Counting d-Step Paths in Extremal Dantzig Figures

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Abstract

The d-step conjecture asserts that every d-polytope P with $2d$ facets has a edge-path of at most d-steps between any two of its vertices. Klee and Walkup showed that to prove the d-step conjecture, it suffices to verify it for all Dantzig figures $(P, \mathbf{w}_1, \mathbf{w}_2)$, which are simple d-polytopes with $2d$ facets together with distinguished vertices $\mathbf{w}_1$ and $\mathbf{w}_2$ which have no common facet, and to only consider paths between $\mathbf{w}_1$ and $\mathbf{w}_2$. This paper counts the number of d-step paths between $\mathbf{w}_1$ and $\mathbf{w}_2$ for certain Dantzig figures $(P, \mathbf{w}_1, \mathbf{w}_2)$ which are extremal in the sense that P has the minimal and maximal vertices possible among such d-polytopes with $2d$ facets, which are $d^2 - d + 2$ vertices (lower bound theorem) and $2\binom{\lfloor \frac{3}{2}d - \frac{1}{2} \rfloor}{\lfloor \frac{d}{2} \rfloor}$ vertices (upper bound theorem), respectively. These Dantzig figures have exactly 2^{d-1} d-step paths.

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1. Introduction

The *(bounded) Hirsch conjecture* asserts that the diameter of the graph of any d-polytope having n facets is at most $n-d$. Let $\Delta(d,n)$ denote the maximal diameter of any (d,n)-polytope; the Hirsch conjecture asserts that

$$\Delta(d,n) \leq n-d \text{ for all } \quad n > d > 0 \ .$$

The *d-step conjecture* is a special case of the Hirsch conjecture and asserts that

$$\Delta(d,2d) = d \ \text{ for all } \quad d \ .$$

Although seemingly less general, the d-step conjecture is known to be equivalent to the Hirsch conjecture. Klee and Walkup [8, Theorem 2.8] showed that $\Delta(d,2d)$ is attained by some simple $(d,2d)$-polytope. They also showed that to prove the Hirsch conjecture it suffices to prove it for d-dimensional Dantzig figures. A d-dimensional Dantzig figure is a triple $(P, \mathbf{w}_1, \mathbf{w}_2)$ where P is a simple $(d,2d)$-polytope with two distinguished vertices $\mathbf{w}_1$ and $\mathbf{w}_2$ which are *antipodal* in the sense that they have disjoint sets of facets incident on them. Klee and Walkup showed that $\Delta(d,2d)$ equals the length of the shortest edge path between the distinguished vertices of some d-dimensional Dantzig figure. Let $\#(P, \mathbf{w}_1, \mathbf{w}_2)$ denote the number of edge-paths of length d between vertices $\mathbf{w}_1$ and $\mathbf{w}_2$ in the d-dimensional Dantzig figure $(P, \mathbf{w}_1, \mathbf{w}_2)$. Then the d-step conjecture and hence the Hirsch conjecture can be restated as:

Hirsch conjecture: *For every $d \geq 1$, and for every d-dimensional Dantzig figure $(P, \mathbf{w}_1, \mathbf{w}_2)$,*

$$\#(P, \mathbf{w}_1, \mathbf{w}_2) \geq 1 \ .$$

For the main results on and several equivalent versions of the Hirsch conjecture, see [8], [9], [10], [15]. Several natural generalizations of the Hirsch conjecture are known to be false. For example, the unbounded version of the Hirsch conjecture [9] as well as the monotone version [14] fail in dimension 4, while the generalization of the dualized Hirsch conjecture for triangulated spheres fails for 11-dimensional spheres [11]. These and other negative results detailed in [8] led to the general belief that the Hirsch conjecture is also false, and perhaps fails in dimension as low as 12, cf. [8, p. 733].

Lagarias, Prabhu and Reeds [10] recently studied the number of d-step paths between vertices of Dantzig figures. They observed that $\#(P, \mathbf{w}_1, \mathbf{w}_2) \leq d!$. (The d-cube shows this bound is sharp.) They presented extensive computational evidence, in dimensions up to 15, which suggested the truth of the following strong form of the d-step conjecture.

Strong d-Step Conjecture. *For every $d \geq 1$, and for every d-dimensional Dantzig figure $(P, \mathbf{w}_1, \mathbf{w}_2)$,*

$$\#(P, \mathbf{w}_1, \mathbf{w}_2) \geq 2^{d-1} \ .$$

This conjecture was proved for $d = 3$ in [10]. Holt and Klee [5] then proved it for $d = 4$, but showed that it is false in all higher dimensions, by constructing d-dimensional Dantzig figures for $d \geq 5$ with $\#(P, \mathbf{w}_1, \mathbf{w}_2) = \frac{3}{4} \cdot 2^{d-1}$.

It is interesting that the Holt-Klee result still leaves open the possibility that $\#(P, \mathbf{w}_1, \mathbf{w}_2) \geq O(2^d)$ as $d \to \infty$, although we see no convincing reason to suspect this is true. We believe that further investigation of the number and structure of d-step paths in d-dimensional Dantzig is warranted, before one concludes that the computational results of [10] were fortuitously misleading. In any eventuality, the results of [10] suggest that "most" Dantzig figures have many d-step paths.

In this paper we continue an investigation of the number and structure of d-step paths between antipodal vertices in d-dimensional Dantzig figures. Specifically, we count the number of d-step paths for certain Dantzig figures which are extremal in the sense of having the maximal or minimal number of vertices possible among $(d, 2d)$-polytopes.

The minimal number of vertices possible for a simple d-polytope having $2d$ facets is $d^2 - 2d + 2$. This is a special case of the Lower Bound Theorem proved by Barnette [1], who also showed that for $d \geq 4$, the set of polytopes attaining the bound are the truncation polytopes — that is, polytopes obtained from a d-simplex by successively truncating vertices, one at a time. There are exponentially many different combinatorial types of truncation polytopes, but in §3 we show that this class of polytopes includes a unique combinatorial type of Dantzig figure. We prove that this Dantzig figure has exactly 2^{d-1} d-step paths between its distinguished vertices. (Theorem 3.1).

The maximal number of vertices possible for any d-polytope having $2d$ facets is $2\binom{\lfloor \frac{3}{2}d - \frac{1}{2} \rfloor}{\lfloor \frac{1}{2}d \rfloor}$. This is a special case of the Upper Bound Theorem of McMullen cf. [12], [15]. All d-polytopes which attain this bound are simple, and they comprise exactly the duals of all simplicial neighborly d-polytopes. (A polytope P is *neighborly* if every set of $[\frac{d}{2}]$ vertices determines a facet of P.) Among such polytopes is the dual $C_d^*(2d)$ of the cyclic polytope $C_d(2d)$ on $2d$ vertices. The d-step conjecture has been verified for all dual cyclic polytopes by Klee [7], but is apparently still open for the class of all duals of simplicial neighborly d-polytopes, see [8, p. 750]. We prove that the Dantzig figure given by $(C_d^*(2d),\ [1, 2, , \ldots, d],\ [d+1, d+2, \ldots, 2d])$ has exactly 2^{d-1} d-step paths between its distinguished vertices. The vertices $[1, 2, \ldots, d]$ and $[d+1, d+2, \ldots, 2d]$ of $C_d^*(2d)$ correspond to the facets of $C_d(2d)$ determined by its (ordered) vertices $1, 2, \ldots, d$ and $d+1, d+2, \ldots, 2d$, respectively. (Theorem 4.1).

The upper bound case differs from the lower bound case in that there are many different combinatorial types of Dantzig figures having $2\binom{\lfloor \frac{3}{2}d - \frac{1}{2} \rfloor}{\lfloor \frac{1}{2}d \rfloor}$ vertices. When d is even, the cyclic polytope $C_d^*(2d)$ itself has other sets of antipodal vertices which give Dantzig figures that are combinatorially distinct from $(C_d^*(2d),\ [1, 2, \ldots, d],\ [d+1, \ldots, 2d])$, and in addition there are such Dantzig figures coming from other dual simplicial neighborly d-polytopes, see [2], [13]. In dimension 4 some of these have 8 d-step paths, while others have 12 d-step paths.

The combinatorial structure of the 2^{d-1} d-step paths are quite different in the minimal vertex case and maximal vertex case; compare Theorem 3.2 and Theorem 4.2.

In Section 2 we present basic definitions and facts about Dantzig figures and d-step paths. Section 3 deals with lower bound Dantzig figures and Section 4 with upper bound Dantzig figures.

2. Dantzig Figures and d-Step Paths

Our terminology on polytopes follows that of Ziegler [15]. All polytopes are bounded. A d-polytope is *simple* if every vertex lies in exactly d facets. A d-polytope is *simplicial* if each facet is a $(d-1)$-simplex. The polar polytope P^* of a polytope P reverses the incidence ordering of all faces; if P is a simplicial polytope then P^* is a simple polytope. The *graph* of a polytope P is the undirected graph giving the incidence structure of its 0-faces (vertices) and 1-faces (edges) of P. Let $conv(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ denote the convex hull of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

The *combinatorial type* of a d-polytope is the set of incidence relations among all its faces. It can be described either by saying which facets contain each face, or, equivalently by specifying which vertices are contained in each face. Two polytopes have the same *combinatorial type* (or are *combinatorially equivalent*) if there is an one-to-one onto, incidence-preserving mapping of faces of one to faces of the other. The *combinatorial type of a Dantzig figure* $(P, \mathbf{w}_1, \mathbf{w}_2)$ is defined similarly, except that incidence-preserving mappings between Dantzig figures are required to take distinguished vertices of one to distinguished vertices of the other. The truth of the strong d-step conjecture for a particular Dantzig figure $(P, \mathbf{w}_1, \mathbf{w}_2)$ depends only on its combinatorial type. Henceforth in this paper the word "polytope" or "Dantzig figure" refers only to its combinatorial type.

We describe the incidence structure of Dantzig figures using facets. Let $H_1,\ H_2, \ldots, H_{2d}$ denote the hyperplanes determined by the $2d$ facets of a Dantzig figure $(P, \mathbf{w}_1, \mathbf{w}_2)$. We address each vertex $\mathbf{v}$ of P by the indices of the facets determining it. The *vertex address*

$$\mathbf{v} := [i_1, i_2, \ldots, i_d]\ , \quad i_1 < i_2 \leq \ldots < i_d\ . \tag{2.1}$$

means that $\mathbf{v} = H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_d}$, with $i_1 < i_2 < \ldots < i_d$. A *normalized Dantzig figure* is one in which $\mathbf{w}_1 = H_1 \cap H_2 \cap \ldots \cap H_d$ and $\mathbf{w}_2 = H_{d+1} \cap H_{d+2} \cap \ldots \cap H_{2d}$, so that

$$\mathbf{w}_1 = [1, 2, \ldots, d] \text{ and } \ \mathbf{w}_2 = [d+1,\ d+2, \ldots, 2d]\ . \tag{2.2}$$

A *distinguished path* is a Dantzig figure $(P, \mathbf{w}_1, \mathbf{w}_2)$ is a d-step path between $\mathbf{w}_1$ and $\mathbf{w}_2$ in the graph of P. We use the following simple characterization of distinguished paths.

Lemma 2.1. *Let $(P, \mathbf{w}_1, \mathbf{w}_2)$ be a normalized Dantzig figure. The following are equivalent:*

(i). *The set of vertices $\mathbf{v}_0 = \mathbf{w}_1,\ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d = \mathbf{w}_2$ is a d-step path in the graph of P, i.e. a distinguished path.*

(ii). The set of vertices $\mathbf{v}_0 = \mathbf{w}_1, \mathbf{v}_1, \ldots, \mathbf{v}_d = \mathbf{w}_2$ of P has each vertex address $\mathbf{v}_{i+1}$ obtained from $\mathbf{v}_i$ by removing one facet index j with $1 \leq j \leq d$ and adding one facet index j' with $d+1 \leq j' \leq 2d$.

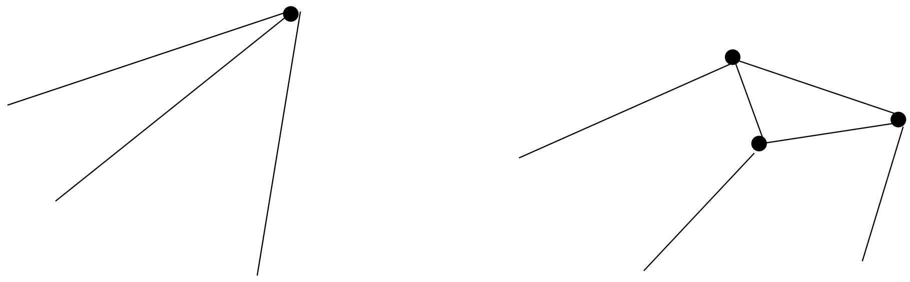
Here (ii) implies that the vertex labels on any such distinguished path are lexicographically increasing. Based on (ii), we sometimes call the change of addresses from $\mathbf{v}_i$ to $\mathbf{v}_{i+1}$ a *move* of j to j'.

Proof. (i) $\Rightarrow$ (ii). An edge deletes one facet, and a new vertex adds exactly one facet. In going from $\mathbf{w}_1$ to $\mathbf{w}_2$ we must add the facets $H_{d+1}, \ldots, H_{2d}$ and we must remove the facets $H_1, \ldots, H_d$. Since there are only d-steps (ii) follows.

(ii) $\Rightarrow$ (i). We need only check that $[\mathbf{v}_i, \mathbf{v}_{i+1}]$ is an edge of P, i.e. is not interior to any k-face of P for $k \geq 2$. But (ii) shows that $conv[\mathbf{v}_i, \mathbf{v}_{i+1}]$ lies in the intersection of $d-1$ facet hyperplanes given by the $d-1$ common facet indices for simplicial polytopes with n vertices. Since P is a simple polytope, any k-face for $k \geq 1$ is contained in exactly $d-k$ facets. $\Box$

3. Lower Bound Dantzig Figures: Truncation Polytopes

Barnette [1] proved that any simple d-polytope having n facets has at least $n(d-1) - (d+1)(d-2)$ vertices. He also proved that for dimension $d \geq 4$ equality is attained only for *truncation polytopes*, which are polytopes recursively constructed from a d-simplex by adding facets, one at a time, each of which cuts off a single vertex of the previous polytope. More precisely, we have a sequence of polytopes $P_0, P_1, \ldots, P_{n-d-1}$, in which P_0 is a d-simplex, and P_i is obtained from P_{i-1} by "truncating" one vertex by a facet that replaces it by d vertices, see Figure 3.1. (This construction is sometimes given in its dual form for simplicial polytopes, and the resulting polytopes are called *stacked polytopes.*)



(i) before truncation (ii) after truncation

Figure 3.1: Truncating a vertex

For simple d-polytopes with $2d$ facets, the lower bound is $d^2 - d + 2$ vertices. There are many combinatorially distinct types of truncation polytopes having $2d$ facets, and most of them contain no antipodal vertices, so don't give rise to any Dantzig figures. We construct below a particular family of truncation polytopes with n facets for $n \geq d+1$, among which the case $n = 2d$ gives a Dantzig figure.

Let $H_1, H_2, \ldots, H_n$ denote the hyperplanes giving the facets of a general truncation polytope P. We label vertex $\mathbf{v}$ of P by the labels of the d hyperplanes determining $\mathbf{v}$, numbered in increasing order. To construct P, we start with the d-simplex P_0 determined by hyperplanes $H_1, H_2, \ldots, H_{d+1}$, so that P_0 has the vertices with addresses $\{[1, 2, \ldots, \hat{i}, \ldots, d{+}1] : 1 \leq i \leq d{+}1\}$, where the symbol $\hat{i}$ means that i is omitted from the address. Among these vertices, the lexicographically minimal vertex is $[1, 2, \ldots, (d-1), d]$, and the lexicographically maximal vertex is $[2, 3, \ldots, d, (d+1)]$. At step k, for $1 \leq k \leq n-d-1$ we truncate polytope P_{k-1} with a hyperplane H_{d+k+1}, to obtain a new truncation polytope P_{k+1}. Then $P = P_{n-d-1}$.

The *maximal truncation polytope* $T_d(n)$ is the truncation polytope with n facets obtained as above, by choosing the lexiographically maximal vertex of P_{k-1} to truncate at step k. Thus at the k-th step, the vertex truncated has address $[k+1, k+2, \ldots, k+d]$ and the d new vertices of P_{k+1} have addresses $\{[k+1, \ldots, \widehat{k+i}, \ldots, k+d, k+d+1] : 1 \leq i \leq d\}$.

For the case $n = 2d$, the $d^2 - d + 2$ vertices of $T_d(2d)$ consist of the two distinguished "end vertices" $\mathbf{w}_1 = [1, 2, \ldots, d]$ and $\mathbf{w}_2 = [d+1, d+2, \ldots, 2d]$ plus the $d^2 - d$ "non-end" vertices which have addresses

$$[i, i+1, , \ldots, \widehat{i+k}, \ldots, i+d], \quad \text{for} \quad 1 \leq k \leq d-1 \quad \text{and} \quad 1 \leq i \leq d\ . \tag{3.1}$$

In particular $(T_d(2d),\ \mathbf{w}_1, \mathbf{w}_2)$ is a normalized Dantzig figure.

Theorem 3.1. *For each $d \geq 2$, the normalized Dantzig figure $(T_d(2d),\ [1,2,\ldots,d],\ [d+1,\ldots,2d])$ with d^2-d+1 vertices has exactly 2^{d-1} d-step paths in its graph between $[1,2,\ldots,d]$ and $[d+1,\ldots,2d]$.*

We obtain this from the following more detailed result.

Theorem 3.2. *Let $\mathbf{v}_i$ denote the vertex of $T_d(2d)$ that is reached from $[1,2,\ldots,d]$ along the edge that deletes H_i, which has the address*

$$\mathbf{v}_i = [1, 2, \ldots, \hat{i}, \ldots, d, d+1]\ , \quad 1 \leq i \leq d\ . \tag{3.2}$$

The number of distinguished paths in $(T_d(2d),\ [1,2,\ldots,d],\ [d+1,\ldots,2d])$ that pass through $\mathbf{v}_i$ is 2^{d-1-i} for $1 \leq i \leq d-1$ and is 1 for $i = d$.

Theorem 3.1 follows directly, since (3.2) runs over all d vertices adjacent to $\mathbf{w}_1$, and $1 + \sum_{j=1}^{d-1} 2^{d-1-j} = 2^{d-1}$.

Proof of Theorem 3.2. We first introduce some additional notation concerning vertex addresses. We add a vertical bar to separate facet numbers at most d from those at least $d+1$. For example, when $d=4$, the vertex label $[1,2,5,7]$ becomes $[1,2|5,7]$.

The "non-end" vertices of $T_d(2d)$ have labels consisting of $d+1$ consecutive facet indices, from i to $d+i$, from which exactly one facet index is omitted, which is neither of the endpoints i or $d+i$. We call the omitted facet index a "bubble," since it interrupts the consecutive sequence of indices of the vertex address.

We call any move between two vertices $\mathbf{v}$ and $\mathbf{v}'$ of $T_d(2d)$ a *legal move* if the vertex address of $\mathbf{v}'$ is obtained from that of $\mathbf{v}$ by removing a facet number $j \leq d$ to the left of the vertical bar and adding a facet number $j' \geq d+1$ to the right of the vertical bar. Lemma 2.1 asserts that every step in a d-step distinguished path is a legal move.

The following claim is verified by inspection, using (3.1).

Claim 1. *The legal moves from a vertex of $T_d(2d)$ to a new vertex depend on the location of the "bubble," and are as follows:*

(i). *If the "bubble" is to the left of the vertical bar, then the only legal move is to replace the index i by $d+i+1$ if the "bubble" is not at $i+1$, and by $d+i+2$ otherwise. In the first case the "bubble" stays where it is, while in the second case the "bubble" moves to the right of the vertical bar, to the position $d+i+1$.*

(ii). *If the "bubble" is to the right of the vertical bar, we can remove any element j to the left of the bar, except i and use it to fill the "bubble." If so, the "bubble" moves to the left of the vertical bar, to position $\hat{j}$. Alternatively, the label i can move to the right of the vertical bar, to position $d+i+1$.*

We now prove Theorem 3.1 by induction on the dimension d. The base case $d = 2$ is easily checked directly. Now suppose that it is true for all dimensions up to $d-1$ and consider dimension d.

First we treat paths which pass through $\mathbf{v}_d = [1, 2, \ldots, d-1, \hat{d} | d+1]$. The "bubble" is to the left of the vertical bar, hence rule (i) of the claim forces a unique legal move for $d-2$ steps, ending at the vertex with address $[d-1, \hat{d} | d+1, \ldots, 2d-1]$. There is now a unique legal final step possible to $\mathbf{w}_2 = [d+1, \ldots, 2d-1, 2d]$ hence there is exactly one d-step distinguished path through $\mathbf{v}_d$.

Next, consider paths which pass through $\mathbf{v}_{d-1} = [1, 2, \ldots, \widehat{d-1}, d | d+1]$. The first $d-2$ legal steps are uniquely forced by rule (i), and end at the vertex with address $[d | d+1, d+2, \ldots, 2d-2, \widehat{2d-1}, 2d]$. Now there is a unique final legal move to $\mathbf{w}_2$, hence there is exactly one d-step distinguished path through $\mathbf{v}_{d-1}$.

For the cases of paths which pass through $\mathbf{v}_i$ for $1 \le i \le d-2$, the first $i-1$ legal steps are uniquely forced by rule (i), with the path reaching the vertex

$$\mathbf{v}^* = [i+1, i+2, \ldots, d | d+1, \ldots, \widehat{d+i}, d+i+1] \ .$$

Here $\mathbf{v}^*$ has the a "bubble" to the right of the vertical bar.

Claim 2. *The $(d-i)$-step legal paths from $\mathbf{v}^*$ to $\mathbf{w}_2$ in $T_d(2d)$ are in one-to-one correspondence with distinguished $(d-i)$-step paths for the normalized Dantzig figure $(T_{d-i}(2d-2i),\ \mathbf{w}_1', \mathbf{w}_2')$ in $\mathbf{R}^{d-i}$.*

The one-to-one correspondence arises by deleting the $2i$ facets labelled $1, 2, \ldots, i$ and $d+1, \ldots, \widehat{d+i}, d+i+1$ and then renumbering the remaining facets in increasing order, denoting them $\bar{1}, \bar{2}, \ldots, \overline{2(d-i)}$, in which $d+i$ is mapped to $\overline{d-i+1}$, and $d+i+2$ to $\overline{d-i+2}$. Then $\mathbf{v}^*$ corresponds to the initial vertex

$$\mathbf{w}_1' = [\bar{1}, \bar{2}, \ldots, \overline{d-1}]$$

of $T_{d-i}(2(d-i))$. We now observe that any sequence of legal moves resulting by applying rules (i) and (ii) starting at the vertex $\mathbf{v}^*$ of $T_d(2d)$ can be exactly matched by the same sequence of legal moves by applying rules (i) and (ii) to $\mathbf{w}_1'$ of $T_{d-1}(2d-2i)$, under the renumbering above, and vice-versa. This holds because deleted vertices can never be moved again in any legal d-step path, by Lemma 2.1. For example, a rule (ii) move of type j to fill the "bubble" of $\mathbf{v}_1$ gives the next vertex

$$\mathbf{v} = [i+1,\ i+2, \ldots, \hat{j}, \ldots, d | d+1, \ldots, d+i, d+i+1] \ ,$$

which corresponds to the vertex

$$\mathbf{v}' = [\bar{1}, \bar{2}, \ldots, \widehat{\overline{j-i}}, \ldots, \overline{d-i} | \overline{d-i+1}]$$

of $T_{d-1}(2d-2i)$. A case (ii) "end move" to

$$\mathbf{v} = [i+2,\ i+3, \ldots, d | d+1, \ldots, \widehat{d+i}, d+i+1,\ d+i+2]$$

corresponds to the "end move"

$$\mathbf{v}' = [\bar{2}, \bar{3}, \ldots, d-i | \widehat{\overline{d-i+1}},\ \overline{d-i+2}] \ ,$$

of $T_{d-i}(2d-2i)$. This proves Claim 2.

The induction hypothesis for $d-i$ gives that the total number of distinguished $(d-i)$-step paths in $T_{d-i}(2d-2i)$ is

$$1 + \sum_{i=1}^{d-i-1} 2^{d-i-1-j} = 2^{d-i-1} \ .$$

Thus Claim 2 shows that there are 2^{d-i-1} d-step paths through $\mathbf{v}_i$ for $1 \le i \le d-2$, which completes the induction step. $\square$

Table 3.1 gives the eight 4-step paths for $(T_4(8), [1234],\ [5678])$, grouped by the value of $\mathbf{v}_i$.

1234	1234	1234	1234
2346	2346	2346	2346
3467	3467	2456	2356
4678	3567	4568	3567
5678	5678	5678	5678
1234	1234	1234	1234
1345	1345	1245	1235
3457	3457	2456	2356
4578	3567	4568	3567
5678	5678	5678	5678

Table 3.1. d-step paths for $T_4(8)$.

We conclude this section with a uniqueness result.

Theorem 3.3. *For $d \geq 4$, there is a unique combinatorial type of d-dimensional Dantzig figure having $d^2 - d + 2$ vertices, which is given by $(T_d(2d),\ [1,2,3,\ldots,d],\ [d+1,\ d+2,\ldots,2d])$.*

Proof. By Barnette's result any such Dantzig figure is a truncation polytope P. Let P be constructed from the d-simplex S_d by truncating $d-1$ successive vertices. By suitably renumbering the facets of S_d and the added facets, we may suppose that the distinguished vertices are $\mathbf{w}_1 = [1,2,\ldots,i,d+1,d+2,\ldots,2d-i]$ and $\mathbf{w}_2 = [i+1,i+2,\ldots,d,2d-i+1,2d-i+2,\ldots,2d]$. In order to create these vertices using $d-1$ truncations, the sequence of truncations leading to $\mathbf{w}_1$ must add facets $H_{d+1},\ldots,H_{2d-i}$ and delete facets $H_{i+1},\ldots,H_d$ in some order, while for $\mathbf{w}_2$ it adds facets $H_{2d-i+1},\ldots,H_{2d}$ and deletes $H_1,\ldots,H_i$, in some order. In particular, no truncation operation for $H_{d+1},\ldots,H_{2d+i}$, ever involves a vertex containing $H_{2d-i+1},\ldots,H_{2d}$, hence these truncation operations mutually commute. Thus we may first make all the truncations $H_{d+1},\ldots,H_{2d-i}$ (in some order), followed by H_{2d-i+1} through H_{2d} (in some order). Next we may relabel these vertices so that they enter in consecutive order $H_{d+1},\ldots,H_{2d-i}$, followed by $H_{2d-i+1},\ldots,H_{2d}$. Finally we may relabel the facets of S_d so that when the facet H_{d+j} enters then the facet H_j leaves. This relabelling is a permutation of $\{1,2,\ldots,i\}$ and $\{i+1,i+2,\ldots,d\}$ separately. The resulting sequence of vertex addresses of the vertices that are truncated are

$$[1,2,\ldots,i,\ i+j+1,\ i+j+2,\ldots,d,d+1,d+2\ldots d+j], \quad 1 \leq j \leq d-i\ ,$$

and

$$[1,2,\ldots,j,\ i+1,\ i+2,\ldots,d,\ 2d-i-1,\ldots,2d-j]\ , \quad 1 \leq j \leq i\ .$$

Now we can explicitly write down all the vertices of the resulting truncation polytope $(P, \mathbf{w}_1, \mathbf{w}_2)$ and check that there is a further permutation σ of the facet labels that sends it to $(T_d(2d),\ [1,2,3,\ldots,d],\ [d+1,\ d+2,\ldots,2d]$.) The permutation σ is given by

$$\sigma(k) = \begin{cases} d+1-k & 1 \le k \le i \ , \\ d-i+k & i+1 \le k \le d \ , \\ i+1-k & d+1 \le k \le 2d-i+1 \ , \\ k-2d+i & 2d-i \le k \le 2d \ . \end{cases}$$

This completes the proof. $\square$

The unique 4-dimensional Dantzig figure with 14 vertices appears in Table 4 of Grünbaum and Sreedharan [4] as $(P_2^8,\ [1367],\ [2458])$.

4. Upper Bound Dantzig Figures: Cyclic Polytopes

We recall the construction of a cyclic polytope $C_d(n)$ on n vertices. The *moment curve* in $\mathbf{R}^d$ is the curve $\{\mathbf{x}(t) = (t, t^2, \ldots, t^d) : t \in \mathbf{R}\}$. A cyclic polytope $C_d(n)$ is the convex hull of n points on the moment curve, i.e., $C_d(n) = conv\{\mathbf{x}(t_1), \ldots, \mathbf{x}(t_n) : t_1 < t_2 < \ldots < t_n\}$. Such polytopes have a well-defined combinatorial type independent of the choice of vertices. The Upper Bound Theorem states that $C_d(n)$ has the maximum number of facets among all d-polytopes having n vertices. The *dual cyclic polytope* $C_d^*(n)$ denotes the combinatorial type of any polytope that is polar to some cyclic polytope $C_d(n)$.

Proposition 4.1. *(Gale's evenness condition) For $n \ge d \ge 2$ the polytope $C_d^*(n)$ is a simple d-polytope. Given a d-subset $S \subseteq \{1, 2, \ldots, n\}$, the point*

$$\mathbf{v} = \bigcap_{i \in S} H_i$$

is a vertex of $C_d^*(n)$ if and only if for any two elements j_1 and j_2 not in S, with $j_1 < j_2$, the number of elements of S between j_1 and j_2 is even, i.e.

$$2 \mid \#\{k : k \in S \ , \quad j_1 < k < j_2\} \ \textit{for} \ \ j_1,\ j_2 \in S \ . \tag{4.1}$$

Equivalently, all maximal blocks of consecutive elements of S which do not contain either 1 or n must contain an even number of elements.

Proof. Gale's evenness condition for facets of the cyclic polytope appears in McMullen and Shepard [12], p. 84, or in Ziegler [15], p. 14. The proposition states the dual form, which follows from the fact that polarity reverses incidence structure of all faces, cf. Ziegler [15], §2.3. $\square$

We apply Gale's evenness condition to derive the following result.

Theorem 4.1. *For each $d \geq 2$, the normalized Dantzig figure $(C_d^*(2d), [1,2,\ldots,d], [d+1,d+2,\ldots,2d])$ with $2\binom{\lfloor \frac{3}{2}d-\frac{1}{2}\rfloor}{\lfloor \frac{1}{2}d\rfloor}$ vertices has exactly 2^{d-1} d-step paths in its graph between $[1,2,\ldots,d]$ and $[d+1,\ldots,2d]$.*

We obtain this from the following more detailed result.

Theorem 4.2. *Let $\mathbf{v}_i$ denote the vertex of $C_d^*(2d)$ reached from $[1,2,\ldots,d]$ along the edge deleting H_i, so that*

$$\mathbf{v}_i = [1,2,\ldots,\hat{i},\ldots,m_i]\ , \quad 1 \leq i \leq d+1 \tag{4.2}$$

where $m_i = d+1$ or $2d$ according as $d-i$ is odd or even. The number of distinguished paths of $(C_d^*(2d), [1,2,\ldots,d],[d+1,\ldots,2d])$ that pass through $\mathbf{v}_i$ is $\binom{d-1}{i-1}$, for $1 \leq i \leq d$.

Theorem 4.1 follows immediately from Theorem 4.2, since $\sum_{i=1}^{d}\binom{d-1}{i-1} = 2^{d-1}$.

Proof of Theorem 4.2. Although $C_d^*(2d)$ has an exponentially large number of vertices, only a polynomial size subset of them appear in the totality of all distinguished paths. We will subsequently show that these are a subset of all vertices having addresses which consist of at most three maximal blocks of consecutive integers; the number of such vertices is $O(d^3)$.

To describe these vertex addresses we introduce a suggestive terminology. Call a consecutive sequence of vertices $Y_1 = \{1,2,\ldots,i\}$ a *(departure) dock* and call a consecutive sequence $Y_2 = \{k+1,k+2,\ldots,2d\}$ an *(arrival) dock.* We call a maximal consecutive sequence of vertices $X = \{r,r+1,\ldots,s\}$ with $1 < r < s < 2d$ a *ship.* A general vertex address has indices that group into either $0,1$ or 2 docks and some number of ships, i.e., $\mathbf{v} = [Y_1,X_1,X_2,\ldots,X_j,Y_2]$, in which we permit Y_1 or Y_2 to be the empty set, and $j=0$ may occur.

Claim 1. *A vertex* $\mathbf{v}$ *occurs in some distinguished path if and only if its address consists of two docks Y_1, Y_2 and a single ship X, in which up to two of X,Y_1,Y_2 may be the empty set, and the ship, if present, has an even number of vertices, and contains at least one of the indices d and $d+1$.*

Call a vertex of the above type a *voyage vertex.* To prove Claim 1 we will need to characterize the legal moves possible from a voyage vertex. Recall that Lemma 2.1 states that on any distinguished path the address of $\mathbf{v}_{i+1}$ is obtained from that of $\mathbf{v}_i$ by replacing one index $j \leq d$ with an index $j' \geq d+1$.

Claim 2. *The legal moves possible from a voyage vertex* $\mathbf{v} = [Y_1, X, Y_2]$ *which remove an index* $j \le d$ *and add an index* $j' \ge d+1$ *are of the following forms.*

(i). *An "initial move" from* $[1, 2, \ldots, d]$ *replaces an index* i $(1 \le i \le d)$ *by* $2d$ *if* $d - i$ *is even, and by* $d+1$ *if* $d-i$ *is odd.*

(ii). *If* $Y_1 = [1, 2, \ldots, i] \neq \emptyset$ *and* $Y_2 = [k+1, \ldots, 2d]$ *or* $\emptyset$*, then a "docking move," removes* i *and adds* k *if* $Y_2 \neq \emptyset$ *and adds* $2d$ *if* $Y_2 = \emptyset$.

(iii). *If the ship* $X = \{r, r+1, \ldots, s\} \neq \emptyset$*, with* $r \le d$ *and* $i \neq d+1$*, a "sailing move" removes* r *and adds* $s+1$*, yielding* $X' = \{r+1,\ r+2, \ldots, s+1\}$.

(iv). *If* $Y_1 = \{1, 2, \ldots, i\} \neq \emptyset$ *with* $i < d$ *and* $Y_2 = \{k+1, \ldots, 2d\}$ *or* $\emptyset$ *and* $1 \le i - 2m < i$*, a "submarine move" removes* $i - 2m$ *and adds* k *if* $Y_2 \neq \emptyset$ *and by* $2d$ *if* $Y_2 = \emptyset$*. This move produces a ship* $X' = \{i - 2m + 1, \ldots, i\}$.

(v). *If the ship* $X = \{r, r+1, \ldots, s\} \neq \emptyset$*, with* $r \le d$*, a "shipwreck move" removes some index* $r + 2m$ *with* $r < r + 2m \le d$ *and adds* $s+1$*. It produces two ships* $X_1' = \{r, \ldots, r+2m-1\}$ *and* $X_2' = \{r + 2m + 1, \ldots, s+1\}$.

In cases (i)–(iii) a legal move from a voyage vertex leads to another voyage vertex. In cases (iv) and (v) the new vertex is not a voyage vertex. In order to prove Claim 1 we must rule out moves of either of these types on a distinguished path.

The terminology "ship" and "dock" is motivated by Claim 2. Assuming that case (iv) and (v) are ruled out, Claim 2 implies that on a distinguished path a ship can be created only at an initial move, and it then "sails" from $X = \{i+1,\ i+2, \ldots, d+1\}$ to $X' = \{d+1,\ d+2, \ldots, 2d-i+1\}$ by a sequence of $d-i$ "sailing" moves. A sequence of $i-1$ "docking moves" removes the departure dock $Y_1 = \{1, 2, \ldots, i-1\}$ and reconstructs it as the arrival dock $Y_2 = \{2d - i + 2, \ldots, 2d\}$. Claim 2 allows this sequence of "sailing" and "docking" moves to occur in any order, and the number of distinguished paths this accounts for is equal to the number of different ordered sequences of $d - i$ red balls and $i - 1$ green balls; this number is $\binom{d-1}{i-1}$. Thus Theorem 4.2 follows from Claim 1 and Claim 2.

We prove Claim 2 first. Gale's evenness criterion (Proposition 4.1) asserts that a sequence of the form $[Y_1, X, Y_2]$ is the address of a vertex of $C_d^*(2d)$ if and only if the ship X contains an even number of elements. All of the permitted moves in cases (i)–(iv) produce a new address that satisfies Gale's evenness criterion, hence they are legal moves. We must show that they are exhaustive. Since Y_2 only contains elements in $\{d+1,\ldots,2d\}$ an element in it is never moved, and cases (i)–(v) cover all possible elements that could be moved from Y_1 and X.

To verify (i), first note that the index i can move only to $d+1$ or $2d$, because any move to $d+1 < j' < 2d$ creates a ship of length 1, violating Gale's evenness criterion. For $1 \le i \le d-1$, moving i creates a ship starting in position $i+1$ which extends at least to d, hence i must move to $d+1$ or $2d$ as necessary to create a ship of even length. Finally for $i = d$ a move to $d+1$ creates a ship of length 1, which is ruled out, so (i) follows.

To verify case (ii), we again argue by contradiction. If the last index i of Y_1 is removed, and isn't replaced by an index j' which is just before the smallest index in Y_2 (which is $2d$ if $Y_2 = \emptyset$), then it either falls adjacent to a ship and changes its length from even to odd, or else it forms a new ship of length 1, both of which contradict Gale's evenness criterion.

To verify case (iii), we note that moving the smallest index i in a ship must always be to the other end $j+1$ of the ship, otherwise the length of the ship changes from even to odd.

To verify case (iv), we note that removing an interior index $1 \le j < i$ of the dock $Y_1 = [1,\ldots,i]$ must have $j = i-2m$. For if $j = i-2m-1$, then the removed index must be replaced by index $j' = i+1$ to create a ship of even length, and then $j' \le d$, contradicting the hypothesis that $j' \ge d+1$. Thus $j = i-2m$, and if it is replaced by j', then $j' = k-1$, otherwise j' is either a ship of length 1 or is adjacent to the ship X and changes its length from even to odd.

The final case (v) concerns removing some index j of a ship X which is not its smallest vertex. This vertex cannot be the largest vertex of X because the length of the ship changes from even to odd. If j is any other index of X, its removal splits the ship into (at least) two ships. The removed vertex $j = r+2m$, in order that the ship $[r,\ r+1,\ldots,r+2m-1]$ has even length, and the added index must be $s+1$ in order that the second ship $[r+2m+1,\ldots,s,+s+1]$ have even length. This establishes Claim 2.

It remains to prove Claim 1. We first show that only voyage vertices occur on any distinguished path. Such a path starts at $[1,2,\ldots,d]$. It suffices to show that of the legal moves in Claim 2, those of type (i), (ii) and (iii) move a voyage vertex to another voyage vertex, and that moves in cases (iv) and (v) never occur on any distinguished path. For cases (i), (ii) and

(iii) it suffices to observe that, if the move takes $[Y_1, X, Y_2]$ to $[Y_1', X', Y_2']$, then either $X' = \emptyset$ or X' contains one of d and $d+1$.

To show that case (iv) move cannot occur, we argue by contradiction. If it did occur, it would produce a vertex $\mathbf{v} = [Y_1', X', Y_2']$ containing a ship $X' = [i-2m+1, \ldots, i]$ whose largest index $i \leq d-1$. Now Lemma 2.1 says that all moves on a distinguished path remove a vertex $j \leq d$ and add a vertex $j' \geq d+1$. In particular, no subsequent vertex on the distinguished path can ever contain either index $i+1$ or $i-2m$, so this ship can never "sail." On a distinguished path all vertices eventually are moved to be larger than d, hence there is a first time on the path after $\mathbf{v}$ that some index j in X' is removed and replaced with $j' \geq d+1$. The resulting address contains the set of indices $X - \{j\}$, and doesn't contain $i+1, j$ and $i-2m$. Since $|X - \{j\}|$ is odd, this address contains a ship with an odd number of elements, which contradicts Gale's evenness criterion.

To show that a case (v) move can never occur, we again argue by contradiction. If it did occur, the vertex reached is $\mathbf{v} = [Y_1', X_1', X_2', Y_2']$, in which $X_1' = \{r, r+1, \ldots, r+2m-1\}$ is a ship with $r-1$ and $r+2m \leq d$. Now the ship X' can never "sail," and we obtain the same contradiction as in case (iv).

We conclude that every distinguished path consists only of voyage vertices, by induction on the number steps in a partial path, using Claim 2.

To show that all voyage vertices actually occur in some distinguished path, it suffices to observe that each move of type (i)–(iii) switches one index $j \leq d$ to an index $j' \geq d+1$, and that each voyage vertex has at least one predecessor and one successor under a move of one of these types. Thus given such a vertex, we can extend it by predecessors back to $[1, 2, \ldots, d]$ and by successors to $[d+1, d+2, \ldots, 2d]$, and the resulting path takes exactly d steps by the index switching property. This establishes Claim 1. $\square$

Table 4.1 gives the eight 4-step paths for $(C_4^*(8), [1234], [5678])$, grouped by the value of $\mathbf{v}_i$.

1234	1234	1234	1234
2345	1348	1348	1348
3456	3478	1458	1458
4567	4578	4578	1568
5678	5678	5678	5678
1234	1234	1234	1234
1245	1245	1245	1238
1458	1458	1256	1278
4578	1568	1568	1678
5678	5678	5678	5678

Table 4.1. d-step paths for $C_4^*(8)$.

There seems to be no obvious pattern to the number of d-step paths for Dantzig figures having the maximal number of vertices, in 4-dimensional examples. Let P_{36}^8 denote the noncyclic polytope with 20 vertices taken from Table 4 of Grünbaum and Sreedharan [4]. Then we have

$$\# \quad (C_4^*(8),\ [1256],\ [3478]) = 12,$$

$$\# \quad (P_{36}^8,\ [2345],\ [1678]) = 8,$$

$$\# \quad (P_{36}^8,\ [1256],\ [3478]) = 12\ .$$

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