generalized Descartes Theorem) *Given a configuration of $n+2$ oriented spheres $S_1, S_2, \ldots S_{n+2}$ in $\mathbb{R}^n$ (allowing hyperplanes), let $\mathbf{M}$ be the $(n+2)\times(n+1)$ matrix whose j-th row entries are the curvature-center coordinates $\mathbf{m}(S_j)$ of the j-th sphere. If this configuration is an oriented Descartes configuration, then*

$$\mathbf{M}^T \mathbf{Q}_n \mathbf{M} = \begin{bmatrix} 0 & 0 \\ 0 & 2I_n \end{bmatrix} = diag(0,\ 2,\ 2, \ldots,\ 2). \tag{3.5}$$

Conversely, any real solution $\mathbf{M}$ to equation (3.5) is the matrix of a unique oriented Descartes configuration.

The curvature-center coordinate matrix $\mathbf{M}$ of an oriented Descartes configuration determines it uniquely even if it contains hyperplanes, because the other spheres in the configuration give enough information to fix the locations of the hyperplanes. This result contains the Soddy-Gossett Theorem as its (1,1)- coordinate. We derive the "if" part of this theorem from the next result. However the converse part of this theorem seems more difficult, and we do not prove it here; see [17, Theorem 2.3].

We proceed to a further generalization, which extends the $(n+2)\times(n+1)$ matrix $\mathbf{M}$ to an $(n+2)\times(n+2)$ matrix $\mathbf{W}$ obtained by adding another column. This augmented matrix incorporates information about two oriented Descartes configurations, the original one and one obtained from it by inversion in the unit sphere, as we now explain. The definition of $\mathbf{W}$ may seem pulled out of thin air, but in §5 we observe that it arises naturally from an analogous result in spherical geometry, which is how we discovered it.

In n-dimensional Euclidean space, the operation of *inversion in the unit sphere* replaces the point $\mathbf{x}$ by $\mathbf{x}/|\mathbf{x}|^2$, where $|\mathbf{x}|^2 = \sum_{j=1}^{n} x_j^2$. Consider a general oriented sphere S with center $\mathbf{x}$ and oriented radius r. Then inversion in the unit sphere takes S to the sphere $\bar{S}$ with center $\bar{\mathbf{x}} = \mathbf{x}/(|\mathbf{x}|^2 - r^2)$ and oriented radius $\bar{r} = r/(|\mathbf{x}|^2 - r^2)$. If $|\mathbf{x}|^2 > r^2$, then $\bar{S}$ has the same orientation as S. In all cases,

$$\frac{\mathbf{x}}{r} = \frac{\bar{\mathbf{x}}}{\bar{r}} \tag{3.6}$$

and

$$\bar{b} = \frac{|\mathbf{x}|^2}{r} - r. \tag{3.7}$$

Definition 3.2. Given an oriented sphere S in $\mathbb{R}^n$, its *augmented curvature-center coordinates* are the $(n+2)$-vector

$$\mathbf{w}(S) := (\bar{b}, b, bx_1, \ldots, bx_n) = (\bar{b}, \mathbf{m}), \tag{3.8}$$

in which $\bar{b} = b(\bar{S})$ is the curvature of the sphere or hyperplane $\bar{S}$ obtained by inversion of S in the unit sphere, and the entries of $\mathbf{m}$ are its curvature-center coordinates. For hyperplanes we define

$$\mathbf{w}(H) := (\bar{b}, 0, h_1, \ldots, h_n) = (\bar{b}, \mathbf{m}), \tag{3.9}$$

where $\bar{b}$ is the oriented curvature of the sphere or hyperplane $\bar{H}$ obtained by inversion of H in the unit sphere.

Augmented curvature-center coordinates provide a global coordinate system: no two distinct oriented spheres have the same coordinates. The only case to resolve is when S is a hyperplane, i.e., $b = 0$. The relation (3.6) shows that $(\bar{b}, bx_1, \ldots, bx_n)$ are the curvature-center coordinates of $\bar{S}$, and if $\bar{b} \neq 0$, this uniquely determines $\bar{S}$; inversion in the unit circle then determines S. In the remaining case, $b = \bar{b} = 0$ and $S = \bar{S}$ is the unique hyperplane passing through the origin whose unit normal is given by the remaining coordinates.

Given a collection $(S_1, S_2, \ldots, S_{n+2})$ of $n+2$ oriented spheres (possibly hyperplanes) in $\mathbb{R}^n$, the *augmented matrix* $\mathbf{W}$ associated with it is the $(n+2) \times (n+2)$ matrix whose j-th row has entries given by the augmented curvature-center coordinates $\mathbf{w}(S_j)$ of the j-th sphere.

The action of inversion in the unit sphere has a particularly simple interpretation in augmented matrix coordinates. If $\mathbf{W}$ is the augmented matrix associated with a Descartes configuration, and if $\mathbf{W}'$ is the augmented matrix associated with its inversion in the unit sphere, then it follows from (3.6) that

$$\mathbf{W} = \mathbf{W}' \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix}. \tag{3.10}$$

Theorem 3.3 (Augmented Euclidean Descartes Theorem) *The augmented matrix* $\mathbf{W}$ *of an oriented Descartes configuration of* $n+2$ *spheres* $\{S_i : 1 \le i \le n+2\}$ *in* $\mathbb{R}^n$ *satisfies*

$$\mathbf{W}^T \mathbf{Q}_n \mathbf{W} = \begin{bmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2I_n \end{bmatrix}. \tag{3.11}$$

Conversely, any real solution $\mathbf{W}$ *to (3.11) is the augmented matrix of a unique oriented Descartes configuration.*

We prove the Augmented Euclidean Descartes Theorem in §5. In the "if" direction it includes as special cases the "if" direction of each of the theorems stated so far, and represents our final stage of generalization of the Descartes Circle Theorem in Euclidean space. In particular, the "if" part of the Euclidean Generalized Descartes Theorem is just (3.11) with the first row and column deleted. However in the converse direction the Augmented Euclidean Descartes Theorem is not as strong as the converse in the Euclidean Generalized Descartes Theorem, nor does it imply the converse to the Soddy-Gossett Theorem; these results require separate proofs.

The augmented Euclidean Descartes Theorem gives a complete parametrization of all oriented Descartes configurations, which shows that the moduli space of all such configurations has the structure of an affine real-algebraic variety.

We discovered the Augmented Euclidean Descartes Theorem while studying analogues of the Descartes Theorem in non-Euclidean geometries. In the next section we formulate and prove an analogue in spherical geometry, and then in §5 deduce the Augmented Euclidean Descartes Theorem from it.

4. Spherical Geometry

The standard model for spherical geometry $\mathbb{S}^n$ is the unit n-sphere S^n embedded in $\mathbb{R}^{n+1}$ as the surface

$$S^n := \{y : y_0^2 + y_1^2 + \cdots + y_n^2 = 1\} \tag{4.1}$$

with the Riemannian metric induced from the Euclidean metric in $\mathbb{R}^{n+1}$ by restriction. In this model, the distance between two points of $\mathbb{S}^n$ is simply the angle α between the radii that join the origin of $\mathbb{R}^{n+1}$ to the representatives of these points on S^n. This distance α always satisfies $0 \le \alpha \le \pi$.

A *sphere* C in this geometry is the locus of points that are equidistant (at distance α say) from a point in $\mathbb{S}^n$ called its *center.* The quantity $\alpha = \alpha(C)$ is the *spherical radius* or *angular radius* of C; it is the angle at the origin $\mathbf{0}$ of $\mathbb{R}^{n+1}$ between a ray from $\mathbf{0}$ to the center of C and a ray from $\mathbf{0}$ to any point of C. There are two choices for the center (and the angular radius) of a given sphere; these two choices form a pair of antipodal points of S^n. The choice of a center amounts to orienting the sphere. In this model the interior of a sphere is a spherical cap, cut off by the intersection of the sphere S^n with a hyperplane in $\mathbb{R}^{n+1}$, so (by abuse of language) we also call an oriented sphere a *spherical cap.*

The two spherical caps determined by a given sphere are called *complementary* and the sum of their angular radii is π. The *interior* of an oriented sphere contains all points of S^n on the same side of the hyperplane as the center of the sphere. If we describe a hyperplane by a linear form

$$\mathbf{F}(y) = \sum_{i=0}^{n} f_i y_i - f, \tag{4.2}$$

normalized by the requirement

$$f_0^2 + \sum_{i=1}^{n} f_i^2 = 1, \tag{4.3}$$

this provides an orientation by defining a positive half-space $F(y) > 0$. The sphere has center $\mathbf{f} := (f_0, f_1, \ldots, f_n)$ and has positive radius if and only if $|f| < 1$. The radius α satisfies $\cos\alpha = f$, and the interior of the spherical cap it determines is the region where the linear form is positive. A spherical cap can be specified either by a pair $(\mathbf{f}, \alpha)$ or by the pair $(-\mathbf{f}, \alpha - \pi)$, while $(-\mathbf{f}, \pi - \alpha)$ determines the complementary spherical cap.

A *spherical Descartes configuration* consists of $n+2$ mutually tangent spherical caps on the surface of the unit n-sphere, such that either (i) the interiors of all spherical caps are mutually disjoint, or (ii) the interiors of all complementary spherical caps are mutually disjoint.

Theorem 4.1 (Spherical Soddy-Gossett Theorem) *Consider a spherical Descartes configuration of $n+2$ mutually tangent spherical caps C_i on the n-dimensional unit sphere S^n embedded in $\mathbb{R}^{n+1}$, with spherical radius α_j subtended by the j-th cap. Then the spherical radii satisfy*

$$\sum_{i=1}^{n+2} (\cot \alpha_i)^2 = \frac{1}{n} (\sum_{i=1}^{n+2} \cot \alpha_i)^2 - 2. \tag{4.4}$$

This theorem was found by Mauldon [23, Theorem 4] in 1962, as part of a more general result allowing non-tangent spheres. He also established a converse: each real solution of (4.4) corresponds to some spherical Descartes configuration, and two spherical Descartes configurations with the same data in (4.4) are congruent configurations in spherical geometry. We will deduce Theorem 4.1 from the next result.

The Spherical Soddy-Gossett Theorem is intrinsic, i.e., it depends only on the Riemannian metric for spherical geometry, and not on the coordinate system used to describe the manifold. However we can establish it as a special case of a result that does depend on a particular choice of coordinate system. If C is a spherical cap with center $\mathbf{y} = (y_0, y_1, y_2, \ldots, y_{n+1})$, and angular radius α, we define its *spherical curvature-center coordinates* $\mathbf{w}_+(C)$ to be the row vector

$$\mathbf{w}_+(C) := (\cot \alpha, \frac{y_0}{\sin \alpha}, \frac{y_1}{\sin \alpha}, \ldots, \frac{y_n}{\sin \alpha}). \tag{4.5}$$

Here the name "curvature-center coordinates" is chosen by analogy with the Euclidean case; the appearance of the "center" in it is clear, and the "bend" is $\cot \alpha$, which can be interpreted as a geodesic curvature; see [13, pp. 180–192]. We can also regard $1/\sin \alpha$ as a kind of curvature in that it is a monotone decreasing functions of the angular radius of the sphere. No two spherical caps have the same coordinates $\mathbf{w}_+$, since α is uniquely determined by the first coordinate, and then the y_j are uniquely determined using the other coordinates.

To any configuration of $n+2$ caps $C_1, \ldots, C_{n+2}$ we associate the $(n+2) \times (n+2)$ *spherical curvature-center coordinate matrix* $\mathbf{W}_+$ whose jth row is $\mathbf{w}_+(C_j)$.

Theorem 4.2 (Spherical Generalized Descartes Theorem) *Consider a configuration of $n+2$ oriented spherical caps C_j that is a spherical Descartes configuration. The $(n+2) \times (n+2)$ matrix $\mathbf{W}_+$ whose j-th row is the spherical curvature-center coordinates of C_j satisfies*

$$\mathbf{W}_+^T \mathbf{Q}_n \mathbf{W}_+ = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2I_n \end{bmatrix} = diag(-2, 2, 2, \ldots, 2). \tag{4.6}$$

Conversely, any real matrix $\mathbf{W}_+$ that satisfies (4.6) is the spherical curvature-center coordinate matrix of some spherical Descartes configuration.

The $(1,1)$-entry of the matrix relation (4.6) is the Spherical Soddy-Gossett Theorem.

This theorem has a remarkably simple proof, which is based on two preliminary lemmas. Let $\mathbf{J}_n$ be the $(n+2) \times (n+2)$ matrix

$$\mathbf{J}_n = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix} = \mathrm{diag}(-1, 1, \ldots, 1). \tag{4.7}$$

Lemma 4.3. *(i) For any $(n+2)$-vector $\mathbf{w}_+$, there is a spherical cap C with $\mathbf{w}_+(C) = \mathbf{w}_+$ if and only if*

$$\mathbf{w}_+ \mathbf{J}_n \mathbf{w}_+^T = 1. \tag{4.8}$$

(ii) The spherical caps C and C' are externally tangent if and only if

$$\mathbf{w}_+(C) \mathbf{J}_n \mathbf{w}_+(C')^T = -1. \tag{4.9}$$

Proof. (i) If $\mathbf{w}_+$ comes from a spherical cap with center $\mathbf{y}$ and angular radius α, then

$$\mathbf{w}_+ \mathbf{J}_n \mathbf{w}_+^T = \frac{-(\cos\alpha)^2 + \sum_{j=0}^n y_j^2}{(\sin\alpha)^2} = \frac{1 - (\cos\alpha)^2}{(\sin\alpha)^2} = 1$$

so (4.8) holds.

Conversely, if (4.8) holds, then one recovers a unique α with $0 < \alpha < \pi$ by setting $\cot\alpha := (\mathbf{w}_+)_1$, and one then defines a vector $\mathbf{y} = (y_0, \ldots, y_{n+1})$ via $y_j := (\mathbf{w}_+)_j / \sin\alpha$, noting that

$\sin\alpha \neq 0$. The equation (4.8) now implies that $|\mathbf{y}|^2 = 1$, so $\mathbf{y}$ lies on the unit sphere, and we have determined a spherical cap that gives the vector $\mathbf{w}_+$.

(ii) Two spherical caps with centers $\mathbf{y}, \mathbf{y}'$ with angular radii α, α' are externally tangent if and only if the angle between their centers, viewed from the origin in $\mathbb{R}^{n+1}$, is $\alpha + \alpha'$. Since $\mathbf{y}$ and $\mathbf{y}'$ are unit vectors, this holds if and only if

$$\mathbf{y}(\mathbf{y}')^T = \cos(\alpha + \alpha').$$

Now

$$\mathbf{w}_+(C)\mathbf{J}_n\mathbf{w}_+(C')^T = \frac{1}{\sin\alpha\sin\alpha'}(-\cos\alpha\cos\alpha' + \mathbf{y}(\mathbf{y}')^T)$$

and this gives (4.9), using $\cos(\alpha+\alpha') = \cos\alpha\cos\alpha' - \sin\alpha\sin\alpha'$. $\square$

Lemma 4.4. *If $\mathbf{A}$, $\mathbf{B}$ are non-singular $n \times n$ matrices and $\mathbf{W}\mathbf{A}\mathbf{W}^T = \mathbf{B}$, then $\mathbf{W}^T\mathbf{B}^{-1}\mathbf{W} = \mathbf{A}^{-1}$.*

Proof. The matrix $\mathbf{W}$ is non-singular since $\mathbf{B}$ is non-singular. Now invert both sides, and multiply on the left by $\mathbf{W}^T$ and on the right by $\mathbf{W}$. $\square$

Proof of the Spherical Generalized Descartes Theorem. If the caps C_j touch externally, Lemma 4.3 ensures that

$$\mathbf{W}_+\mathbf{J}_n\mathbf{W}_+^T = 2\mathbf{I}_{n+2} - \mathbf{1}_{n+2}\mathbf{1}_{n+2}^T = 2\mathbf{Q}_n^{-1}. \tag{4.10}$$

Then applying Lemma 4.4 (with $\mathbf{A} = \mathbf{J}_n$ and $\mathbf{W} = \mathbf{W}_+$) we obtain

$$\mathbf{W}_+^T\mathbf{Q}_n\mathbf{W}_+ = 2{\mathbf{J}_n}^{-1} = 2\mathbf{J}_n. \tag{4.11}$$

Conversely, (4.11) implies (4.10) by Lemma 4.4. Looking at the diagonal elements of $\mathbf{W}_+\mathbf{J}_n\mathbf{W}_+^T$, which are all 1's, Lemma 4.3(i) guarantees that the j-th row of $\mathbf{W}_+$ is a vector $\mathbf{w}_+(C_j)$ for some (uniquely determined) spherical cap C_j. Since the off-diagonal elements are all -1, Lemma 4.3(ii) ensures that the caps touch externally pairwise, so they form a spherical Descartes configuration. $\square$

Remark. Wilker [35, pp. 388-390] came tantalizingly close to obtaining the Spherical Generalized Descartes Theorem. He termed a spherical Descartes configuration a "cluster", and introduced spherical curvature-center coordinates. In a remark he noted our Lemma 4.3 and stated equation (4.10). However he did not invert his formula, via Lemma 4.4, and so failed to formulate a result in terms of the Descartes quadratic form.

5. Stereographic Projection and the Augmented Euclidean Descartes Theorem

We derive the Augmented Euclidean Descartes Theorem from the Spherical Generalized Descartes Theorem, using stereographic projection. The resulting derivation is reversible, so the Spherical Generalized Descartes Theorem and the Augmented Euclidean Descartes Theorem may be viewed as equivalent results.

Consider the unit sphere in $\mathbb{R}^{n+1}$, given by $\sum_{i=0}^{n} y_i^2 = 1$. Points on this sphere can be mapped into the plane $y_0 = 0$ by stereographic projection from the "south pole" $(-1, 0, \ldots, 0)$; see Figure 4. (We use the hyperboloid in the figure later.)

This mapping $(y_0, \ldots, y_n) \to (x_1, \ldots, x_n)$ is given by

$$x_j = \frac{y_j}{1 + y_0}, \quad 1 \le j \le n.$$

The spherical cap C with center $(p_0, \ldots, p_n)$ and angular radius α is the intersection of the unit sphere with the plane

$$\sum_{j=1}^{n} p_j y_j = \cos\alpha.$$

The sterographic projection of this cap in the hyperplane $y_0 = 0$ is the (Euclidean) sphere S with center $(x_1, \ldots, x_n)$ and radius r, where

$$x_j = \frac{p_j}{p_0 + \cos\alpha}, \quad 1 \le j \le n, \qquad \text{and} \qquad r = \frac{\sin\alpha}{p_0 + \cos\alpha}.$$

If the boundary of the cap C contains the south pole, the corresponding sphere S has infinite radius, i.e., it is a hyperplane.

Proof of the Augmented Euclidean Descartes Theorem. The spherical coordinates of the spherical cap C are given by the row vector

$$\mathbf{w}_+(C) = (\cot\alpha, \frac{p_0}{\sin\alpha}, \frac{p_1}{\sin\alpha}, \ldots, \frac{p_n}{\sin\alpha}),$$

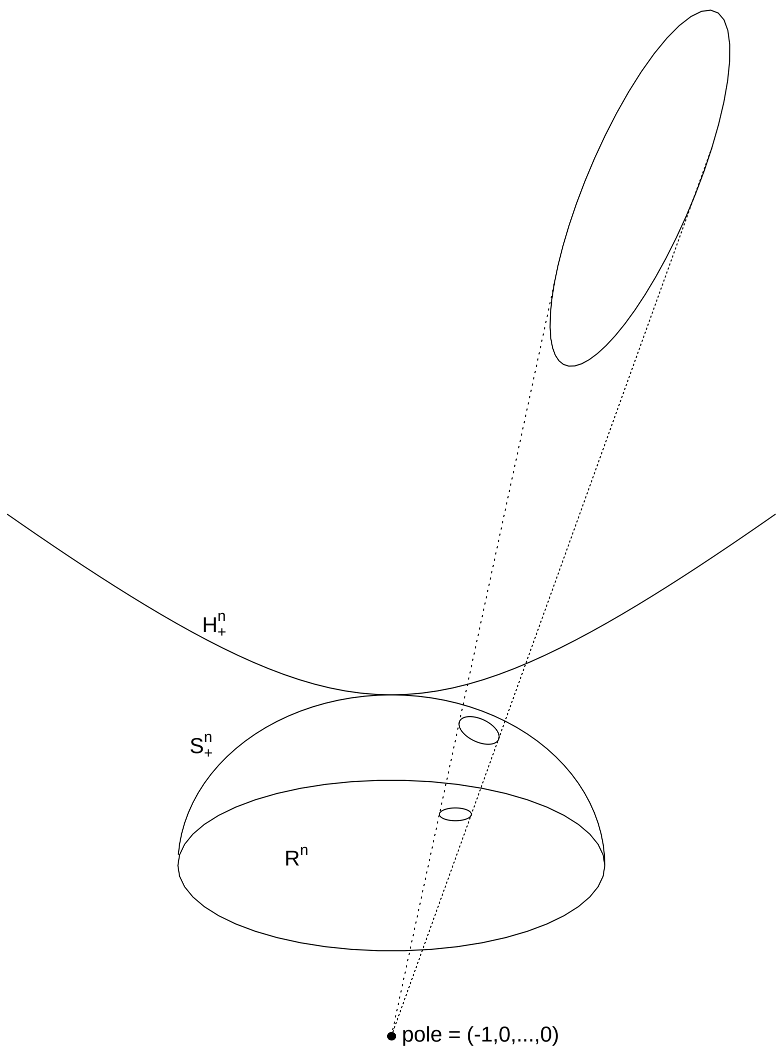

Figure 4: Stereographic projection-hyperplane, sphere and hyperboloid

We relate this to the augmented Euclidean curvature-center coordinate vector $\mathbf{w}(S)$ associated with the corresponding projected sphere S in the plane $y_0 = 0$, given by (3.8). We have $x_j/r = p_j/\sin\alpha$, $b = 1/r = \cot\alpha + p_0/\sin\alpha$, and we find

$$\bar{b} = \cot\alpha - \frac{p_0}{\sin\alpha}.$$

Thus

$$\mathbf{w}(S) = (\cot\alpha - \frac{p_0}{\sin\alpha}, \cot\alpha + \frac{p_0}{\sin\alpha}, \frac{p_1}{\sin\alpha}, \ldots, \frac{p_n}{\sin\alpha}) = \mathbf{w}_+(C)\mathbf{G}, \tag{5.1}$$

where

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix}. \tag{5.2}$$

In fact (5.1) can be used to *define* (Euclidean) augmented curvature-center coordinates in terms of spherical curvature-center coordinates; this is how we found them. The matrix (5.2) is uniquely determined by requiring (5.1) produce the curvature-center coordinates of S in its last $n+1$ positions.

Suppose we have a configuration of $n+2$ spherical caps $C_1, \ldots, C_{n+2}$ on the unit sphere. These project stereographically into a configuration of Euclidean spheres $S_1, \ldots, S_{n+2}$ in the equatorial plane $y_0 = 0$, and conversely every configuration of Euclidean spheres lifts to a configuration of spherical caps. The map sends spherical Descartes configurations to Euclidean Descartes configurations. We assemble the corresponding rows $\mathbf{w}_+(C_j)$, $\mathbf{w}(S_j)$ into matrices $\mathbf{W}_+$ and $\mathbf{W}$, respectively. Then

$$\mathbf{W} = \mathbf{W}_+\mathbf{G}, \tag{5.3}$$

and, using the Spherical Generalized Descartes Theorem 4.2, we have

$$\mathbf{W}^T\mathbf{Q}_n\mathbf{W} = \mathbf{G}^T\mathbf{W}_+^T\mathbf{Q}_n\mathbf{W}_+\mathbf{G} = \mathbf{G}^T\mathrm{diag}(-2, 2, \ldots, 2)\mathbf{G} = \begin{bmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2I_n \end{bmatrix},$$

which proves the Augmented Euclidean Descartes Theorem 3.3. $\square$

6. Hyperbolic Geometry

There are many models of hyperbolic space $\mathbb{H}^n$, of which the three most common are the (Poincaré) unit ball model, the half space model, and the hyperboloid model. (In two dimensions we say "unit disk" and "half-plane" for the first two models.) The unit ball and half-space models are described in [3] and [5, Chapter 19]. The hyperboloid model, which is less well known, but is in some ways simpler than the others, is described in [3, Section 3.7], [25], and [26]. The unit ball and half-space models are embedded in $\mathbb{R}^n$, though with different metrics, while the hyperboloid model is embedded in $\mathbb{R}^{n+1}$, endowed with a Minkowski metric. Here we need only the unit ball and hyperboloid models. A *sphere* in hyperbolic n-space $\mathbb{H}^n$ is the locus of points that are equidistant (in the hyperbolic metric) from some fixed point in $\mathbb{H}^n$, the center.

The unit ball model consists of the points $(y_1, \ldots, y_n)$ in $\mathbb{R}^n$ with $\sum_{j=1}^n y_i^2 < 1$, with the ideal boundary being $\sum_{j=1}^n y_j^2 = 1$. In this model, the hyperbolic metric is

$$ds^2 = (dy_1^2 + \cdots + dy_n^2)/(1 - \sum_{j=1}^n y_j^2)^2.$$

and the hyperbolic distance between two points $\mathbf{y}, \mathbf{y}'$ satisfies

$$\cosh(d(\mathbf{y}, \mathbf{y}')) = \left((1 + \sum_{j=1}^n y_j^2)(1 + \sum_{j=1}^n {y'_j}^2) - 4 \sum_{j=1}^n y_j y'_j \right) \Big/ \left((1 - \sum_{j=1}^n y_j^2)(1 - \sum_{j=1}^n {y'_j}^2) \right). \quad (6.1)$$

In this model a hyperbolic sphere (of finite radius) is a Euclidean sphere contained strictly inside the unit ball; however, its hyperbolic center and hyperbolic radius usually differ from the Euclidean ones.

Points in the hyperboloid model are represented in $\mathbb{R}^{n+1}$ as points on the upper sheet H_+^n ($u_0 > 0$) of the two-sheeted hyperboloid $H_\pm^n$ cut out by the equation

$$u_0^2 = 1 + u_1^2 + \cdots + u_n^2,$$

where $H_\pm^n = H_+^n \cup H_-^n$ with $H_-^n = -H_+^n$. However, the metric on $H_\pm^n$ is not that induced from the Euclidean metric on $\mathbb{R}^{n+1}$, but rather is that induced from the space $\mathbb{R}^{n,1}$ endowed with the Minkowski metric

$$ds^2 = -du_0^2 + du_1^2 + \cdots + du_n^2;$$

see [3, p. 49]. A Minkowski metric is not Riemannian, but the induced metric on the hyperboloid is Riemannian, and the formula for the *hyperbolic distance* $d(\mathbf{u}, \mathbf{u}')$ in this metric is given by

$$\cosh(d(\mathbf{u}, \mathbf{u}')) = u_0 u_0' - u_1 u_1' - \cdots - u_n u_n'; \tag{6.2}$$

see [25, (6.10)]. One can go between the hyperboloid model and the ball model by the change of variables

$$y_j = \frac{u_j}{1 + u_0}, \qquad \text{for } \ 1 \le j \le n,$$

and in the opposite direction by

$$u_0 = \frac{2}{\Delta} - 1 \qquad \text{and} \qquad u_j = \frac{2y_j}{\Delta}, \ \ 1 \le j \le n,$$

where

$$\Delta = 1 - \sum_{j=1}^{n} u_j^2.$$

From (6.2) we see that in this model a hyperbolic sphere is represented by the intersection of H_+^n with a hyperplane $\mathbf{G}(u) = 0$, where

$$\mathbf{G}(u) = g_0 u_0 - \sum_{i=1}^{n} g_i u_i - g, \tag{6.3}$$

and where $\mathbf{g} := (g_0, g_1, \ldots, g_n)$ is the center of the hyperbolic sphere and lies on H_+^n, so it satisfies

$$g_0^2 = 1 + \sum_{i=1}^{n} g_i^2 \tag{6.4}$$

and $g_0 > 0$. Its radius s has $g = \cosh s$, hence $g > 1$ so that the (oriented) hyperbolic radius is real. We define the "bend" associated with a hyperbolic sphere to be $\coth s$; it has an interpretation as a geodesic curvature. As in the spherical case, we define the *interior* of the hyperbolic sphere to be the region on the same side of the plane $\mathbf{G}(u) = 0$ as the center, i.e., the region where $\mathbf{G}(u) > 0$. Note that as geometrical figures in $\mathbb{R}^{n+1}$ all hyperbolic spheres are $(n-1)$-dimensional Euclidean ellipsoids on the hyperboloid H_+^n.

We also allow degenerate hyperbolic spheres, which consist of the *horoballs* that touch the ideal boundary (absolute) in the ball model of hyperbolic geometry, as well as the ideal boundary itself. In the two-dimensional case, parabolas on the upper sheet correspond to *horocycles*;

in the unit disc model, these are (Euclidean) circles that are tangent to the bounding circle. They have infinite radius. In the disc model their centers are on the bounding circle, and in the hyperboloid model their centers are at infinity. The boundary of the disc model (absolute) corresponds to a circle at infinity in the hyperboloid model. A degenerate oriented hyperbolic sphere together with its interior is specified as the intersection of the hyperboloid with a closed half-space; the degenerate sphere itself is the intersection of H^n_+ with the hyperplane giving the boundary of the half-space, but the linear equation defining it cannot be normalized to satisfy (6.4). We define the "bend" of a degenerate oriented hyperbolic sphere to be ± 1, with $+1$ chosen if the interior is inside the horoball and is all of H^n_+ for the absolute, and -1 otherwise.

An *oriented hyperbolic Descartes configuration* is any set of $n+2$ mutually tangent oriented hyperbolic $(n-1)-spheres$ in $\mathbb{H}^n$, having the property that either (i) all interiors of the spheres are disjoint, or (ii) the interiors of each pair of spheres intersect in a nonempty open set. We also allow the hyperbolic spheres to include the degenerate cases.

Theorem 6.1 (Hyperbolic Soddy-Gossett Theorem) *The oriented hyperbolic radii $\{s_j \;:\; 1 \le j \le n+2\}$ of an oriented Descartes configuration of $n+2$ spheres in hyperbolic space $\mathbb{H}^n$ satisfy*

$$\sum_{j=1}^{n+2} (\coth s_j)^2 = \frac{1}{n}\Big(\sum_{j=1}^{n+2} \coth s_j\Big)^2 + 2. \tag{6.5}$$

This result was found by Mauldon [23]. The Hyperbolic Soddy-Gossett Theorem is intrinsic, depending only on the hyperbolic metric. We derive it as a special case of a result that does depend on a specific coordinate system, namely that for the hyperboloid model.

If S is a hyperbolic sphere in H^n_+ with center $\mathbf{u} = (u_0, u_1, u_2, \ldots, u_n)$, and hyperbolic radius s_j, we define its *hyperbolic curvature-center coordinates* $\mathbf{w}_-(S)$ to be the row vector

$$\mathbf{w}_-(S) := \Big(\ \coth s, \frac{u_0}{\sinh s}, \frac{u_1}{\sinh s}, \ldots, \frac{u_n}{\sinh s}\Big). \tag{6.6}$$

Once again the name "curvature-center coordinates" is chosen by analogy with the Euclidean case; it contains a "center" and the "bend" is $\coth s$, which can be interpreted as a geodesic curvature; see [13, p. 284] and [33, p. 190]. To a configuration of $n+2$ hyperbolic spheres $S_1, \ldots, S_{n+2}$ we associate the $(n+2) \times (n+2)$ matrix $\mathbf{W}_-$ whose jth row is $\mathbf{w}_-(S_j)$.

Theorem 6.2 (Hyperbolic Generalized Descartes Theorem) *Consider a configuration of $(n+2)$ oriented hyperbolic spheres that is a hyperbolic Descartes configuration. The associated matrix $\mathbf{W}_-$ whose rows are the hyperbolic curvature-center coordinates of the spheres satisfies*

$$\mathbf{W}_-^T \mathbf{Q}_n \mathbf{W}_- = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2I_n \end{bmatrix} = diag(2, -2, 2, \ldots, 2). \tag{6.7}$$

The converse of Theorem 6.2 does not hold, because some matrices $\mathbf{W}_-$ that satisfy (6.7) do not correspond to proper hyperbolic Descartes configurations. However, one can obtain a converse by defining a generalized notion of "virtual Descartes configuration" that uses both sheets of the hyperboloid, based on stereographic projection from the spherical model.

To do this we first define a notion of "virtual oriented hyperbolic sphere". Take any oriented spherical cap C on S^n, together with its interior, and define the associated *virtual oriented hyperbolic sphere* $S = S(C)$ to be its image on the two-sided hyperboloid $H_\pm^n = H_+^n \cup H_-^n$ under stereographic projection through the "south pole" $(-1, 0, 0, \ldots, 0)$ in $\mathbb{R}^{n+1}$, together with its interior. Every virtual hyperbolic sphere $S(C)$ (with its interior) is the intersection of the two-sheeted hyperboloid with some closed half-space, and the spherical cap C is uniquely determined by $S(C)$. We define the *hyperbolic curvature-center coordinates* of $S(C)$ to be

$$\mathbf{w}_-(S) := \mathbf{w}_+(C) \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix}. \tag{6.8}$$

If the spherical cap C lies entirely in the open upper half-sphere $S^{n+} = \{(u_0, u_1, \ldots, u_n) \in S^n : u_0 > 0$, the image $S(C)$ is a genuine hyperbolic sphere plus its interior, and all genuine oriented hyperbolic spheres arise this way. One can check that the formula (6.8) for hyperbolic curvature-center coordinates agrees with the definition (6.6); this can be proved along the lines of §5. In terms of the hyperplane defining (6.3) and (6.4) defining S we have

$$\mathbf{w}_-(S) = \left(\frac{g}{\sqrt{g^2-1}}, \frac{g_0}{\sqrt{g^2-1}}, \ldots, \frac{g_n}{\sqrt{g^2-1}}\right).$$

If that C lies in the closed upper-half sphere and touches the boundary $u_0 = 0$, the image $S(C)$ is a horoball, while if C is an oriented spherical cap with boundary consisting of the part of S^n with $u_0 = 0$, then the image $S(C)$ is the absolute. The hyperbolic curvature-center coordinates (6.8) of any virtual hyperbolic sphere that is not a genuine hyperbolic sphere are always real, and one may reverse-engineer a hyperbolic radius and center using (6.6), but s and the centers are then non-real complex numbers.

One now defines a *virtual oriented hyperbolic Descartes configuration* as the set of virtual oriented hyperbolic spheres resulting from stereographic projection on the two-sheeted hyperboloid $H^{\pm}$ of any spherical Descartes configuration. (We view genuine hyperbolic Descartes configurations as a special kind of virtual hyperbolic Descartes configurations.) If the associated hyperbolic curvature-center coordinate matrix $\mathbf{W}_-$ is defined in (6.8), and if $\mathbf{W}_+$ contains the spherical Descartes coordinates associated with the spherical Descartes configuration, then they are related by

$$\mathbf{W}_- = \mathbf{W}_+ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix}. \tag{6.9}$$

Given this relation, one immediately deduces Theorem 6.2 from Theorem 4.2, plus a converse if "virtual Descartes configurations" are included; these allow projections onto the full two-sheeted hyperboloid.

In applying stereographic projection, the locus of a genuine hyperbolic $(n-1)$-sphere on the hyperboloid is mapped to the locus of a spherical $(n-1)$-sphere on the unit n-sphere, and also to the locus of a Euclidean $(n-1)$-sphere in the plane $x_0 = 0$. The hyperbolic center, the spherical center of the associated spherical cap, and the Euclidean center of the Euclidean sphere are typically all different in the strong sense that they usually lie on three different lines through the "south pole" $(-1, 0, \ldots, 0)$ in $\mathbb{R}^{n+1}$.

7. Apollonian Packings

Using stereographic projection we have a recipe to pass between Euclidean, spherical, and hyperbolic Descartes configurations. It gives a one-to-one correspondence between configurations $\mathbf{W}, \mathbf{W}_+$, and $\mathbf{W}_-$ given by (5.2) and (6.9), namely

$$\mathbf{W} = \mathbf{W}_+ \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix} = \mathbf{W}_- \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix}, \tag{7.1}$$

where we use "virtual Descartes configurations" in the hyperbolic case; they are viewed as lying on the full two-sheeted hyperboloid. This recipe clearly lifts to Apollonian packings.

Since the Spherical and Hyperbolic Soddy-Gossett Theorems involve quadratic forms, an analogue of the relation (2.1) holds in spherical and hyperbolic geometry. This permits easy calculation of the "bends" $\cot \alpha$ (respectively $\coth s$) of circles in spherical (respectively hyperbolic) packings. There is a notion of "integral Apollonian circle packing" for such "bends" that makes sense in spherical and hyperbolic geometry. Furthermore, analogues of the relation (2.4) hold in spherical and hyperbolic geometry as well, which permit easy calculation of the centers in spherical (respectively hyperbolic) Apollonian packings.

Thus the standard Euclidean Apollonian packing pictured in Figure 3, with center at the origin, has a corresponding hyperbolic packing obtained by stereographic projection in which the $\coth s$'s are all integers, but *not* the same integers as in the Euclidean packing, calculated using $\mathbf{W}_-$ in (7.1). The bounding outer circle is assigned the "bend" -1; see Figure 5.
The circles that are tangent to the bounding circle are known as *horocycles*, and have infinite hyperbolic radius, so the corresponding value of $\coth s$ is 1. This explains the large number of circles assigned the value 1 in Figure 5, namely all those that touch the outer circle.

Similarly, in the spherical packing associated with the standard Euclidean packing in Figure 3, the $\cot \alpha$'s are all integers, different from both the Euclidean and hyperbolic cases, starting from $(0,1,1,2)$; see Figure 6.

One may notice interesting numerical relations among the integers in these three packings. Consider a "loxodromic sequence" of spheres as studied in [9] and [11], where each sphere is produced by Descartes reflection of the largest sphere of the preceding Descartes configuration. For the curvatures, one obtains for the Euclidean packing the infinite sequence $E : (-1,2,2,3,15,38,\ldots)$, for the spherical packing $S : (0,1,1,2,8,21,\ldots)$, and for the hyperbolic packing $H : (-1,1,1,1,7,17,\ldots)$. Note that $S + H = E$, since this is so for the initial values, and each sequence satisfies the same fourth-order linear recurrence relation, $x_{n+1} = 2x_n + 2x_{n-1} + 2x_{n-2} - x_{n-3}$, by (2.1).

In the Euclidean case, infinitely many different kinds of Apollonian packings have integer curvatures for all circles; see [18]. The same occurs for both hyperbolic and spherical Apollonian circle packings. In the hyperbolic case we also include among such integer hyperbolic circle packings some packings that are "virtual packings". Figure 5 is generated from the basic configuration having coth's $(-1,1,1,1)$, and the next simplest hyperbolic case is $(-2,3,5,6)$.

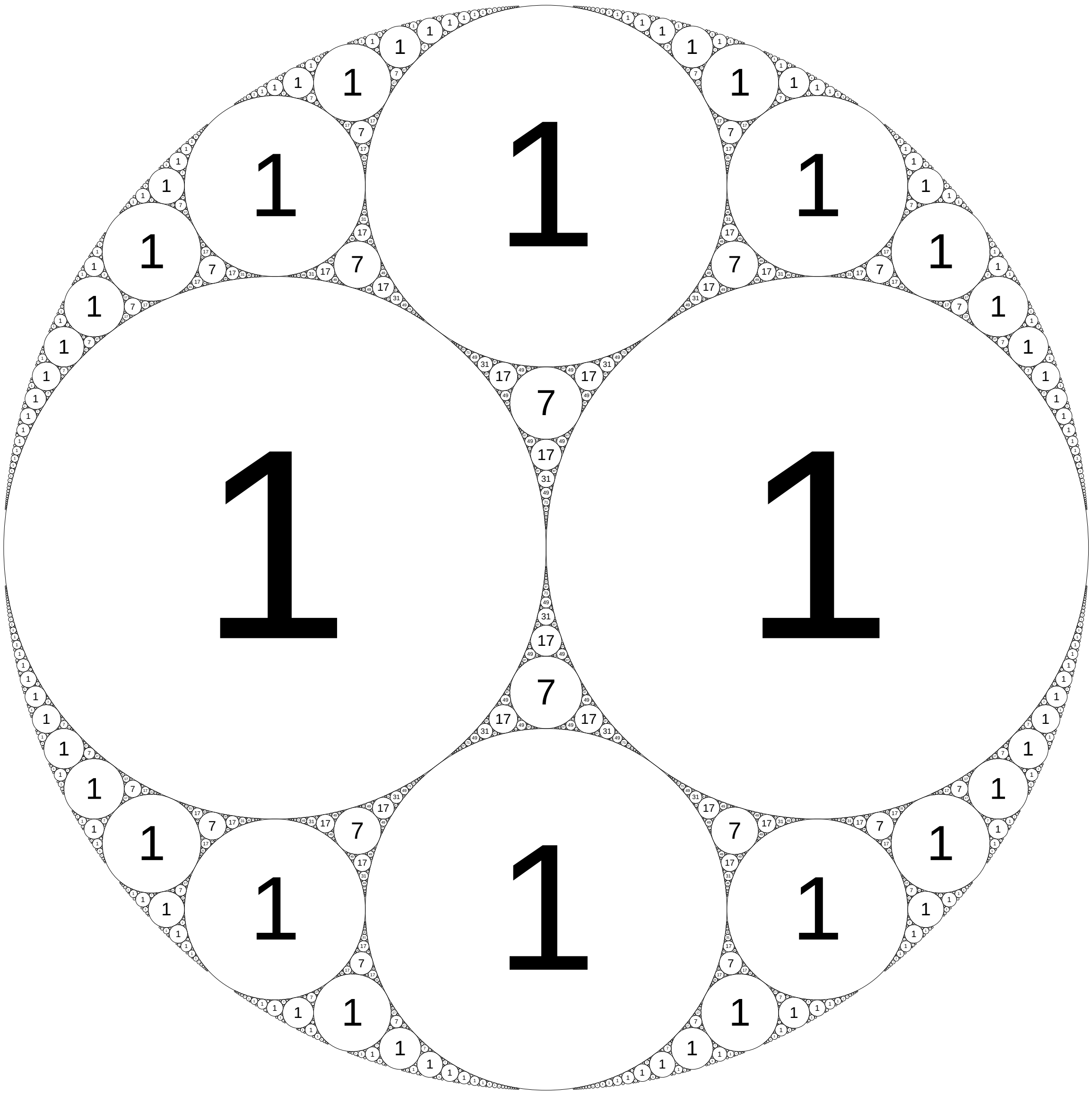

Figure 5: A hyperbolic Apollonian packing

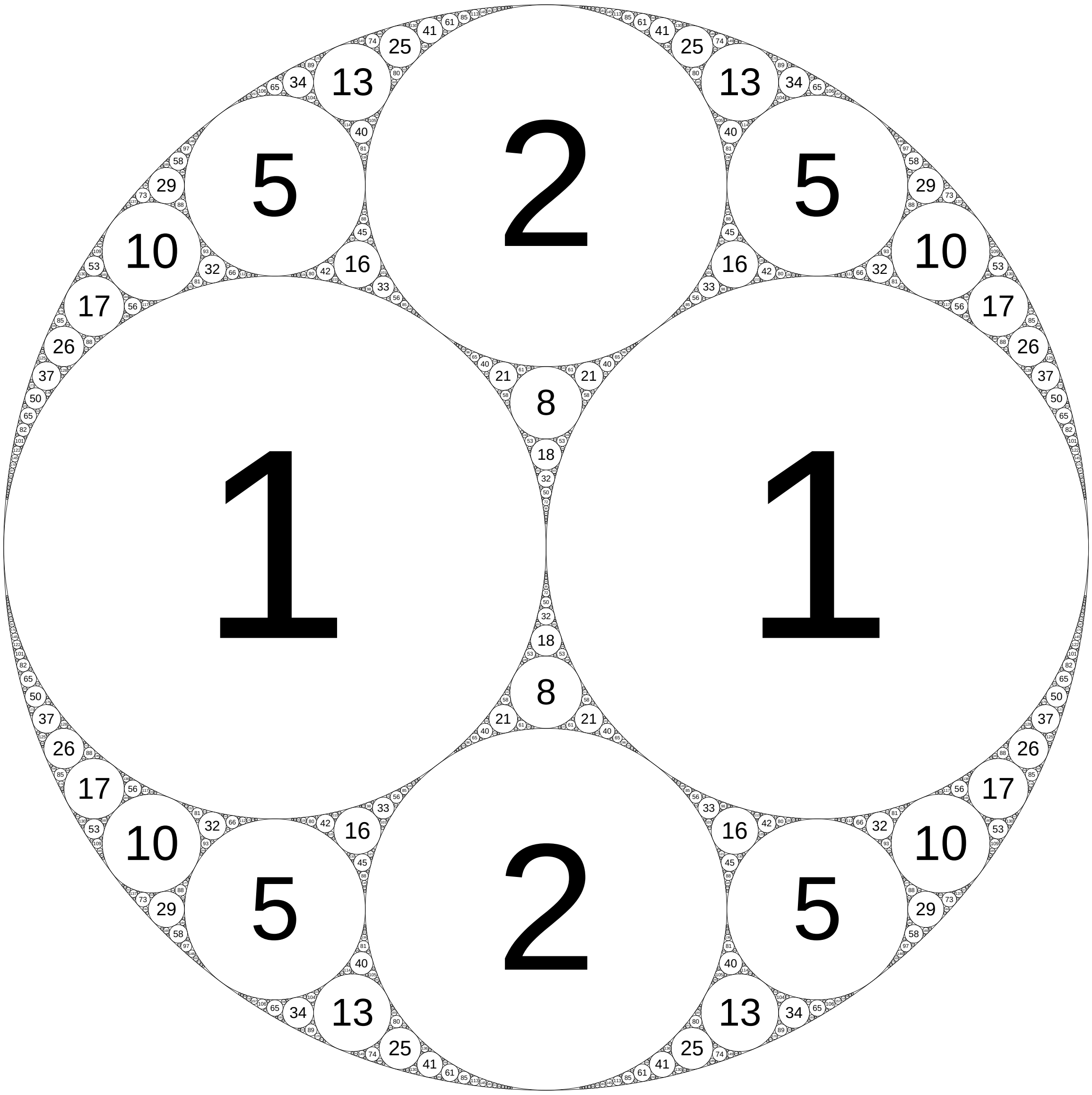

Figure 6: A spherical Apollonian packing

The Apollonian construction works also in higher dimensions, but gives sphere packings only in dimensions two and three; in dimensions four and higher we do not get proper packings; after several steps the spheres overlap; see [6]. However "Apollonian sphere ensembles" continue to exist in all dimensions as collections of Descartes configurations; see [17].

There is a considerable amount of mathematics devoted to circle packings; Kenneth Stephenson's bibliography of circle-packing papers lists over 90 papers since 1990 [31]. For further relations of Apollonian packings and the relation of integer Apollonian circle packings to the integer Lorentz group $O(1,3,\mathbb{Z})$, see [1], [29], [15], [16], [17], and [18].

8. Conclusion

We have extended the Descartes Circle Theorem, well known for n-dimensional Euclidean space, to n-dimensional spherical and hyperbolic spaces. We presented matrix generalizations of the Descartes Circle Theorem, which characterize Descartes configurations in all three geometries and require for their formulation the use of a particular coordinate system in each of these geometries. Mauldon [23] generalized the Soddy-Gossett Theorem in all three geometries to apply to sets of $n+2$ equally inclined spheres, as measured by an inclination parameter γ, with $\gamma = -1$ for touching spheres; our matrix theorems can be extended to the case of arbitrary γ as well.

All these theorems have one-dimensional analogues. For the Euclidean case in one dimension a "circle" consists of two points bounding an interval, and two "circles" are tangent if they have one point in common. The one-dimensional Descartes form is

$$\mathbf{Q}_1 := I_3 - \mathbf{1}_3\mathbf{1}_3^T = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}. \tag{8.2}$$

A *one-dimensional Euclidean Descartes configuration* consists of two touching intervals, and a third "interval" that is the complement of their union, so that the three intervals cover the line $\mathbb{R}$. Call the third "interval" the *infinite interval,* and define its "length" to be the negative of the length of its complement, which is the union of the first two intervals. The radius is half the "length". The radii r_1, r_2, r_3 of the three intervals then satisfy

$$r_1 + r_2 + r_3 = 0,$$

which is equivalent to the Descartes relation

$$Q_1(\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}) = -\frac{2}{r_1 r_2} - \frac{2}{r_1 r_3} - \frac{2}{r_2 r_3} = 0. \tag{8.3}$$

The value of curvature×center of the infinite interval is defined to be the curvature×center of the finite interval obtained by reflection sending $x \to 1/x$. This describes a positively oriented Descartes configuration; a negatively oriented one is obtained by reversing all signs. One can now define a 3×3 augmented matrix $\mathbf{W}$ exactly as in the Augmented Euclidean Descartes Theorem, and one finds that

$$\mathbf{W}^T \mathbf{Q}_1 \mathbf{W} = \begin{bmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{8.4}$$

Conversely, every solution $\mathbf{W}$ to (8.4) corresponds to a one-dimensional Descartes configuration. There is even a notion of Apollonian packing in dimension $n = 1$, but it consists of a single Descartes configuration because only one circle is tangent to a pair of tangent one-dimensional circles. That is, the Descartes equation (8.3) is linear in each curvature variable $a_i = 1/r_i$ separately, instead of quadratic, hence there is no Descartes reflection operation that generates new circles to add to the Apollonian packing. Finally, there are one-dimensional spherical and hyperbolic analogues of these results, defined via (7.1) (taking $n = 1$), which can be established by stereographic projection.

The main results in this paper are theorems in *inversive geometry*, as described in [35], [2], and [27]. Inversive geometry preserves spheres and their incidences, and consists of the study of geometric properties preserved by the group Möb(n) of conformal transformations of the space $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\} \approx S^n$. The set of inwardly oriented Descartes configurations form a single orbit under the action of the conformal group, and this group appears in our results as a subgroup of index 2 in the (real) automorphism group

$$Aut(Q_n) := \{\mathbf{N} \in M_{n+2,n+2}(\mathbb{R}) \ : \ \mathbf{N}^T \mathbf{Q}_n \mathbf{N} = \mathbf{Q}_n\}$$

of the Descartes quadratic form Q_n, which is a real Lie group that is isomorphic to $O(1, n+1)$; see [35, Corollary, p. 390] for the isomorphism. The three generalized Descartes theorems given here are invariant under the action of $Aut(Q_n)$. One can ask the question: Is there a "natural" conformal geometric characterization of the global coordinate systems (curvature-center coordinates) used in the generalized Descartes Circle Theorems presented here?

We summarize our results in the following verse, whose first stanza is taken from Soddy [28], to be read in the Queen's English.

The Complex Kiss Precise

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

Yet more is true: if all four discs
Are sited in the complex plane,
Then centers over radii
Obey the self-same rule again.

Suppose the circles now appear
Upon the surface of a sphere.
Then if by "bend" we mean to say
Cotan of radius, no more,
Then square of sum of "bends" becomes
Two times the sum of squares, plus four.

Now in the hyperbolic plane,
We try to make it work again.
It turns out now by "bend" is meant
The hyperbolic cotangent.
And if we square the sum of those,
Twice sum of squares, less four, it goes.

And more such wonders can be found
In n dimensions, if allowed.
Rene Descartes would have been proud.

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