Orthonormal Bases of Exponentials for the n-Cube

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Abstract

A compact domain Ω in $\mathbb{R}^n$ is a *spectral set* if there is some subset Λ of $\mathbb{R}^n$ such that $\{\exp(2\pi i\langle\lambda, x\rangle) : \lambda \in \Lambda\}$ when restricted to Ω gives an orthogonal basis of $L^2(\Omega)$. The set Λ is called a *spectrum* for Ω. We give a criterion for Λ being a spectrum of a given set Ω in terms of tiling Fourier space by translates of a suitable auxiliary set D. We apply this criterion to classify all spectra for the n-cube by showing that Λ is a spectrum for the n-cube if and only if $\{\lambda + [0,1]^n : \lambda \in \Lambda\}$ is a tiling of $\mathbb{R}^n$ by translates of unit cubes.

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1. Introduction

A compact set Ω in $\mathbb{R}^n$ of positive Lebesgue measure is a *spectral set* if there is some set of exponentials

$$\mathcal{B}_\Lambda := \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\} , \tag{1.1}$$

which when restricted to Ω gives an orthogonal basis for $L^2(\Omega)$, with respect to the inner product

$$\langle f, g\rangle_\Omega := \int_\Omega \overline{f(x)}g(x)dx . \tag{1.2}$$

Any set Λ that gives such an orthogonal basis is called a *spectrum* for Ω. Only very special sets Ω in $\mathbb{R}^n$ are spectral sets. However when a spectrum exists, it can be viewed as a generalization of Fourier series, because for the n-cube $\Omega = [0,1]^n$ the spectrum $\Lambda = \mathbb{Z}^n$ gives the standard Fourier basis of $L^2([0,1]^n)$.

The main object of this paper is to relate the spectra of sets Ω to tilings in Fourier space. We develop such a relation and apply it to geometrically characterize all spectra for the n-cube $\Omega = [0,1]^n$.

Theorem 1.1. *The following conditions on a set Λ in $\mathbb{R}^n$ are equivalent.*

(i) *The set $\mathcal{B}_\Lambda = \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ when restricted to $[0,1]^n$ is an orthonormal basis of $L^2([0,1]^n)$.*

(ii) *The collection of sets $\{\lambda + [0,1]^n : \lambda \in \Lambda\}$ is a tiling of $\mathbb{R}^n$ by translates of unit cubes.*

This result was conjectured by Jorgensen and Pedersen [6], who proved it in dimensions $n \le 3$. We note that in high dimensions there are many "exotic" cube tilings. There are aperiodic cube tilings in all dimensions $n \geq 3$, while in dimensions $n \geq 10$ there are cube tilings in which no two cubes share a common $(n-1)$-face (Lagarias and Shor [8]).

In the theorem above the n-cube $[0,1]^n$ appears in both conditions (i) and (ii), but the n-cube in (i) lies in the space domain $\mathbb{R}^n$ while the n-cube in (ii) lies in the Fourier domain $(\mathbb{R}^n)^*$, so that they transform differently under linear change of variables. Thus Theorem 1.1 is equivalent to the following result.

Theorem 1.2. *For any invertible linear transformation $A \in GL(n, \mathbb{R})$, the following conditions are equivalent.*

(i) $\Lambda \subset \mathbb{R}^n$ *is a spectrum for* $\Omega_A := A([0,1]^n)$.

(ii) *The collection of sets* $\{\lambda + D_A : \lambda \in \Lambda\}$ *is a tiling of* $\mathbb{R}^n$, *where* $D_A = (A^T)^{-1}([0,1]^n)$.

Our main result in §3 gives a necessary and sufficient condition for a general set Λ to be a spectrum of Ω in terms of a tiling of $\mathbb{R}^n$ by $\Lambda + D$ where D is a specified auxiliary set in Fourier space. The applicability of this result is restricted to cases where a suitable auxiliary set D exists. Theorem 1.2 is then proved in §4.

Spectral sets were originally studied by Fuglede [2], who related them to the problem of finding commuting self-adjoint extensions in $L^2(\Omega)$ of the set of differential operators $-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}$ defined on the common dense domain $C_c^\infty(\Omega)$. Our definition of spectrum differs from his by a factor of 2π. Fuglede showed that for sufficiently nice regions Ω each spectrum Λ of Ω (in our sense) has $2\pi\Lambda$ as a joint spectrum of a set of commuting self-adjoint extensions of $-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}$, and conversely; we state his result precisely in Appendix B. He also showed that only very special sets Ω are spectral sets. Much recent work on spectral sets is due to Jorgensen and Pedersen, see [4]–[6] and [12], [13].

Fuglede [2, p. 120] made the following conjecture.

Spectral Set Conjecture. *A set Ω in $\mathbb{R}^n$ is a spectral set if and only if it tiles $\mathbb{R}^n$ by translation.*

This conjecture concerns tilings by Ω in the space domain; in contrast Theorem 1.2 above describes spectra Λ for the n-cube in terms of tilings in the Fourier domain by an auxiliary set D. In general there does not seem to be any simple relation between sets of translations T used to tile Ω in the space domain and the set of spectra Λ for Ω, see [5].[9],[13]. However our main results in §3 indicate a relation between the Spectral Set Conjecture and tilings in the Fourier domain — this is discussed at the end of §3.

Theorem 1.2 also implies a result concerning sampling and interpolation of certain classes of entire functions. Given a compact set Ω of nonzero Lebesgue measure, let $B_2(\Omega)$ denote the set of *band-limited functions* on Ω, which are those L^2 functions on $\mathbb{R}^n$ whose Fourier transform has compact support contained in Ω. Such a function is necessarily the restriction to $\mathbb{R}^n$ of an entire function $f : \mathbb{C}^n \to \mathbb{C}$ of restricted growth, see Stein and Weiss [15, Theorem 4.9]. A set Λ is a *set of sampling* for $B_2(\Omega)$ if there is a positive constant C such that each $f \in B_2(\Omega)$ satisfies

$$||f||^2 \geq C \sum_{\lambda \in \Lambda} ||f(\lambda)||^2. \tag{1.3}$$

A set Λ is a *set of interpolation* for $B_2(\Omega)$ if for each set of complex values $\{c_\lambda : \lambda \in \Lambda\}$ with $\sum |c_\lambda|^2 < \infty$ there is at least one function $f \in B_2(\Omega)$ such that

$$f(\lambda) = c_\lambda, \quad \text{for each} \quad \lambda \in \Lambda\ . \tag{1.4}$$

It is clear that a spectrum Λ of a spectral set Ω is both a set of sampling and a set of interpolation for $B_2(\Omega)$. So Theorem 1.2 immediately yields:

Theorem 1.3. *Given a linear transformation A in $GL(n, \mathbb{R})$, set $\Omega_A = A([0,1]^n)$ and $D_A = (A^T)^{-1}([0,1]^n)$. If $\Lambda + D_A$ is a tiling of $\mathbb{R}^n$, then Λ is both a set of sampling and a set of interpolation for $B_2(\Omega_A)$.*

Note that the set Λ has density exactly the Nyquist rate $|\det(A)|$, as is required by results of Landau ([10], [11]) for sets of sampling and interpolation. In this connection see also Gröchenig and Razafinjatovo [3].

Theorem 1.2 also can be viewed as providing a collection of "nonharmonic Fourier series" expansions for L^2-functions on an affine image of the n-cube; see Young [16].

We end this introduction with three remarks. First, in comparison with other spectral sets, the n-cube $[0,1]^n$ has an enormous variety of spectra Λ. It seems likely that a "generic" spectral set has a unique spectrum, up to translations.[2] Second, the tiling result in §3 applies to more general sets Ω than linearly transformed n-cubes $\Omega_A = A([0,1]^n)$; a one-dimensional example is $\Omega = [0,1] \cup [2,3]$. Third, there are open questions remaining in explicitly describing

[2] It can be shown that "generic" fundamental domain Ω of a full rank lattice L in $\mathbb{R}^n$ has a unique spectrum $\Lambda = L^*$, the dual lattice.

the commuting self-adjoint extensions of $-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}$ in $L^2([0,1]^n)$ that correspond to cube tilings; see Appendix B.

Appendix A to the paper addresses the question of whether an orthogonal cube packing in $\mathbb{R}^n$ can be extended to a cube tiling; Appendix B describes the connection of spectral sets and commuting partial differential operators.

Notation. For $x \in \mathbb{R}^n$, let $\|x\|$ denote the Euclidean length of x. We let

$$B(x;T) := \{y : \|y - x\| \leq T\}$$

denote the ball of radius T centered at x. The Lebesgue measure of a set Ω in $\mathbb{R}^n$ is denoted $m(\Omega)$. The Fourier transform $\hat{f}(u)$ is normalized by

$$\hat{f}(u) := \int_{\mathbb{R}^n} e^{-2\pi i \langle u,x\rangle} f(x)dx \ .$$

Throughout the paper we let

$$e_\lambda(x) := e^{2\pi i\langle \lambda, x\rangle}, \quad \text{for} \quad x \in \mathbb{R}^n \ . \tag{1.5}$$

Note that other authors ([2] [6]) define $e_\lambda(x)$ without the factor 2π.

2. Orthogonal Sets of Exponentials and Packings

We consider packings and tilings in $\mathbb{R}^n$ by compact sets Ω of the following kind.

Definition 2.1. A compact set Ω in $\mathbb{R}^n$ is a *regular region* if it has positive Lebesgue measure $m(\Omega) > 0$, is the closure of its interior Ω°, and has a boundary $\partial\Omega = \Omega \setminus \Omega^\circ$ of measure zero.

Definition 2.2. If Ω is a regular region, then a discrete set Λ is a *packing set* for Ω if the sets $\{\Omega + \lambda : \lambda \in \Lambda\}$ have disjoint interiors. It is a *tiling set* if in addition the union of the sets $\{\Omega + \lambda : \lambda \in \Lambda\}$ covers $\mathbb{R}^n$. In these cases we say $\Lambda + \Omega$ is a *packing* or *tiling* of $\mathbb{R}^n$ by Ω, respectively.

To a vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ in $\mathbb{R}^n$ we associate the exponential function

$$e_\lambda(x) := e^{2\pi i\langle \lambda, x\rangle}, \quad \text{for} \quad x \in \mathbb{R}^n \ . \tag{2.1}$$

Given a discrete set Λ in $\mathbb{R}^n$, we set

$$\mathcal{B}_\Lambda := \{e_\lambda(x) : \lambda \in \Lambda\} \ . \tag{2.2}$$

Now suppose that $\mathcal{B}_\Lambda$ restricted to a regular region Ω gives an orthogonal set of exponentials in $L^2(\Omega)$. We derive conditions that the points of Λ must satisfy. Let

$$\chi_\Omega(x) = \begin{cases} 1 & \text{for} \quad x \in \Omega \\ 0 & \text{for} \quad x \notin \Omega \end{cases} \tag{2.3}$$

be the characteristic function of Ω, and consider its Fourier transform

$$\hat{\chi}_\Omega(u) = \int_{\mathbb{R}^n} e^{-2\pi i \langle u, x\rangle} \chi_\Omega(x) dx, \quad u \in \mathbb{R}^n . \tag{2.4}$$

Since Ω is compact the function $\hat{\chi}_\Omega(u)$ is an entire function of $u \in \mathbb{C}^n$. We denote the set of real zeros of $\hat{\chi}_\Omega(u)$ by

$$Z(\Omega) := \{u \in \mathbb{R}^n : \hat{\chi}_\Omega(u) = 0\} . \tag{2.5}$$

Lemma 2.1. *If Ω is a regular region in $\mathbb{R}^n$ then a set Λ gives an orthogonal set of exponentials $\mathcal{B}_\Lambda$ in $L^2(\Omega)$ if and only if*

$$\Lambda - \Lambda \subseteq Z(\Omega) \cup \{0\} . \tag{2.6}$$

Proof. For distinct λ, $\lambda' \in \Lambda$ we have

$$\begin{aligned} \hat{\chi}_\Omega(\lambda - \lambda') &= \int_{\mathbb{R}^n} e^{-2\pi i \langle \lambda - \lambda', x\rangle} \chi_\Omega(x) dx \\ &= \int_\Omega e^{-2\pi i \langle \lambda, x\rangle} e^{2\pi i \langle \lambda', x\rangle} dx \\ &= \langle e_\lambda, e_{\lambda'}\rangle_\Omega . \end{aligned} \tag{2.7}$$

If (2.6) holds, then $\langle e_\lambda, e_{\lambda'}\rangle_\Omega = 0$, and conversely. ∎

This lemma implies that the points of Λ cannot be too close together. Since $\hat{\chi}_\Omega(0) = m(\Omega) > 0$, the continuity of $\hat{\chi}_\Omega(u)$ implies that there is some ball $B(0; R)$ around 0 that includes no point of $Z(\Omega)$, hence $|\lambda - \lambda'| \geq R$ for all $\lambda, \lambda' \in \Lambda$, $\lambda \neq \lambda'$.

Definition 2.3. Let Ω be a regular region in $\mathbb{R}^n$. A regular region D is said to be an *orthogonal packing region* for Ω if

$$(D^\circ - D^\circ) \cap Z(\Omega) = \emptyset . \tag{2.8}$$

Lemma 2.2. *Let Ω be a regular region in $\mathbb{R}^n$ and let D be an orthogonal packing region for Ω. If a set Λ gives an orthogonal set of exponentials $\mathcal{B}_\Lambda$ in $L^2(\Omega)$ then Λ is a packing set for D.*

Proof. If $\lambda \neq \lambda' \in \Lambda$ then Lemma 2.1 gives $\lambda - \lambda' \in Z(\Omega)$. By definition of an orthogonal packing region we have $D^\circ \cap (D^\circ + u) = \emptyset$ for all $u \in Z(\Omega)$ hence

$$D^\circ \cap (D^\circ + \lambda - \lambda') = \emptyset ,$$

as required. ∎

As indicated above, each regular region Ω has an orthogonal packing region D given by a ball $B(0;T)$ for small enough T. The larger we can take D, the stronger the restrictions imposed on Λ.

Lemma 2.3. *If Ω is a spectral set, and D is an orthogonal packing region for Ω, then*

$$m(D)m(\Omega) \leq 1 . \tag{2.9}$$

Proof. Let Λ be a spectrum for Ω. Then Λ is a set of sampling for $B_2(\Omega)$, so the density results of Landau [10] (see also Gröchenig and Razafinjatovo [3]) give

$$\mathbf{d}(\Lambda) = \liminf_{n\to\infty} \frac{1}{(2T)^n} \#(\Lambda \cap [-T,T]^n) \geq m(\Omega) . \tag{2.10}$$

Now $\Lambda + D$ is a packing of $\mathbb{R}^n$, hence if $R = \operatorname{diam}(D)$, we have

$$\begin{aligned} \frac{m(D)}{(2T)^n}\#(\Lambda \cap [-T,T]^n) &= \frac{1}{(2T)^n} m\left(\left\{\bigcup_\lambda (\lambda + D) : \lambda \in \Lambda \cap [-T,T]^n\right\}\right) \\ &\leq \frac{m([-T+R, T+R]^n)}{(2T)^n} = \left(1 + \frac{R}{2T}\right)^n . \end{aligned} \tag{2.11}$$

Letting $T \to \infty$ and taking the $\liminf$ yields

$$m(D)\mathbf{d}(\Lambda) \leq 1, \tag{2.12}$$

which with (2.10) yields (2.9). ∎

In §3 we give a self-contained proof of Lemma 2.3. The inequality (2.9) of Lemma 2.3 does not hold for general sets Ω. In fact the set $\Omega = [0,1] \cup [2, 2+\theta]$ for suitable irrational θ has a Fourier transform $\hat{\chi}_\Omega(\xi)$ which has no real zeros, so $Z(\Omega) = \emptyset$, and any regular region D is an orthogonal packing region for Ω.

In view of Lemma 2.3 we introduce the following terminology.

Definition 2.4. An orthogonal packing region D for a regular region Ω is *tight* if

$$m(D) = \frac{1}{m(\Omega)} . \tag{2.13}$$

Lemma 2.4. *Let D be a tight orthogonal packing region for a regular region Ω. Then for any $A \in GL(n, \mathbb{R})$ the set $(A^T)^{-1}(D)$ is a tight orthogonal packing region for $A(\Omega)$.*

Proof. Since $\hat{\chi}_{A(\Omega)}(u) = |\det(A)|\hat{\chi}_\Omega(A^T u)$, $Z(A(\Omega)) = (A^T)^{-1}Z(\Omega)$. Hence $(A^T)^{-1}(D)$ is an orthogonal packing region for $A(\Omega)$. It is tight because

$$m((A^T)^{-1}D) = \frac{1}{|\det(A^T)|}\, m(D) \;=\; \frac{1}{|\det(A^T)|\; m(\Omega)} = \frac{1}{m(A(\Omega))} \ . \quad \blacksquare$$

There are many spectral sets which have tight orthogonal packing regions. For our main result in §4 we show that if $\Omega_A = A([0,1]^n)$ is an affine image of the n-cube, with $A \in GL(n, \mathbb{R})$, then

$$D := (A^T)^{-1}([0,1]^n) \tag{2.14}$$

is a tight orthogonal packing region for Ω_A. Another example in $\mathbb{R}^1$ is the region

$$\Omega = [0,1] \cup [2,3] \ . \tag{2.15}$$

In this case we can take

$$D = \left[0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right] \ . \tag{2.16}$$

Indeed $\chi_\Omega(x)$ is the convolution of $\chi_{[0,1]}(x)$ with the sum of two delta functions $\delta_0 + \delta_2$. Thus

$$\hat{\chi}_\Omega(x) = (1 + e^{-4\pi i x})\hat{\chi}_{[0,1]}(x) \ . \tag{2.17}$$

From this it is easy to check that the zero set is given by

$$Z(\Omega) = (\mathbb{Z} \setminus \{0\}) \cup \left(\frac{1}{4} + \mathbb{Z}\right) \cup \left(-\frac{1}{4} + \mathbb{Z}\right) \ , \tag{2.18}$$

that D is an orthogonal packing region for Ω, and, since $m(D) = \frac{1}{2} = \frac{1}{m(\Omega)}$, that D is tight. A spectrum for Ω is $\Lambda = \mathbb{Z} \cup \left(\mathbb{Z} + \frac{1}{4}\right)$.

Lemma 2.3 together with the spectral set conjecture lead us to propose:

Conjecture 2.1. *If Ω tiles $\mathbb{R}^n$ by translations, and D is an orthogonal packing region for Ω, then*

$$m(\Omega)m(D) \le 1 \ . \tag{2.19}$$

3. Spectra and Tilings

A main result of this paper is the following criterion which relates spectra to tilings in the Fourier domain.

Theorem 3.1. *Let Ω be a regular region in $\mathbb{R}^n$, and let Λ be such that the set of exponentials $\mathcal{B}_\Lambda$ is orthogonal for $L^2(\Omega)$. Suppose that D is a regular region with*

$$m(D)m(\Omega) = 1 \tag{3.1}$$

such that $\Lambda + D$ is a packing of $\mathbb{R}^n$. Then Λ is a spectrum for Ω if and only if $\Lambda + D$ is a tiling of $\mathbb{R}^n$.

Proof. $\Rightarrow$. Suppose first that Λ is a spectrum for Ω. Pick a "bump function" $\gamma(x) \in C_c^\infty(\Omega)$, and set

$$\gamma_t(x) = e^{-2\pi i \langle t,x\rangle}\gamma(x), \quad \text{for} \quad t \in \mathbb{R}^n \ .$$

By hypothesis $\mathcal{B}_\Lambda = \{e_\lambda(x) : \lambda \in \Lambda\}$ is orthogonal and complete for $L^2(\Omega)$. Thus, on Ω, we have

$$\gamma_t(x) \sim \sum_{\lambda\in\Lambda} \frac{\langle e^{2\pi i\langle\lambda,x\rangle}, \gamma_t(x)\rangle_\Omega}{\|e_\lambda\|_2^2} \ e^{2\pi i\langle\lambda,x\rangle} \ , \tag{3.2}$$

with coefficients

$$\begin{aligned} \frac{\langle e^{2\pi i\langle\lambda,x\rangle}, \gamma_t(x)\rangle_\Omega}{\|e_\lambda\|_2^2} &= \frac{1}{m(\Omega)}\int_\Omega e^{-2\pi i\langle\lambda,x\rangle}\gamma_t(x)dx \\ &= \frac{1}{m(\Omega)}\int_{\mathbb{R}^n} e^{-2\pi i\langle\lambda+t,x\rangle}\gamma(x)dx \\ &= \frac{1}{m(\Omega)}\hat{\gamma}(\lambda+t) \ , \end{aligned} \tag{3.3}$$

where $m(\Omega)$ is the Lebesgue measure of Ω. Since γ is a smooth function,

$$\|\hat{\gamma}(u)\| \le C_\gamma \|u\|^{-n-2}, \quad \text{for} \quad u \in \mathbb{R}^n \quad \text{with} \quad \|u\| \ge 1 \ . \tag{3.4}$$

This fact, plus the "well-spaced" property of Λ shows that the right side of (3.2) converges absolutely and uniformly on $\mathbb{R}^n$, for fixed t, to a continuous function. Since $\gamma_t(x)$ is continuous, we have

$$\gamma_t(x) = \frac{1}{m(\Omega)}\sum_{\lambda\in\Lambda}\hat{\gamma}(\lambda+t)e^{2\pi i\langle\lambda,x\rangle}, \quad \text{for all} \quad x \in \Omega \ . \tag{3.5}$$

This yields, for all $t \in \mathbb{R}^n$, that

$$\begin{aligned} \gamma(x) &= e^{2\pi i\langle t,x\rangle}\gamma_t(x) \\ &= \frac{1}{m(\Omega)}\sum_{\lambda\in\Lambda}\hat{\gamma}(\lambda+t)e^{2\pi i\langle\lambda+t,x\rangle}, \quad \text{for all} \quad x \in \Omega \ . \end{aligned} \tag{3.6}$$

The series on the right side of (3.6) converges absolutely and uniformly for all $x \in \mathbb{R}^n$ and for t in any fixed compact subset of $\mathbb{R}^n$, but is only guaranteed to agree with $\gamma(x)$ for $x \in \Omega$.

We now integrate both sides of (3.6) in t over all $t \in D$ to obtain:

$$\begin{aligned} m(D)\gamma(x) &= \frac{1}{m(\Omega)} \int_D \sum_{\lambda\in\Lambda} \hat{\gamma}(\lambda+t) e^{2\pi i \langle \lambda+t, x\rangle} dt \\ &= \frac{1}{m(\Omega)} \sum_{\lambda\in\Lambda} \int_D \hat{\gamma}(\lambda+t) e^{2\pi i \langle \lambda+t, x\rangle} dt \\ &= \frac{1}{m(\Omega)} \int_{\Lambda+D} \hat{\gamma}(u) e^{2\pi i \langle u, x\rangle} du, \quad \text{for all} \quad x \in \Omega \ . \end{aligned} \tag{3.7}$$

In the last step we used the fact that the translates $\lambda + D$ overlap on sets of measure zero, because $\Lambda + D$ is a packing of $\mathbb{R}^n$. Since $m(D) = \frac{1}{m(\Omega)}$, (3.7) yields

$$\gamma(x) = \int_{\mathbb{R}^n} \hat{\gamma}(u) h(u) e^{2\pi i \langle u, x\rangle} du, \quad \text{for all} \quad x \in \Omega \ , \tag{3.8}$$

where

$$h(u) = \begin{cases} 1 & \text{if} \ \ u \in \Lambda + D \\ 0 & \text{otherwise} \ . \end{cases}$$

Define $k \in L^2(\mathbb{R}^n)$ by $\hat{k} = h\hat{\gamma}$, so (3.8) asserts that $\gamma(x) = k(x)$ for almost all $x \in \Omega$. Plancherel's theorem on $L^2(\mathbb{R}^n)$ applied to k, together with (3.8), gives

$$\begin{aligned} ||\hat{\gamma}||_2^2 &\geq ||h\hat{\gamma}||_2^2 = ||k||_2^2 \\ &\geq \int_\Omega |k(x)|^2 dx = \int_\Omega |\gamma(x)|^2 dx = ||\gamma||_2^2 \ . \end{aligned} \tag{3.9}$$

Since Plancherel's theorem also gives $||\hat{\gamma}||_2^2 = ||\gamma||_2^2$, we must have

$$||\hat{\gamma}||_2^2 = ||h\hat{\gamma}||_2^2 \ . \tag{3.10}$$

We next show that this equality implies that $h(u) = 1$ almost everywhere on $\mathbb{R}^n$. To do this we show that $\hat{\gamma}(u) \neq 0$ a.e. in $\mathbb{R}^n$. Since γ has compact support, the Paley-Wiener theorem states that $\hat{\gamma}(u)$ is the restriction to $\mathbb{R}^n$ of an entire function on $\mathbb{C}^n$ that satisfies an exponential growth condition at infinity, see Stein and Weiss [15], Theorem 4.9. Thus $\hat{\gamma}(u)$ is real-analytic on $\mathbb{R}^n$ and is not identically zero, hence

$$Z := \{u \in \mathbb{R}^n : \hat{\gamma}(u) = 0\}$$

has Lebesgue measure zero. Together with (3.10) this yields

$$h(u) = 1 \quad \text{a.e.} \quad \text{in } \mathbb{R}^n \ . \tag{3.11}$$

Thus $\Lambda + D$ covers all of $\mathbb{R}^n$ except a set of measure zero.

Finally we show that $\Lambda + D$ covers all of $\mathbb{R}^n$. By the well-spaced property of Λ and the compactness of D, the set $\Lambda + D$ is locally the union of finitely many translates of D, hence $\Lambda + D$ is closed. Thus the complement of $\Lambda + D$ is an open set. But the complement of $\Lambda + D$ has zero Lebesgue measure, hence it is empty, so $\Lambda + D$ is a tiling of $\mathbb{R}^n$.

$\Leftarrow$. Suppose $\Lambda + D$ tiles $\mathbb{R}^n$. By hypothesis $\mathcal{B}_\Lambda$ is an orthogonal set in $L^2(\Omega)$, and to show that Λ is a spectrum it remains to show that it is complete in $L^2(\Omega)$. Let S be the closed span of $\mathcal{B}_\Lambda$ in $L^2(\Omega)$. We will show that $C_c^\infty(\Omega)$ is contained in S. Since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$ this implies $S = L^2(\Omega)$.

For each $\gamma \in C_c^\infty(\Omega)$ set

$$\gamma_t(x) = e^{-2\pi i \langle t,x\rangle}\gamma(x), \quad \text{for} \quad t \in \mathbb{R}^n .$$

Since the elements of $\mathcal{B}_\Lambda$ are orthogonal, Bessel's inequality gives

$$\begin{aligned} \|\gamma_t\|^2 &\geq \sum_{\lambda\in\Lambda} \frac{|\langle e_\lambda, \gamma_t\rangle|^2}{\|e_\lambda\|^2} \\ &= \frac{1}{m(\Omega)} \sum_{\lambda\in\Lambda} |\hat{\gamma}(\lambda + t)|^2 \ , \end{aligned} \tag{3.12}$$

where the last series converges uniformly on compact sets by the rapid decay of $\hat{\gamma}$ at infinity. Integrating this inequality over $t \in D$ yields

$$\int_D \|\gamma_t\|^2 dt \geq \frac{1}{m(\Omega)} \int_D \sum_{\lambda\in\Lambda} |\hat{\gamma}(\lambda + t)|^2 dt \ .$$

Since $\|\gamma_t\| = \|\gamma\|$ for all t, and since $\Lambda + D$ is a tiling, we obtain $m(D)\|\gamma\|^2 \geq \|\hat{\gamma}\|^2/m(\Omega)$. But $m(D) = 1/m(\Omega)$ and $\|\gamma\|^2 = \|\hat{\gamma}\|^2$, so equality must hold in (3.12) for almost all t:

$$\|\gamma\|^2 = \sum_{\lambda\in\Lambda} \frac{|\langle e_\lambda, \gamma_t\rangle|^2}{\|e_\lambda\|_2^2} \ . \tag{3.13}$$

Now the right side of (3.13) converges uniformly on compact sets in t to a continuous function of t, and the left side is a constant, so (3.13) holds for all t, including $t = 0$. Hence $\gamma \in S$. ∎

At first glance this proof of Theorem 3.1 appears "too good to be true" because it only uses functions $\gamma_t(x)$ supported on a tiny part of Ω. In fact all of Ω is used in the formula (3.6) which is required to be valid for all $x \in \Omega$.

The proof of Theorem 3.1 yields a direct proof of Lemma 2.3. If D is an orthogonal packing set, then (3.7) holds for it, hence $m(D)m(\Omega)\gamma(x)$ agrees with $k(x)$ on Ω, hence

$$m(D)m(\Omega)\|\gamma\|_2 \leq \|k\|_2 \leq \|\gamma\|_2$$

hence (2.9) holds.

The following result is an immediate corollary of Theorem 3.1, which we state as a theorem for emphasis.

Theorem 3.2. *Let Ω be a regular region in $\mathbb{R}^n$, and suppose that D is a tight orthogonal packing region for Ω. If Λ is a spectrum for Ω, then $\Lambda + D$ is a tiling of $\mathbb{R}^n$.*

Proof. The assumption that D is a tight orthogonal packing region guarantees that $\Lambda + D$ is a packing for all spectra Λ, so Theorem 3.1 applies. ∎

Theorem 3.2 sheds some light on Fuglede's conjecture that every spectral set Ω tiles $\mathbb{R}^n$.

Definition 3.1. A pair of regular regions $(\Omega, \hat{\Omega})$ are a *tight dual pair* if each is a tight orthogonal packing region for the other.

In §4 we show that $(A([0,1]^n),\ (A^T)^{-1}([0,1]^n))$ are a tight dual pair of regions. The sets $([0,1] \cup [2,3],\ [0,\frac{1}{4}] \cup [\frac{1}{2},\frac{3}{4}])$ are a tight dual pair in $\mathbb{R}^1$.

If $(\Omega, \hat{\Omega})$ are a tight dual pair, then Theorem 3.1 states that if one of $(\Omega, \hat{\Omega})$ is a spectral set, say Ω, then the other set $\hat{\Omega}$ tiles $\mathbb{R}^n$. If $\hat{\Omega}$ were also a spectral set (as Fuglede's conjecture implies) then Theorem 3.1 would show that Ω tiles $\mathbb{R}^n$. This raises the question whether the current evidence in favor of Fuglede's conjecture is mainly based on sets Ω which are part of a tight dual pair $(\Omega, \hat{\Omega})$. At present we can only say that there are many nontrivial examples of tight dual pairs.

To clarify matters, we formulate two conjectures.

Conjecture 3.1. (*Spectral Set Duality Conjecture*) *If $(\Omega, \hat{\Omega})$ is a tight dual pair of regular regions, and Ω is a spectral set, then $\hat{\Omega}$ is also a spectral set.*

In this case Theorem 3.2 would imply that both Ω and $\hat{\Omega}$ tile $\mathbb{R}^n$.

The following is a tiling analogue of the conjecture above.

Conjecture 3.2. (*Weak Spectral Set Conjecture*) *If $(\Omega, \hat{\Omega})$ are a tight dual pair of regular regions, and one of them tiles $\mathbb{R}^n$, then so does the other, and both Ω and $\hat{\Omega}$ are spectral sets.*

4. Spectra for the n-cube and Cube Tilings

We now prove Theorem 1.2, using the results of §3.

The next two lemmas show that if $\Omega_A = A([0,1]^n)$ is an affine image of the n-cube, with $A \in GL(n,\mathbb{R})$, then

$$D := (A^T)^{-1}([0,1]^n) \tag{4.1}$$

is a tight orthogonal packing region for Ω_A.

Lemma 4.1. $\mathcal{B}_\Lambda := \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ *gives a set of orthogonal functions in* $L^2([0,1]^n)$ *if and only if for any distinct* λ, $\mu \in \Lambda$,

$$\lambda_j - \mu_j \in \mathbb{Z} \setminus \{0\} \quad \text{for some} j, \qquad 1 \le j \le n \ . \tag{4.2}$$

Proof. For $\Omega = [0,1]^n$ and $u \in \mathbb{R}^n$,

$$\hat{\chi}_\Omega(u) = \int_{[0,1]^n} e^{-2\pi i\langle u,x\rangle} dx = \prod_{j=1}^n h_0(u_j) \ ,$$

where $h_0(\omega) := (1 - e^{-2\pi i\omega})/(2\pi i\omega)$, $\omega \in \mathbb{R}$, and $h_0(0) := 1$. Note that $h_0(\omega) = 0$ if and only if $\omega \in \mathbb{Z} \setminus \{0\}$. Hence $\hat{\chi}_\Omega(u) = 0$ if and only if $u_j \in \mathbb{Z} \setminus \{0\}$ for some j, $1 \le j \le n$. The lemma now follows immediately from Lemma 2.1. ∎

Lemma 4.2. *Let* $A \in GL(n,\mathbb{R})$ *and* $\Omega = A([0,1]^n)$. *Then* $D = (A^T)^{-1}([0,1]^n)$ *is a tight orthogonal packing region for* Ω.

Proof. By Lemma 2.4 we need only check this for $\Omega = [0,1]^n$. Lemma 4.1 implies that $D = [0,1]^n$ is an orthogonal packing region for Ω_1 and it is clearly tight. ∎

We will also use the following basic result of Keller [7].

Proposition 4.1. (*Keller*) *If* $\Lambda + [0,1]^n$ *is a tiling of* $\mathbb{R}^n$, *then each* λ, $\lambda' \in \Lambda$ *has*

$$\lambda_i - \lambda'_i \in \mathbb{Z} \setminus \{0\} \quad \text{for some} \quad i, \quad 1 \le i \le n \ . \tag{4.3}$$

Proof. This result was proved by Keller [7] in 1930. A detailed proof appears in Perron [14], Satz 9. ∎

Proof of Theorem 1.2. (i) $\Rightarrow$ (ii). Suppose that Λ is a spectrum for $D = A([0,1]^n)$. Lemma 4.2 gives that $D = (A^T)^{-1}([0,1]^n)$ is a tight orthogonal packing set for Ω. By Theorem 3.2 $\Lambda + D$ is a tiling of $\mathbb{R}^n$.

(ii) $\Rightarrow$ (i). It suffices to prove this direction for the n-cube $\Omega = [0,1]^n$, since the general case follows by a linear change of variables. We take $D = [0,1]^n$, so $m(\Omega)m(D) = 1$. Let $\Lambda + D$ be a cube-tiling. Now Proposition 4.1 shows that $\mathcal{B}_\Lambda$ is an orthogonal set in $L^2([0,1]^n)$, by the criterion of Lemma 2.4. The hypotheses of Theorem 3.1 hold, and we conclude that Λ is a spectrum because $\Lambda + D$ is a tiling. ∎

Appendix A. Extending Cube Packings to Cube Tilings

This appendix addresses the problem of when a cube packing in $\mathbb{R}^n$ can be extended to a cube tiling by adding extra cubes.

Definition A.1. A cube packing $\Lambda + [0,1]^n$ is *orthogonal* if for distinct $\lambda, \mu \in \Lambda$,

$$\lambda_j - \mu_j \in \mathbb{Z} \setminus \{0\} \quad \text{for some } j, \qquad 1 \le j \le n. \tag{A.1}$$

Keller's theorem (Proposition 4.1) shows that a necessary condition for a cube packing to be extendible to a cube tiling is that it be orthogonal. A natural question is: Can every orthogonal cube packing in $\mathbb{R}^n$ be completed to a cube tiling of $\mathbb{R}^n$? The answer is "yes" in dimensions 1 and 2, as can be easily checked. However, we show that it is "no" in dimensions 3 and above.

Theorem A.1. *In each dimension $n \ge 3$ there is an orthogonal cube packing that does not extend to a cube tiling of $\mathbb{R}^n$.*

Proof. In dimension 3, consider the set of four cubes $\{v^{(i)} + [0,1]^3 : 1 \le i \le 4\}$ in $\mathbb{R}^4$, given by

$$\begin{aligned} v^{(1)} &= \left(-1, \quad 0, \quad -\frac{1}{2}\right) \\ v^{(2)} &= \left(-\frac{1}{2}, \quad -1, \quad 0\right) \\ v^{(3)} &= \left(0, \quad -\frac{1}{2}, \quad -1\right) \\ v^{(4)} &= \left(\frac{1}{2}, \quad \frac{1}{2}, \quad \frac{1}{2}\right) \end{aligned}$$

The orthogonality condition (4.2) is easily verified. The cubes corresponding to $v^{(1)}$ through $v^{(3)}$ contain $(0,0,0)$ on their boundary and create a corner $(0,0,0)$. Any cube tiling that extended $\{v^{(i)} + [0,1]^3 : 1 \le i \le 3\}$ would have to fill this corner by including the cube $[0,1]^3$. However $[0,1]^3$ has nonempty interior in common with $v^{(4)} + [0,1]^3$.

This construction easily generalizes to $\mathbb{R}^n$ for $n \ge 3$. ■

Appendix B. Commuting Self-Adjoint Partial Differential Operators

B. Fuglede [2] studied the problem of finding commuting self-adjoint extensions of the operators $-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}$ to suitable regions in $L^2(\Omega)$. Note that each operator $-i\frac{\partial}{\partial x_i}$ is a "Dirac operator" in the sense that it is the "square root" of the "Laplace operator" $\frac{\partial^2}{\partial x_i^2}$.

Definition B.1 A *Nikodym region* Ω in $\mathbb{R}^n$ is an open set such that every distribution u on Ω such that $D_j u \in L^2(\Omega)$ for $1 \le j \le n$ necessarily has $u \in L^2(\Omega)$.

Any bounded open subset of $\mathbb{R}^n$ of finite measure which is star-shaped with respect to some interior point is a Nikodym region [1, p. 332]. Thus the open unit cube $(0,1)^n$ is a Nikodym region.

Let D_j denote the operator $\frac{\partial}{\partial x_j}$ extended to its maximal domain in $L^2(\Omega)$, given by

$$\mathrm{dom}(D_j) := \{u \in L^2(\Omega) : D_j u \in L^2(\Omega)\}\ , \tag{B.1}$$

where D_j acts in the sense of distributions on $L^2(\Omega)$. Fuglede [2, Theorem 1], proved the following[3] result.

Theorem B.1. (Fuglede) *Suppose that $\Omega \subset \mathbb{R}^n$ is a Nikodym region.*

(i) *Let $H = (H_1, \ldots, H_n)$ denote a commuting family (if any) of self-adjoint restrictions H_j of D_j on $L^2(\Omega)$. Then H has a discrete joint spectrum $\sigma(H)$ in which each point $2\pi\lambda \in \sigma(H)$ is a simple eigenvalue with eigenspace $\mathbb{C}e_\lambda$, and if $\Lambda = \frac{1}{2\pi}\sigma(H)$ then $\mathcal{B}_\Lambda = \{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal basis of $L^2(\Omega)$.*

(ii) *Conversely, let Λ be a subset (if any) of $\mathbb{R}^n$ such that $\{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal basis for $L^2(\Omega)$. Then there exists a unique commuting family $H = (H_1, \ldots, H_n)$ of self-adjoint restrictions H_j of D_j on $L^2(\Omega)$ such that $\{e_\lambda : \lambda \in \Lambda\} \subset \mathrm{dom}(H)$, or equivalently that $\Lambda = \frac{1}{2\pi}\sigma(H)$.*

We apply this theorem to the special case where $\Omega = [0,1]^n$ is the n-cube. Theorem 1.1 classified all orthogonal bases of exponentials for Ω, and the result above shows that there is a unique commuting family H_Λ associated to each cube tiling Λ. Can one give a precise description of H_Λ in terms of the data Λ?

[3] Note that our exponential e_λ corresponds to Fuglede's exponential $e_{2\pi\lambda}$.

The self-adjoint extensions of $-i\frac{\partial}{\partial x_j}$ acting on $C^\infty([0,1]^n)$ inside the Hilbert space $L^2([0,1]^n)$ may be thought of as being specified by boundary conditions; this is described in Jorgensen and Pedersen [6, Lemma 3.1]. The boundary conditions for $-i\frac{\partial}{\partial x_j}$ are imposed on the two opposite $(n-1)$-faces of the cube $H_j^{(0)}$ and $H_j^{(1)}$ given by

$$H_j^{(k)} := \{x \in [0,1] : x_j = k\} \quad \text{for} \quad k = 0, 1 \ . \tag{B.2}$$

Each self-adjoint extension V_j of $-i\frac{\partial}{\partial x_j}$ corresponds to a partial isometry

$$U_{V_j} : \mathcal{D}_+^{(j)} \longrightarrow \mathcal{D}_-^{(j)} \ ,$$

in which $\mathcal{D}_+^{(j)} \subseteq L^2(H_j^{(0)})$ and $\mathcal{D}_-^{(j)} \subseteq L^2(H_j^{(1)})$ are suitable dense subspaces. Can the boundary condition operators $(U_{V_1}, \ldots U_{V_n})$ for $H = H_\Lambda$ be explicitly constructed for a tiling Λ?

As one example, consider the translated Fourier basis $\Lambda = \mathbb{Z} + t$. If we identify each $L_2(H_j^{(i)})$ with $L^2([0,1]^{n-1})$ in the obvious way, then the corresponding boundary conditions are given by

$$U_{V_j}(f) = e^{2\pi i t_j} f, \quad 1 \leq j \leq n \ . \tag{B.3}$$

Here the domain and range of U_{V_j} are all of $L^2([0,1]^{n-1})$.

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