Hilbert Spaces of Entire Functions and Dirichlet L-Functions

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Summary. We describe connections between the de Branges theory of Hilbert spaces of entire functions and the Riemann hypothesis for Dirichlet L-functions. Assuming the Riemann hypothesis, for each L-function there exists a particular de Branges space with interesting properties, and conversely. These de Branges spaces are shown to have associated "Hilbert-Polya" operators of a classical kind.

1 Introduction

The object of this paper is to formulate a connection between Hilbert spaces of entire functions and the Riemann hypothesis for various L-functions. Louis de Branges has long advocated the applicability of his theory of Hilbert spaces of entire functions to the Riemann hypothesis. He has proposed several approaches to the problem, including [11], [12]. The approach we present here is different from these. We associate to the Riemann zeta function the entire function

$$E_{\chi_0}(z) := \xi(\frac{1}{2} - iz) + \xi'(\frac{1}{2} - iz)$$

in which $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ is the Riemann ξ-function, and ξ' denotes its derivative with respect to the s-variable. More generally, to each Dirichlet L-function with a primitive character χ we associate the entire function

$$E_{\chi}(z) := \xi_{\chi}(\frac{1}{2} - iz) + \xi_{\chi}^{'}(\frac{1}{2} - iz),$$

in which $\xi_{\chi}(s)$ denotes the Dirichlet L-function completed with its archimedean factors, multiplied by a certain constant of modulus one which makes $\xi(s)$ real on the critical line $\Re(s) = \frac{1}{2}$. Our main observation is that for each χ, $E_{\chi}(z)$ is the structure function of a de Branges space $\mathcal{H}(E_{\chi}(z))$ if and only if the Riemann hypothesis holds for $L(s,\chi)$. Furthermore it gives a strict de Branges space (defined below) if and only if the Riemann hypothesis holds for $L(s,\chi)$

and all its non-trivial zeros are simple zeros. Thus, assuming the Riemann hypothesis holds, these associated de Branges spaces exist, and we then explore what de Branges's theory implies about such spaces.

The de Branges theory associates to each de Branges space an integral transform which we term here the *de Branges transform.* For the spaces above this transform produces a "Hilbert-Polya" differential operator together with self-adjoint boundary conditions that give an eigenvalue interpretation of the zeros of these L-functions. The Riemann hypothesis is interpretable as a positivity property of the coefficient functions of this "Hilbert-Polya" operator. This allows the possibility of approaching the Riemann hypothesis by finding a direct construction of this operator.

The approach described here differs from those taken in de Branges [11], [12]. These papers present certain general theorems that conclude that any de Branges space that satisfy their hypotheses necessarily has a structure function $E(z)$ that has all its zeros on the horizontal line $\Im(z) = -\frac{1}{2}$. His hope was that these results might apply to the de Branges space $\mathcal{H}(E)$ with structure function $E(z) = \xi(1 - iz)$, where the conclusions of his theorems would yield the Riemann hypothesis. However Conrey and Li [13] later showed that the de Branges space with $E(z) = \xi(1-iz)$ fails to satisfy the hypotheses of these general theorems.

This paper presents one theorem and sketches consequences of it. An expanded version of this paper, with additional results, is in preparation [20]. In particular the approach extends to automorphic L-functions, i.e. principal L-functions for GL_n. The research in this paper was done while the author worked at AT&T Labs.

2 Hilbert Spaces of Entire Functions

We give a brief review of the de Branges theory of Hilbert spaces of entire functions. This review formulates some of de Branges's results in [10] into an operator-theoretic language, using the terminology of canonical differential systems as in Remling [22] or Sakhnovich [23], rather than in terms of integral equations as in de Branges's formulation. In places it makes some simplifying assumptions, and for technically precise statements of the results, see [10] as indicated. The complex variable used is $z = x + iy$ and x, y always denote real variables.

A *structure function* or a *de Branges function* $E(z)$ is an entire function having the property that

$$|E(z)| > |E(\bar{z})| \text{ when } \Im(z) > 0. \tag{1}$$

This property implies that $E(z)$ has no zeros in the upper half plane. We say it is a *strict de Branges function* if $E(z)$ has no zeros on the real axis. This class of functions has a long history, see Chapter VII of Levin [21], who uses the term Hermite-Biehler functions, and M. Krein [17, Theorems 9 and 11]. The de Branges theory makes use of a decomposition of the structure function $E(z)$ into entire functions that are pure real and pure imaginary on the real axis, namely

$$E(z) = A(z) - iB(z),$$

given by

$$A(z) = \frac{1}{2}\left(E(z) + \overline{E(\bar{z})}\right),$$

$$B(z) = \frac{1}{2i}\left(E(z) - \overline{E(\bar{z})}\right).$$

A crucial observation of de Branges is that the condition (1) implies: *For a de Branges function $E(z) = A(z) - iB(z)$, the functions $A(z)$ and $B(z)$ have only real zeros, and these zeros interlace. If $E(z)$ is a strict de Branges function, then all the zeros are simple zeros* (de Branges [7, Lemma 5]).

One associates to any de Branges function $E(z)$ a de Branges Hilbert space $\mathcal{H}(E)$ of entire functions, as follows. The Hilbert space scalar product is

$$\langle f, g\rangle_E = \int_{-\infty}^{\infty} \frac{f(x)\overline{g(x)}}{|E(x)|^2} dx. \tag{2}$$

(conjugate-linear in the second factor). The entire functions $f(z)$ that belong to the space are those which have a finite norm $||f||_E$ and whose growth with respect to $E(z)$ is controlled in the upper half-plane $\mathbb{C}^+ := \{z : \Im(z) > 0\}$ and in the lower half-plane. We require that $\frac{f(z)}{E(z)}$ and $\frac{\overline{f(\bar{z})}}{E(z)}$ be of bounded type and nonpositive mean type in $\mathbb{C}^+$. A function $h(z)$ is of *bounded type* if it can be written as a quotient of two bounded analytic functions in $\mathbb{C}^+$ and it is of nonpositive mean type if it grows no faster than $e^{\epsilon y}$ for each $\epsilon > 0$ as $y \to \infty$ on the positive imaginary axis $\{iy \ : y > 0\}$. One can show there always exist such functions, so the space $\mathcal{H}(E)$ is always nontrivial. There are examples where it is a finite-dimensional Hilbert space, but in the cases we will consider here it will always be infinite-dimensional.

A de Branges space $\mathcal{H}(E)$ is a *reproducing kernel Hilbert space*, with a kernel function $K_E(w, z)$ having the property that for each $f(z) \in \mathcal{H}(E)$, there holds

$$f(w) = \langle f(z), K(w, z)\rangle_E \text{ for all } w \in \mathbb{C}.$$

That is, evaluation of a function in $\mathcal{H}(E)$ at the point w is a continuous linear functional on $\mathcal{H}(E)$ and is therefore represented by a scalar product with some function $g_w(z) \in \mathcal{H}(E)$ and we have $K(w, z) := g_w(z)$. (Only values of z on the real axis are used in computing the scalar product.) The reproducing kernel is

$$K(w, z) = \frac{\overline{A(w)}B(z) - A(z)\overline{B(w)}}{\pi(z - \bar{w})}.$$

If we consider the de Branges space to be determined by its reproducing kernel, then there is some freedom in the choice of de Branges functions $E(z)$.

For $k \in \mathbb{R}^+$ the function $E_k(z) = kA(z) - \frac{i}{k}B(z)$ gives the same reproducing kernel, and for $0 \le \theta < 2\pi$ so does $E_\theta(z) = e^{i\theta}E(z) := A_\theta(z) - iB_\theta(z)$. This gives an $SL(2,\mathbb{R})$-action on structure functions that preserve the reproducing kernel. In the case of strict de Branges functions, we can remove this ambiguity by requiring that $E(0) = 1$, and $E'(0) \in i\mathbb{R}$, i.e. $A(0) = 1$, $A'(0) = 0$, and $B(0) = 0$. It proves convenient to only partially remove this ambiguity and call a structure function *normalized* if $E(0) = 1$, with no condition imposed on $A'(0)$.

There is an additional degree of freedom in that one can remove zeros on the real axis from the structure function without changing the de Branges space in an essential way. Indeed if $E(z)$ has a zero on the real axis, at $z = x_0$, say, then the form of the Hilbert space norm in (2) shows that every function in $\mathcal{H}(E(z)$ must have a zero at the same location, so we can divide all functions in the space by $z - x_0$ and obtain a new Hilbert space of entire functions having structure function $\mathcal{H}(\frac{E(z)}{z-x_0})$, preserving the Hilbert space inner product. The reproducing kernel changes, with the new reproducing kernel obtained from the old by dividing by $(z-x_0)(\bar{w}-x_0)$. In this way we can in principle reduce to the case of a *strict de Branges space*, one where the structure function $E(z)$ is a strict de Branges function.

There is an abstract theory of de Branges spaces. An *(abstract) de Branges space* is a nonzero Hilbert space $\mathcal{H}$ whose elements are entire functions, such that $\mathcal{H}(E)$ satisfies the axioms:

(H1) Whenever $f(z)$ is in the space and has a non-real zero z_0 then $g(z) := f(z)\frac{z-\bar{z}_0}{z-z_0}$ is in the space and has the same norm as $f(z)$.

(H2) For every nonreal number $w \in \mathbb{C}$, the linear functional on $\mathcal{H}$ defined by $f(z) \mapsto f(w)$ is continuous.

(H3) If $f(z) \in \mathcal{H}$ then $f^*(z) := \overline{f(\bar{z})}$ belongs to $\mathcal{H}$ and has the same norm as $f(z)$.

Two abstract de Branges spaces are *isomorphic* if there is an isometry between them that preserves properties (H1)–(H3). Then any (abstract) de Branges space is isomorphic to some de Branges space $\mathcal{H}(E)$ ([10, Theorem 23]). Each such space is isomorphic to a de Branges space $\mathcal{H}(E(z))$ for which $E(z)$ is a strict de Branges function that is normalized, i.e. $E(0) = 1$.

A de Branges space $\mathcal{H}(E)$ comes with an unbounded operator $(M_z, \mathcal{D}(M_z))$ in which M_z is "multiplication by z" and its domain is

$$\mathcal{D}(M_z) = \{f(z) \in \mathcal{H}(E) : \ zf(z) \in \mathcal{H}(E)\}.$$

This domain is either dense in $\mathcal{H}(E)$ (the "dense" case) or has closure of codimension 1 in $\mathcal{H}(E)$ (the "non-dense" case). We are interested here only in the "dense" case; the property of being "dense" can be read off from properties of $E(z)$ on the real axis. The operator M_z is symmetric and closed (i.e. its graph is closed in $\mathcal{H}(E) \oplus \mathcal{H}(E)$). In the "dense" case the operator has deficiency indices $(1,1)$, and so has a family of self-adjoint extensions parametrized by the group $U(1) = \{e^{i\theta} : 0 \le \theta < 2\pi\}$.

One interpretation of the de Branges theory is that it supplies a "canonical model" for a particular subset of closed symmetric operators with deficiency indices $(1,1)$. This class of operators contains the class of "entire operators" introduced by M. Krein (see [15]). The "canonical model" allows various properties of the operator M_z to be read off by inspection.

First, from the normalized structure function $E(z)$ we obtain a description of all self-adjoint extensions of the operator M_z. These extensions all have discrete, simple spectra, as we describe below.

Second, the "canonical model" exhibits a complete (and unique) chain of invariant subspaces for the operator, which consists of a nested family of subspaces which are themselves de Branges spaces. Associated to this chain of invariant subspaces is an integral transform somewhat like the Fourier transform, which we will call here the *de Branges transform*, with a corresponding inverse de Branges transform. The de Branges transform gives an isometry of a de Branges transform Hilbert space $\mathcal{K}(M)$ (defined below) onto the Hilbert space $\mathcal{H}(E)$, with inverse transform going the opposite direction. (See [10, Theorem 44].) The inverse de Branges transform takes the multiplication operator M_z is to a (generalized) linear differential operator [1] D_t acting on a system of 1×2 vectors of functions, whose dependent variable t runs over an interval of the real line $\mathbb{R}$, which can be taken to be $(0, b]$ with b finite, parametrizing the chain of invariant subspaces.

A major theorem of de Branges used in the construction of this transform is the *total ordering theorem* which says if two de Branges spaces $\mathcal{H}(E_1)$ and $\mathcal{H}(E_2)$ are isometrically embedded in a de Branges space $\mathcal{H}(E)$ (i.e. their reproducing kernels are obtained by restriction) then either $\mathcal{H}(E_1) \subset \mathcal{H}(E_2)$ or $\mathcal{H}(E_2) \subset \mathcal{H}(E_1)$ ([10, Theorem 35]). The order type of the resulting chain of subspaces can be either discrete, where the dimension jumps by 1 at some points, or continuous, or some mixture of discrete and continuous. In any case we can embed such an order type in an interval, and write the family as $\mathcal{H}(E_t)$ where $0 \le t \le b$, say, with $\mathcal{H}(E_{t_1}) \subset \mathcal{H}(E_{t_2})$ if $t_1 < t_2$. In a parametrization by an interval, some values of t correspond to members of the chain, and other values of t do not, being filler values to permit parametrization of the chain by an interval on the real axis. For this de Branges introduces notions of "regular" and "singular" values of a parametrization ([10, p. 136]) in which a "regular" value of t apparently corresponds to belonging to the chain. For the discussion here we shall suppose that we are dealing with a pure continuous case, and also that it is legitimate to differentiate and obtain the canonical system (3) below.

[1] More accurately, the de Branges theory uses a 2×2 matrix integral equation in the parameter t. If the integral equation could be differentiated, then one obtains the canonical differential system (3) (4) given below, see for example Dym [14, p. 396]. The matrix $M(t)$ in (4) is related to de Branges's symmetric 2×2 matrix $m(t)$ with entries $(\alpha(t), \beta(t), \gamma(t))$ given in [10, Theorem 38] by $M(t) = \frac{d}{dt} m(t)$. The canonical differential equation formalism works more generally by allowing $M(t)dt = dm(t)$ to be a 2×2 matrix-valued measure, see Remling [22].

For a given normalized strict structure function $E(z)$ it is possible to find a family of normalized strict structure functions of the de Branges chain $\mathcal{H}(E_t)$, denoted by $E_t(z) := A(t,z) - iB(t,z)$ parametrized with $0 < t \le b$, that satisfy [2] a "canonical differential system" (see [23]) for each $z \in \mathbb{C}$,

$$\frac{d}{dt}\begin{bmatrix} A(t) \\ B(t) \end{bmatrix} = zJM(t)\begin{bmatrix} A(t) \\ B(t) \end{bmatrix}, \tag{3}$$

in which

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad M(t) = \begin{bmatrix} \tilde{\alpha}(t) & \tilde{\beta}(t) \\ \tilde{\beta}(t) & \tilde{\gamma}(t) \end{bmatrix}. \tag{4}$$

such that at the right endpoint $E_b(z) = E(z)$ for all $z \in \mathbb{C}$, and at the left endpoint satisfies

$$\lim_{t \to 0^+} A(t,z) = 1 \quad \text{and} \quad \lim_{t \to 0^+} B(t,z) = 0, \tag{5}$$

see [10, Theorem 40]. A principal feature is that for (almost all) t the matrix $M(t)$ is real, symmetric and positive semidefinite. The right endpoint is "regular" in a sense defined by de Branges [10, p. 136]. The Hilbert space $\mathcal{K}(M)$ consists of vector-valued functions $[A(t), B(t)]^T$ on an interval, say $[0,b]$, with norm

$$||(f(t), g(t))||_M^2 = \int_0^b [f(t),\ g(t)]M(t)\begin{bmatrix} \overline{f(t)} \\ \overline{g(t)} \end{bmatrix} dt.$$

The *de Branges transform* $\mathcal{T} : \mathcal{K}(M) \to \mathcal{H}(E)$ is:

$$V(t) := (f(t), g(t)) \longmapsto \mathcal{T}(V)(z) := \frac{1}{\pi}\int_0^b [f(t)\ g(t)]M(t)\begin{bmatrix} \overline{A(t,\bar{z})} \\ \overline{B(t,\bar{z})} \end{bmatrix} dt, \tag{6}$$

see [10, Theorems 43 and 44]. Note that $\overline{A(t,\bar{z})} = A(t,z)$ and $\overline{B(t,\bar{z})} = B(t,z)$.

The *de Branges direct theorem* asserts that: Any canonical differential system (3) with "initial condition" (5) with the property that the 2×2 matrix function $M(t)$ is measurable and positive semi-definite symmetric for all $t \in (0,b]$, and which is integrable over the interval, has solutions $\{(A(t,z), B(t,z)) : 0 < t < b, \text{ all } z \in \mathbb{C}\}$ such that each $E(t,z) = A(t,z) - iB(t,z)$ with t constant and $z \in \mathbb{C}$ is a strict, normalized de Branges structure function. This is proved ([10, Theorem 41]) provided some growth conditions are imposed on the coefficients of $M(t)$. These growth conditions characterize those (strict, normalized) de Branges functions belonging to the Polya class. The Polya class consists of those de Branges functions whose modulus is nondecreasing on each vertical line in the upper half-plane, see [10, Sect. 7]. The de Branges functions which are entire functions of exponential type are exactly the class for which the canonical differential equation

[2] A a differential equation of the general form (3), (4) for a fixed z is called a canonical differential equation. A "canonical differential system" is a family of such equations where z varies.

(3) is regular at its left endpoint for each $z \in \mathbb{C}$. For de Branges functions of faster growth this endpoint is singular.

The *de Branges inverse theorem* asserts the following: For any strict normalized de Branges structure function $E(z) = A(z) - iB(z)$ in the Polya class, there exists a set of real coefficient functions $(\tilde{\alpha}(t), \tilde{\beta}(t), \tilde{\gamma}(t))$ such that the real matrix $M(t)$ is positive semidefinite for almost all t in a finite half-open interval $(0, b]$ and the solutions to the canonical differential system (3) at the left-endpoint satisfy (5) and at the right endpoint $t = b$ have

$$A(b,z) = A(z) \quad \text{and} \quad B(b,z) = B(z). \tag{7}$$

In addition the right endpoint is a "regular" value in de Branges's sense. The precise assertion (see [10, Theorem 40]), is an extremely strong inverse spectral theorem which subsumes many known inverse spectral theorems, see Krein [18] and Remling [22, Theorem 7.3]. The canonical differential system (3) can be made essentially unique by reparametrizing it to have $Tr(M(t)) \equiv 1$ almost everywhere. With this reparametrization the interval may become infinite to the left, with an endpoint at $-\infty$.

We now describe the self-adjoint extensions of the operator M_z, assuming that we are in the "dense" case. The structure function $E(z)$ specifies two particular self-adjoint extensions associated to $A(z)$ and $B(z)$, respectively. Recall that $A(z)$ and $B(z)$ have real zeros and these zeros interlace. The self-adjoint extension M_z associated to $A(z)$, denoted $\tilde{M}_z$ or $M_z(A)$, has pure discrete simple spectrum located at those zeros of $A(z)$ that have multiplicity exceeding that of $B(z)$ at the same point, and for each such zero ρ an eigenfunction $f_\rho(z) = \frac{A(z)}{(z-\rho)^j}$, where $A(z)$ has a zero of order j at $z = \rho$. The domain $\mathcal{D}(M_z(A)) = \mathcal{D}(M_z) \oplus \mathbb{C}[f_{\rho_0}]$ for any single function f_{ρ_0}. We obtain all self-adjoint extensions of M_z by considering instead $\{A_\theta(z) : 0 \le \theta < 2\pi\}$, obtained from $E_\theta(z) = e^{i\theta}E(z)$. Now suppose that $E(z)$ is a strict de Branges function, in which case $j = 1$ always, and the functions $\{f_\rho(z) = \frac{A(z)}{z-\rho} : A(\rho) = 0\}$ form an orthogonal basis of $\mathcal{H}(E(z))$ ([10, Theorem 22]). This orthogonal basis gives rise to a "summation formula" expressing the Hilbert space norm of an arbitrary function $f(z) \in \mathcal{H}(E)$,

$$||f(z)||_E^2 = \int_{-\infty}^{\infty} \left| \frac{f(x)}{E(x)} \right|^2 dx = \sum_\rho \frac{\pi}{\phi'(\rho)} \left| \frac{f(\rho)}{E(\rho)} \right|^2, \tag{8}$$

in which the phase function $\phi(t)$ is given by $E(x) = e^{-i\phi(x)}E_0(x)$, with $E_0(x)$ real-valued.

The operator D_t on the de Branges transform space $\mathcal{K}(M)$ having deficiency indices (1, 1) can be formally written as

$$D_t := M(t)^{-1}J^{-1}\frac{d}{dt} = M(t)^{-1}\begin{bmatrix} 0 & \frac{d}{dt} \\ -\frac{d}{dt} & 0 \end{bmatrix},$$

under the extra assumption that $M(t)$ is invertible everywhere. Theorem 45 of [10] gives a description of the range of the symmetric operator D_t. In the de Branges transform space $\mathcal{K}(M)$ is a corresponding orthogonal basis of eigenfunctions $V_\rho(t) := [A_\rho(t), B_\rho(t)]^T$ of $\tilde{D}_t$ defined indirectly by $\mathcal{T}(V_\rho) = \frac{A(z)}{z-\rho}$, where $A(\rho) = 0$. Expanding members of the Hilbert space $\mathcal{K}(M)$ in this basis is an "eigenfunction expansion" associated with the de Branges theory.

There is further content to the de Branges theory not covered here. Some of this is to be discussed in [20].

3 de Branges Spaces associated to Dirichlet L-Functions

Associated to the Riemann zeta function is the Riemann ξ-function, given by

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s).$$

It is an entire function, real on the real axis and on the critical line $\Re(s) = \frac{1}{2}$, satisfies the functional equation $\xi(s) = \xi(1-s)$ and its zeros are exactly the nontrivial zeros of the Riemann zeta function, those in the critical strip $0 < \Re(s) < 1$. We write $\xi(s) := \xi_{\chi_0}(s)$ where χ_0 is the identity Dirichlet character.

We can similarly associate to each Dirichlet L-function $L(s, \chi)$ with a primitive character χ of conductor q a corresponding ξ-function $\xi_\chi(s)$. We define the completed L-function

$$\hat{L}_\chi(s) := (\frac{\pi}{q})^{-\frac{s+k}{2}}\Gamma(\frac{s+k}{2})L(s, \chi),$$

in which $k = 0$ if $\chi(-1) = 1$ and $k = 1$ if $\chi(-1) = -1$. This is an entire function which satisfies the functional equation

$$\hat{L}_\chi(s) = \epsilon(\chi)\hat{L}_{\bar{\chi}}(1-s),$$

in which $\bar{\chi}$ is the complex conjugate of the character χ and $\epsilon(\chi) := i^k \frac{\tau(\chi)}{q^{\frac{1}{2}}}$ is a constant of absolute value 1, see for example Davenport [6, Chap. 9]. The fact that $\hat{L}_\chi(s)$ transforms under complex conjugation as $\hat{L}_\chi(s) = \overline{\hat{L}_\chi(\bar{s})}$ together with the functional equation implies that $\hat{L}_\chi(s)$ has constant modulus $\pmod{\pi}$ on the critical line, i.e. there is a constant $e^{i\theta}$ such that $\hat{L}_\chi(\frac{1}{2}+it) = e^{i\theta}g_\chi(t)$ for some continuous real-valued function $g_\chi(t)$. There remains an ambiguity of a sign in the choice of $e^{i\theta}$ which is removed by requiring that $\xi_\chi(s)$ be positive on the critical line in the upper half-plane just above $s = \frac{1}{2}$. We then define the modified function

$$\xi_\chi(s) := e^{-i\theta}\hat{L}(s, \chi), \tag{9}$$

which is real-valued on the critical line, and satisfies the functional equation

$$\xi_\chi(s) = \xi_{\bar{\chi}}(1-s).$$

The zeros of the function $\xi_\chi(s)$ are exactly the non-trivial zeros of the Dirichlet L-function in the critical strip, counting multiplicities.

In the following result the notation $f'(s)$ denotes the derivative with respect to the s-variable, following standard usage in number theory.

Theorem 1. *For each primitive Dirichlet character χ including the trivial character χ_0, set $E_\chi(z) = A_\chi(z) - iB_\chi(z)$ with*

$$A_\chi(z) = \xi_\chi(\frac{1}{2} - iz), \quad B_\chi(z) = i\ \xi_\chi^{'}(\frac{1}{2} - iz).$$

Then these functions are real on the real axis, and the following holds.

(i) $E_\chi(z)$ is a de Branges function if and only if the Riemann hypothesis holds for $L(s,\chi)$.

(ii) $E_\chi(z)$ is a strict de Branges function if and only if the Riemann hypothesis holds for $L(s,\chi)$ and all its nontrivial zeros are simple zeros.

Proof. The function $A_\chi(z)$ is real on the real axis since $\xi_\chi(s)$ is real on the critical line. Then $B_\chi(z)$ inherits this property under differentiation.

(i) If $E_\chi(z)$ is a de Branges function then by de Branges' lemma both $A_\chi(z)$ and $B_\chi(z)$ have only real zeros, which interlace. The reality of zeros of $A_\chi(z)$ is the Riemann hypothesis for $\xi_\chi(s)$.

Now assume that the Riemann hypothesis holds for $\xi_\chi(s)$. Then $A_\chi(z)$ has real zeros. Since $A_\chi(z)$ is an entire function of order 1 (and infinite type), Laguerre's theorem ([16, Theorem 5.7]) applies to show that $B_\chi(z) = -\frac{d}{dz}A_\chi(z)$ has real zeros and they interlace with those of $A_\chi(z)$.

We show that the Riemann hypothesis for $L(s,\chi)$ implies that

$$\Re\left(\frac{\xi_\chi'(s)}{\xi_\chi(s)}\right) > 0 \quad \text{for} \quad \Re(s) > \frac{1}{2}. \tag{10}$$

This fact is well known for the Riemann ξ-function, see Lagarias [19]. Starting from the Hadamard product factorization

$$\xi_\chi(s) = e^{A+Bs}\prod_\rho \left(1 - \frac{s}{\rho}\right)e^{\frac{s}{\rho}},$$

and the logarithmic derivative we obtain

$$g_\chi(s) := \frac{\xi_\chi'(s)}{\xi_\chi(s)} = B + \sum_\rho(\frac{1}{s-\rho} + \frac{1}{\rho}) = (B + \sum_\rho \frac{1}{\rho}) + \sum_\rho(\frac{1}{s-\rho}),$$

where the sum is not absolutely convergent in the last equality and must be viewed as taken over $|\rho| < T$ and then letting $T \to \infty$. Taking the real part of this sum, one can check term by term that $\Re(\frac{1}{s-\rho}) > 0$ whenever $\Re(s) > \Re(\rho)$. We also have $\Re(B + \sum_\rho \frac{1}{\rho}) = 0$, which is deduced from the functional equation $g_\chi(s) = -g_\chi(1-s)$. By hypothesis $\Re(\rho) \le \frac{1}{2}$, and (10) follows.

Now (10) gives $\Im\left(\frac{B_\chi(z)}{A_\chi(z)}\right) = \Im\left(i\frac{\xi'(\frac{1}{2}-iz)}{\xi(\frac{1}{2}-iz)}\right) > 0$ when $\Im(z) > 0$. Then

$$|\frac{B_\chi(z)}{A_\chi(z)} + i| > |\frac{B_\chi(z)}{A_\chi(z)} - i| = |\frac{B_\chi(\bar{z})}{A_\chi(\bar{z})} + i|$$

under the same condition. Thus for $\Im(z) > 0$,

$$\begin{aligned} |E_\chi(z)| &= |-iA_\chi(z)||i + \frac{B_\chi(z)}{A_\chi(z)}| \\ &> |-iA_\chi(\bar{z})||i + \frac{B_\chi(\bar{z})}{A_\chi(\bar{z})}| = |E_\chi(\bar{z})|, \end{aligned}$$

so that $E_\chi(z)$ is a de Branges function.

(ii) This is straightforward. $\square$

Assuming the Riemann hypothesis, it can be shown that the functions $E_\chi(z)$ belong to the Polya class, and that the associated de Branges spaces fall in the "dense" case. We now consider the consequences of having a strict de Branges space, where we can make use of the de Branges transform.

First, the self-adjoint extension $\tilde{M}_z$ of M_z corresponding to the function $A_\chi(z) = \xi_\chi(\frac{1}{2} - iz)$ has a complete orthogonal set of eigenfunctions given by

$$f_\rho(z) := \frac{\xi_\chi(\frac{1}{2} - iz)}{z - \gamma}, \text{ with } \rho = \frac{1}{2} + i\gamma,$$

where ρ runs over all the zeros (assumed simple) of $\xi_\chi(s)$.

Second, the de Branges summation formula applied to this set of orthogonal eigenfunctions gives for all $f(z) \in \mathcal{H}(E_\chi)$, putting $F(\frac{1}{2} - iz) = f(z)$,

$$||f(z)||^2_{E_\chi} = \pi \sum_{\{\rho:\xi_\chi(\rho)=0\}} \frac{|F(\rho)|^2}{|\xi'_\chi(\rho)|^2},$$

The right side of this formula resembles the spectral side of the "explicit formula" of prime number theory. Viewed this way, the positivity of the Hilbert space norm appears to encode "Weil positivity," compare [1, Sec. 4].

Third, the associated de Branges transform gives an encoding of the Riemann hypothesis plus simplicity of the zeros as a positivity property. To show a given $E_\chi(z)$ is a strict de Branges function, it suffices to show that a corresponding normalized function $E^N_\chi(z) = k_0 E(z)$ (with constant k_0 chosen so that $E^N_\chi(1) = 1$) is a normalized strict de Branges function. The de Branges inverse theorem then says there exists data

$$M(t) = \begin{bmatrix} \tilde{\alpha}(t) & \tilde{\beta}(t) \\ \tilde{\beta}(t) & \tilde{\gamma}(t) \end{bmatrix}.$$

which is real, symmetric and positive semidefinite, and whose canonical differential system, suitably parametrized, produces on the interval $(0, b]$ the function

$$E_{\chi}^{N}(z)=A_{\chi}^{N}(b,z)-iB_{\chi}^{N}(b,z)$$

at its right endpoint. (The $Tr(M)\equiv 1$ reparametrization would necessarily be on an infinite interval $(-\infty,b]$ in this case.) If these coefficient functions are found, then the de Branges direct theorem certifies that $E_{\chi}^{N}(z)$ is a strict de Branges function, so that $E_{\chi}(z)$ is as well, whence $A(b,z)=\xi_{\chi}(\frac{1}{2}-iz)$ has real simple zeros. Thus the Riemann hypothesis plus simple zeros is encoded as the positive semidefiniteness property of the coefficient matrix $M(t)$ on $(0,b]$. It seems reasonable to expect that for these particular de Branges spaces the matrix $M(t)$ will always be positive definite. We note that the fact that $E_{\chi}^{N}(z)$ is not a bandlimited function implies that the canonical differential system for it will necessarily be singular at the left endpoint $t=0$, with $\gamma(t)\to\infty$ as $t\to 0$.

Fourth, the de Branges transform produces a “Hilbert-Polya” operator, by which we mean a self-adjoint differential operator on a Hilbert space whose eigenvalues encode the zeta zeros. We take the operator $\tilde{D}_t$ to be the self-adjoint extension of the differential operator D_t on $\mathcal{K}(M)$ that corresponds to the extension $\tilde{M}_z$ of the de Branges operator M_z under the de Branges transform. It is possible to describe this operator and its domain more concretely. There are particularly interesting forms for it in the case of a real primitive character χ, the self-dual case.

According to the de Branges theory there has been so far no loss of information. That is, if the Riemann hypothesis plus simple zeros holds, then the objects above all exist, if properly interpreted as integral equations rather than differential equations, and conversely. Some inferences on what the coefficient functions of $M(t)$ might look like for the Riemann zeta function case $\mathcal{H}(E_{\chi_0})$ can be obtained by analogy with those of certain Sonine spaces of entire functions, cf. de Branges [8], [9], Burnol [3], [4]. Burnol has also studied some other Hilbert spaces associated to the zeta function and Dirichlet L-functions [2] [5].

4 Conclusions

We have formulated the Riemann hypothesis for Dirichlet L-functions in terms of the existence of particular de Branges spaces. This provides a possible approach to the Riemann hypothesis plus simplicity of the zeta zeros, namely to construct these hypothetical spaces directly in a way that certifies they are de Branges spaces with the correct structure function.

There are at least three ways to construct a de Branges space. The first way is to find a structure function $E(z)$ for the space, and directly prove $E(z)$ has the defining property (2). The second way is to obtain the de Branges transform data $\{M(t):0\leq t\leq b\}$, verify that each 2×2 matrix $M(t)$ is real and positive semi-definite symmetric, and integrable over the specified interval. The third way is to construct in some fashion a Hilbert space of entire functions and show directly that it satisfies the axioms (H1)–(H3), without obtaining either the structure function or the de Branges transform. This last approach can sometimes be taken using a weighted Mellin transform, as in de Branges [9] and Burnol [3]. In following the latter two approaches, an additional necessary task is to establish that the resulting de Branges space has the desired structure function $E(z)$.

The usefulness of this reformulation of the Riemann hypothesis will likely depend on whether information coming from number theory, either from arithmetical algebraic geometry, automorphic representations, or from some other source entirely, can be applied to show the existence of these particular (hypothetical) de Branges spaces.

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