Local Complexity of Delone Sets and Crystallinity

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Abstract

This paper characterizes when a Delone set X in $\mathbb{R}^n$ is an ideal crystal in terms of restrictions on the number of its local patches of a given size or on the heterogeneity of their distribution. For a Delone set X, let $N_X(T)$ count the number of translation-inequivalent patches of radius T in X and let $M_X(T)$ be the minimum radius such that every closed ball of radius $M_X(T)$ contains the center of a patch of every one of these kinds. We show that for each of these functions there is a "gap in the spectrum" of possible growth rates between being bounded and having linear growth, and that having sufficiently slow linear growth is equivalent to X being an ideal crystal.

Explicitly, for $N_X(T)$, if R is the covering radius of X then either $N_X(T)$ is bounded or $N_X(T) \geq T/2R$ for all $T > 0$. The constant $1/2R$ in this bound is best possible in all dimensions.

For $M_X(T)$, either $M_X(T)$ is bounded or $M_X(T) \geq T/3$ for all $T > 0$. Examples show that the constant $1/3$ in this bound cannot be replaced by any number exceeding $1/2$. We also show that every aperiodic Delone set X has $M_X(T) \geq c(n)\,T$ for all $T > 0$, for a certain constant $c(n)$ which depends on the dimension n of X and is $> 1/3$ when $n > 1$.

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1. Introduction

The discovery in 1984 of quasicrystalline materials, which are strongly ordered aperiodic structures, has generated renewed interest in precisely delineating the boundary between discrete sets that are fully periodic (ideal crystals) and those with less global order, see Radin [24]. This paper studies questions in discrete geometry, investigating how strong the restrictions on local features of discrete point sets in $\mathbb{R}^n$ must be to enforce crystallinity. Such questions originally arose in geometric crystallography, see [9]. The questions we study also have connections with the ergodic theory of $\mathbb{R}^n$-actions [17]; there are parallel questions in the combinatorics of words, in one or several dimensions, which we briefly consider.

We study the patch-counting function $N_X(T)$ and the repetitivity function $M_X(T)$ of Delone sets X, where patches are identified only up to translation equivalence. (These concepts were studied in [17] and are defined in §2.) We show that both functions have a "gap in the spectrum" of their possible growth rates: either they are bounded, which happens precisely when X is an ideal crystal, or they grow at least linearly in the radius T of the patch. Growth rates like $\log T$ or $\sqrt{T}$ are impossible for these functions. These results can be viewed as characterizations of ideal crystals: when X is an ideal crystal both functions are bounded, but nevertheless sufficiently slow linear growth of either of these functions is also enough to characterize ideal crystals. They can alternatively be viewed as giving linear lower bounds on the growth rates of these functions for Delone sets X that are not ideal crystals.

The problem of characterizing ideal crystals in terms of restrictions on allowed types of local patches was first studied for "regular point systems" in 1976 by Delone et al. [6] for isometry-equivalence classes of local patches. They identified a radius T (depending on the dimension and the Delone constants of X) such that every Delone set X whose patches of radius T are all isometry-equivalent is a regular point system and therefore an ideal crystal. Dolbilin et al. [7] gave an extension of this result to "multiregular point systems", which comprise all ideal crystals: for each $k \geq 1$ they gave a radius T (depending on the dimension, the Delone constants of X and on k) such that if there are at most k isometry-equivalence types of patches of this radius, then X is an ideal crystal with at most k isometry-equivalence types of patches of *any* radius. This gives an effective way of deciding, for a given k, whether X has degree of regularity $\leq k$. It does not, however, give a way of deciding whether a given set X has some finite degree of regularity; that is, whether X is an ideal crystal. This latter drawback is unavoidable, because with any finite sample of X there is no way to tell that it is not part of an ideal crystal whose unit cell is larger than the size of the sample. However, the dependence of T on k in [7] is linear, and it follows that there is a constant c (depending only on the dimension and the Delone constants) such that if there is *some* T with at most cT isometry patches of radius T then X is an ideal crystal.

For the patch-counting function $N_X(T)$, our result (Theorem 2.1) is an analogue of this "small linear growth rate" characterization of ideal crystals, in which patches are identified only up to translation-equivalence. This is a finer equivalence relation on patches than isometry-equivalence, and stronger lower bounds hold for non-crystals. The corresponding coefficient of T we obtain for translation-equivalence types of patches (for a linear growth rate of $N_X(T)$ in the gap between crystals and non-crystals) depends only on the covering radius of X, not on its packing radius or dimension, and we show that it is best possible in all dimensions. We conjecture that certain super-linear polynomial growth rates of $N_X(T)$, while not guaranteeing that X is an ideal crystal, are sufficient to ensure that X has some periods—more precisely that if, for a sufficiently small c, $N_X(T) < cT^{n-j+1}$ for all large T then X has j independent periods—but no cases other than $j = n$ have yet been proved.

For the repetitivity function $M_X(T)$, which is unaffected by changes of scale of the Delone set X, the coefficient of T we obtain in Theorem 2.2 for a linear growth rate in the gap between crystals and non-crystals is the absolute constant $\frac{1}{3}$, independent of dimension. We show by examples that this cannot be improved beyond $\frac{1}{2}$. In Theorem 2.3 we also show that a coefficient slightly larger than $\frac{1}{3}$ (but depending on the dimension n) in the linear growth rate of $M_X(T)$ is enough to ensure that X has a non-zero period when $n > 1$. This coefficient depends on the packing-covering constant $\kappa(n)$ for n-dimensional Delone sets, a concept originally studied by Ryshkov [26], which is defined in §2. In the final section we study various properties of this constant.

The results of this paper are analogous to results in the combinatorics of words (or symbolic dynamics) on the lattice $\mathbb{Z}^n$. They can even be viewed as generalizing such results, because symbolic words on $\mathbb{Z}^n$ can be encoded as Delone sets by placing points near the lattice points in $\mathbb{Z}^n$ using small dispacements to distinguish the symbol types. A well known result in the combinatorics of words on the lattice $\mathbb{Z}$—the one-dimensional case—given in Morse and Hedlund [21] is that if $N_S(m)$ is the number of different words of length m in a two-sided infinite sequence S in which A distinct symbols occur then either S is periodic and $N_S(m) \leq P$ (the length of the period) for all m or

$$N_S(m) \geq m + A - 1 \text{ for all } m. \tag{1.1}$$

Associated to any such symbol sequence is a dynamical system with a $\mathbb{Z}$-action (the closure of its orbit under the shift) and this result can be interpreted as a condition for this dynamical system to be finite. In the context of higher dimensional symbolic dynamical systems, with symbols on the lattice $\mathbb{Z}^n$ and a corresponding $\mathbb{Z}^n$-action, the question has also been raised to what extent growth restrictions on the number of local symbol patterns (in rectangular patches) enforce periodicity, see for example Sander and Tijdeman [27, 28, 29] and Berthé and Vuillon [2, 3] for a discussion of the two dimensional case. Symbolic dynamics analogues of the patch-counting function are called "permutation numbers" or "complexity functions" in this context, cf. [11, 18]. The approach of Theorem 2.1 can be used to show that a low enough complexity bound on the number of rectangular patches of such systems enforces full-dimensional periodicity. For square (or, more generally, cubic) patches, however, one can directly obtain an optimal result in n dimensions, given as Theorem 4.1 in §4; the proof is a straightforward extension of the proof of Morse and Hedlund [21, Theorem 7.4] of (1.1) in one dimension.

The analogue in symbolic dynamics of the repetitivity function is the *recurrence function* $R_S(m)$, introduced by Morse and Hedlund [21, 22] in 1938. Given a two-sided infinite symbol sequence S, $R_S(m)$ is the shortest length such that every word of length $R_S(m)$ contains a copy of every word of length m that occurs in S. To aid comparison with our repetitivity function we shall describe results in terms of a related function $M_S(m) = R_S(m) - m$, which represents the maximum distance between the leading symbols of any two successive identical words of length m. Put in terms of $M_S(m)$, Morse and Hedlund showed [21, Theorem 7.5] that for any aperiodic one-dimensional repetitive sequence S in which A distinct symbols occur

$$M_S(m) \geq m + A - 1 \text{ for all } m \geq 1 \tag{1.2}$$

and [22, p.2] that

$$\limsup_{m\to\infty} \frac{M_S(m)}{m} \geq \tau + 1, \tag{1.3}$$

where $\tau = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio. Moreover (1.3) holds with equality when S is the Fibonacci sequence, so represents an optimal "gap in the spectrum" result. This suggests the possibility that the limiting constant $\tau + 1$ in the analogous Delone set inequality on the right of (6.9) in Theorem 6.2 may also be optimal. The inequality (1.2) follows from (1.1) and the trivial estimate $M_S(m) \geq N_S(m)$ (the analogue for symbol sequences of Theorem 4.1 of [17]) yet the coefficient of m in (1.2) is 1, which is significantly larger than the $\frac{1}{3}$ in our Theorem 2.2, the analogous result for Delone sets. This seems largely due to m being restricted to integer values in the symbolic case.

As mentioned, some of the bounds in this paper depend on the n-dimensional Delone packing-covering constant $\kappa(n)$, defined in §2, which was introduced and studied by Ryshkov [26] in 1974. In the final section we present basic properties of this constant and the related n-dimensional lattice packing-covering constant $\kappa_L(n)$, also introduced by Ryshkov. Most of the results in that section are not new, but we recall them for completeness and in order to raise the following open question: Is the function $\kappa_L(n)$ unbounded as $n \to \infty$? This question is of great interest, because in any dimension n with $\kappa_L(n) > 2$ no lattice packing can be a densest sphere packing.

2. Statements of Results

Definition 2.1 A *Delone set*, or (r, R)*-set*, is a discrete set X in $\mathbb{R}^n$ that is *uniformly discrete* (i.e. its packing radius r by equal balls is positive) and *relatively dense* (i.e. its covering radius R by equal balls is finite). We call the values r and R the *Delone constants* of the set X.

Clearly $r \leq R$, with equality only when X is a one-dimensional set of equally spaced points. Our definition of the Delone constant r differs slightly from other authors (see [6, 7, 26], for example) who take for this value the infimum $2r$ of the inter-point distances in place of the packing radius r.

Definition 2.2 A Delone set X in $\mathbb{R}^n$ is an *ideal crystal* if it has a full rank lattice of translation symmetries, i.e. if $X = \Lambda + F$, where Λ is a full rank lattice in $\mathbb{R}^n$ and F is a finite set.

Note that an *aperiodic set* is one with no global translation symmetries, while a *non-crystalline set* may have some translation symmetries, but not a full rank set of them.

Definition 2.3 For a Delone set X, a T*-patch* centered at the point $\mathbf{x} \in X$ is

$$\mathcal{P}_X(\mathbf{x}; T) := X \cap B(\mathbf{x}; T) ,$$

where $B(\mathbf{x}; T)$ is the open ball with center $\mathbf{x}$ and radius T, and the *patch-counting function* $N_X(T)$ is the number (possibly infinite) of translation-inequivalent T-patches centered at points $\mathbf{x}$ of X. We use the notation $\mathcal{P}(\mathbf{x}; T) \sim \mathcal{P}(\mathbf{y}; T)$ to mean translation equivalence of patches, i.e. that $\mathcal{P}(\mathbf{y}; T) = \mathcal{P}(\mathbf{x}; T) + \mathbf{y} - \mathbf{x}$.

The function $N_X(T)$ is a non-decreasing function of T and $N_X(T) = 1$ for $T \leq 2r$. It is possible that $N_X(T)$ may be infinite for large T.

Definition 2.4 A Delone set X with $N_X(T)$ finite for all T is said to have *finite local complexity.*

Delone sets with finite local complexity are studied in [15, 16, 17], where they are called Delone sets of finite type. The growth rate of the patch-counting function $N_X(T)$ provides a quantitative measure of the complexity of X. The slowest possible growth rate for $N_X(T)$ is to be eventually constant, which occurs when X is an ideal crystal, as we review in §3. Our first main result is that except for ideal crystals the slowest possible growth rate is at least linear in T. Dolbilin et al. [7, Theorem 1.3] have already shown, in the context of counting isometry classes of patches, that $N_X(T) < c(n, r, R)T$ for a single value of T implies that X is a crystal, where

$$c(n, r, R) = \frac{1}{2(n^2+1)R\log_2(\frac{R}{r}+2)},$$

and this holds *a fortiori* for our function $N_X(T)$ which counts the more numerous translation classes of patches. But with our finer classification of patches we can obtain a larger value of the constant, which is independent of the dimension.

Theorem 2.1 *If a Delone set X in $\mathbb{R}^n$ with covering radius R has a single value of $T > 0$ such that*

$$N_X(T) < \frac{T}{2R}, \tag{2.1}$$

then X is an ideal crystal.

The constant $\frac{1}{2R}$ here is optimal, in the sense that for any $c > \frac{1}{2R}$ there are Delone sets $X \in \mathbb{R}^n$ with $N_X(T) < cT$, for some $T > 0$, that are not ideal crystals, as we show in §6.

In [17, Conjecture 2.2] we put forward the following conjecture, which says that slow growth of $N_X(T)$ implies that X has many independent periods:

Period Conjecture. *For each integer $j = 1, \dots, n$ there is a positive constant $c_j(n, r, R)$ such that any Delone set X in $\mathbb{R}^n$ with Delone constants r, R that satisfies*

$$N_X(T) < c_j(n, r, R) T^{n-j+1} \textit{ for all } T > T_0(X)$$

has j linearly independent periods.

Weaker forms of this conjecture would be to allow c_j to depend on X or merely to assert that $T^{j-n-1} N_X(T) \to 0$ as $T \to \infty$ implies that X has j linearly independent periods. Theorem 2.1 is a strong form of the case $j = n$ of the conjecture. The only other case that has been proved to date is a strong form of the case $n = 2$, $j = 1$ in the context of doubly infinite arrays of symbols instead of Delone sets [10].

The basis of the conjecture is the feeling that a pattern with less than j independent periods should have "$(j-1)$-dimensional recognizable features" at arbitrarily large scales, causing $N_X(T)$ to be of order at least T^{n-j+1} for some arbitrarily large values of T; but concrete evidence for it is limited. We formulate it in order to focus attention on the problem, rather than to strongly assert its truth. The formulation above is the strongest consistent with currently known constraints. In particular, the conjecture cannot be strengthened to assert the conclusion when the inequality is satisfied for only a single value of T, as Theorem 2.1 does: in [17, Theorem 2.2] we give examples, for any $n \geq 3$ and $\epsilon > 0$, of aperiodic sets $X \subset \mathbb{R}^n$ for which there exist arbitrarily large radii T with $N_X(T) < T^{\lceil (n+1)/2 \rceil + \epsilon}$. Also there is no corresponding conjecture (except possibly for $n = 2$, $j = 1$) generalizing the result of [7] for the function that counts patches up to isometry: [8] gives examples of aperiodic sets $X \subset \mathbb{R}^n$, for $n \geq 3$, whose isometry-patch-counting functions are $O(T^{1+\epsilon})$ for any $\epsilon > 0$.

We next consider sets that have restrictions on the distribution of patches of a given type.

Definition 2.5 For a Delone set X, the *repetitivity function* $M_X(T)$ is the least M (possibly infinite) such that every closed ball $\bar{B}$ of radius M contains the center of a T-patch of every kind that occurs in X. That is, for every T-patch $\mathcal{P}$ of X, $\bar{B}$ contains a point of X which is the center of a T-patch of X that is a translate of $\mathcal{P}$. (This T-patch may extend outside the ball $\bar{B}$.)

Another way of expressing this definition is to say that $M_X(T)$ is the largest of the covering radii of the sets of centers of patches of X translation-equivalent to $\mathcal{P}$, taken over all T-patches $\mathcal{P}$ of X. The function $M_X(T)$ is a non-decreasing function of T and $M_X(T) = R$ for $T \leq 2r$. For large T, $M_X(T)$ may become infinite.

Definition 2.6 (i) A Delone set X with $M_X(T)$ finite for all T is said to be *repetitive.*[1]

(ii) A Delone set X is *linearly repetitive* if there is a constant c such that $M_X(T) < cT$ for all $T > 0$.

Repetitive Delone sets necessarily have finite local complexity, since the definition implies that $N_X(T)$ is finite for all T. The repetitivity function $M_X(T)$ of a Delone set provides a second quantitative measure of its complexity, supplementing $N_X(T)$. Ideal crystals are repetitive and have $M_X(T)$ bounded.

Our second main result is that except for ideal crystals the slowest possible growth rate of $M_X(T)$ is at least linear in T.

Theorem 2.2 *If a Delone set X in $\mathbb{R}^n$ has a single value of $T > 0$ such that*

$$M_X(T) < \frac{1}{3}\,T, \tag{2.2}$$

then X is an ideal crystal.

The constant $\frac{1}{3}$ here is independent of the Delone constants r and R (which is made possible by the fact that the repetitivity function is independent of scale) and it is even independent of the dimension n. It cannot be increased to be larger than $\frac{1}{2}$ (for any n) as we show in §6.

For $n \geq 2$ we can show that a slightly larger value of the constant in Theorem 2.2, while not necessarily guaranteeing that X is an ideal crystal, at least gives a sufficient condition for X to have a non-zero period. Before stating this result we need to define some constants.

Definition 2.7 (i) For a Delone set X we call the ratio $\kappa_X = R/r$ the *packing-covering ratio* of X.

(ii) The *Delone packing-covering constant* $\kappa(n)$, for dimension n, is the infimum of κ_X over all Delone sets X in $\mathbb{R}^n$.

(iii) The *lattice packing-covering constant* $\kappa_L(n)$, for dimension n, is the infimum of κ_Λ over all lattices Λ in $\mathbb{R}^n$.

These constants were intoduced by Ryshkov [26], who actually studied the quantities $\frac{1}{2}\kappa(n)$ and $\frac{1}{2}\kappa_L(n)$ (corresponding to taking $2r$ instead of r as the first Delone constant, as mentioned after Definition 2.1). If we now put

$$c(n) := \frac{\kappa(n)}{\kappa(n)+2}$$

we have:

Theorem 2.3 *If a Delone set X in $\mathbb{R}^n$ has a single value of $T > 0$ such that*

$$M_X(T) < c(n)\,T, \tag{2.3}$$

then X has a non-zero period.

[1] There is a parallel concept for tilings, where various terms are used: "almost periodic" in [32], "tilings with local isomorphism" in [13, 25], and "repetitive" in [30]. For symbolic dynamical systems the term "uniformly recurrent" is used.

Clearly $\kappa(1) = 1$ (so $c(1) = \frac{1}{3}$, reflecting the fact that in dimension 1 having a non-zero period is equivalent to being an ideal crystal) while $\kappa(n) > 1$ for $n \geq 2$. In §7 we present some results about $\kappa(n)$ and $\kappa_L(n)$, including the result of Ryshkov [26] that $\kappa(n) \leq 2$, which implies that $\frac{1}{3} < c(n) \leq \frac{1}{2}$ for $n \geq 2$. Thus Theorem 2.3 gives at most a very slight dimension-dependent improvement on Theorem 2.2. In Theorem 6.3 in §6 we show that one cannot increase $c(n)$ in Theorem 2.3 to any value exceeding $\frac{1}{2}\kappa(n) \leq 1$.

3. Bounded Patch-Counts and Ideal Crystals

A Delone set X is an ideal crystal if and only if its patch-counting function $N_X(T)$ is bounded. We have the following more precise result.

Theorem 3.1 *If a Delone set X in $\mathbb{R}^n$ has $N_X(T)$ bounded with maximum value N, then X is a union of N cosets of a lattice in $\mathbb{R}^n$.*

Proof. For each $T > 0$ we can classify the points of X according to their T-patches. If $\mathbf{x}$ and $\mathbf{y}$ have different T-patches then they have different T'-patches for every $T' > T$, so increasing T refines the classification. If $N_X(T)$ has maximum value N there is a U with $N_X(U) = N$ and classifying points of X according to their U-patches gives a partition

$$X = X_1 \cup X_2 \cup \cdots \cup X_N$$

of X such that points in the same class have identical T-patches for all T and points in different classes have different T-patches for all $T \geq U$. There is no loss of generality in translating X so that $\mathbf{0} \in X_1$. For any two points $\mathbf{x}$ and $\mathbf{y}$ we have $B(\mathbf{0}; U) \subseteq B(\mathbf{y}; U + \|\mathbf{y}\|)$ and $B(\mathbf{x} - \mathbf{y}; U) \subseteq B(\mathbf{x}; U + \|\mathbf{y}\|)$. When $\mathbf{x}$ and $\mathbf{y}$ are both in X_1 their $(U + \|\mathbf{y}\|)$-patches are equal, so the translation by $\mathbf{x} - \mathbf{y}$ that takes $B(\mathbf{y}; U + \|\mathbf{y}\|)$ to $B(\mathbf{x}; U + \|\mathbf{y}\|)$ maps the points of X in $B(\mathbf{0}; U)$ one-to-one onto the points of X in $B(\mathbf{x} - \mathbf{y}; U)$. Thus $\mathbf{x} - \mathbf{y} \in X$ and $\mathcal{P}_X(\mathbf{x} - \mathbf{y}; U) \sim \mathcal{P}_X(\mathbf{0}; U)$. Hence $\mathbf{x} - \mathbf{y} \in X_1$. This shows that X_1 is an additive subgroup of $\mathbb{R}^n$.

Next take any $\mathbf{x}_1 \in X_1$ and $\mathbf{x}_2 \in X_2$ (if $N > 1$). Then $\mathcal{P}_X(\mathbf{x}_1; U + \|\mathbf{x}_2\|) \sim \mathcal{P}_X(\mathbf{0}; U + \|\mathbf{x}_2\|)$, and in a similar way we can use the translation by $\mathbf{x}_2$ to show that $\mathbf{x}_1 + \mathbf{x}_2 \in X$ and has the same U-patch as $\mathbf{x}_2$. Hence $\mathbf{x}_1 + \mathbf{x}_2 \in X_2$ and $X_2 \supseteq X_1 + \mathbf{x}_2$. Finally, for any other point $\mathbf{y}_2 \in X_2$ we have $\mathcal{P}_X(\mathbf{y}_2; U + \|\mathbf{x}_2\|) \sim \mathcal{P}_X(\mathbf{x}_2; U + \|\mathbf{x}_2\|)$ and the translation by $-\mathbf{x}_2$ shows that $\mathbf{y}_2 - \mathbf{x}_2 \in X_1$. Hence $X_2 - \mathbf{x}_2 \subseteq X_1$ so $X_2 = X_1 + \mathbf{x}_2$ is a coset of X_1 in $\mathbb{R}^n$. In the same way, each X_i is a coset of X_1.

Since the Delone set X is the union of finitely many translates of X_1, X_1 itself is a Delone set and, being a subgroup of $\mathbb{R}^n$, is a full rank lattice. $\square$

Corollary 3.1 *For Delone sets X,*

$$\begin{aligned} N_X(T) = 1 \text{ for all } T &\iff X \text{ is a translate of a lattice, and} \\ N_X(T) \text{ is bounded} &\iff X \text{ is an ideal crystal.} \end{aligned}$$

Proof. The implications in one direction follow from Theorem 3.1. For the other direction, if $X = \Lambda + F$ is an ideal crystal then the orbits of X under its group of translation symmetries are cosets of a lattice containing the lattice Λ and the number of orbits is at most the cardinality of the finite set F. Clearly points in the same orbit have identical T-patches for all T. $\square$

4. Linear Patch-Counts and Ideal Crystals

The object of this section is to show that if X is not an ideal crystal then there is a linear lower bound on the growth rate of $N_X(T)$. The main step in doing this is the following lemma, which is analogous to the result in the combinatorics of words that if the word-counting function of an infinite sequence of symbols is the same for two consecutive word lengths then it remains the same for all greater word lengths (cf. our proof of Theorem 4.1 below).

Lemma 4.1 *If X is a Delone set with covering radius R and there are radii $U > 0$ and $V > U + 2R$ such that $N_X(V) = N_X(U)$ then $N_X(T) = N_X(U)$ for all $T \geq U$, and X is an ideal crystal.*

Proof. If $\mathbf{x}$ and $\mathbf{y}$ are points of X with $\mathcal{P}_X(\mathbf{x}; U) \not\sim \mathcal{P}_X(\mathbf{y}; U)$ then clearly $\mathcal{P}_X(\mathbf{x}; V) \not\sim \mathcal{P}_X(\mathbf{y}; V)$. So if $N_X(V) = N_X(U)$ then, for each $\mathbf{x} \in X$, $\mathcal{P}_X(\mathbf{x}; V)$ is determined uniquely by $\mathcal{P}_X(\mathbf{x}; U)$.

We next show that $\mathcal{P}_X(\mathbf{x}; 2V - U - 2R)$ is also determined uniquely by $\mathcal{P}_X(\mathbf{x}; U)$. Take any $\mathbf{x}_0 \in X$, fixed for the moment, and consider a point $\mathbf{c} \in B(\mathbf{x}_0; 2V - U - 2R)$. We can find a point $\mathbf{d} \in B(\mathbf{x}_0; V - U - R)$ with $\|\mathbf{d} - \mathbf{c}\| < V - R$ and, by the definition of R, a point $\mathbf{x}$ of X in $B(\mathbf{d}; R)$. Then $B(\mathbf{x}; U) \subset B(\mathbf{x}_0; V)$ and $\mathbf{c} \in B(\mathbf{x}; V)$. So $\mathcal{P}_X(\mathbf{x}_0; V)$ determines $\mathcal{P}_X(\mathbf{x}; U)$ which in turn determines $\mathcal{P}_X(\mathbf{x}; V)$. Since $\mathbf{c}$ was an arbitrary point in $B(\mathbf{x}_0; 2V - U - 2R)$,

$$B(\mathbf{x}_0; 2V - U - 2R) \subset \bigcup_{\mathbf{x}} B(\mathbf{x}; V)$$

where the union is over all points $\mathbf{x} \in \mathcal{P}_X(\mathbf{x}_0; V - U)$. Thus the points of X in $B(\mathbf{x}_0; 2V - U - 2R)$ are determined uniquely by $\mathcal{P}_X(\mathbf{x}_0; V)$. Since $\mathbf{x}_0$ was an arbitrary point of X,

$$N_X(2V - U - 2R) = N_X(V) = N_X(U).$$

Iteration of this argument successively increases V by $V - U - 2R$ and shows that $N_X(T) = N_X(U)$ for all $T \geq U$. $\square$

Theorem 2.1 is an immediate consequence of this lemma.

Proof of Theorem 2.1. If X is not an ideal crystal then, by Corollary 3.1, $N_X(T)$ is unbounded. Certainly $N_X(\epsilon_0) \geq 1$ for any $\epsilon_0 > 0$. Since $N_X(T)$ is unbounded,

$$N_X(2R + \epsilon_0 + \epsilon_1) > N_X(\epsilon_0) \text{ for any } \epsilon_1 > 0,$$

by Lemma 4.1, where R is the covering radius of X, by Lemma 4.1. Hence $N(2R + \epsilon_0 + \epsilon_1) \geq 2$. Repeating this argument we find $N(4R + \epsilon_0 + \epsilon_1 + \epsilon_2) \geq 3, \ldots, N(2mR + \epsilon_0 + \epsilon_1 + \cdots + \epsilon_m) \geq m + 1$ for any integer $m \geq 0$, where $\epsilon_0, \ldots, \epsilon_m$ can be chosen arbitrarily small. It follows that $N_X(T) \geq T/2R$ for every $T > 0$. $\square$

One can prove an analogous result in multi-dimensional symbolic dynamics, obtaining a condition for full periodicity of a symbol pattern on the lattice $\mathbb{Z}^n$. We consider symbolic words S on the lattice $\mathbb{Z}^n$, drawn from a finite alphabet $\mathcal{A}$, i.e. $S \in \mathcal{A}^{\mathbb{Z}^n}$.

Theorem 4.1 *Let S be a symbolic word on the lattice $\mathbb{Z}^n$ in which A different symbols occur and let $N_S(m)$ count the number of different symbol patterns in S over all lattice cubes with m lattice points along each side. If, for some integer $m \geq 1$,*

$$N_S(m) < m + A - 1, \tag{4.1}$$

then S is fully periodic, with a full rank period lattice $\Lambda \subset \mathbb{Z}^n$.

Proof. Suppose that $N_S(m_0 + 1) = N_S(m_0)$ for some m_0. Then whenever two cubes of side m_0 with centers at $\mathbf{x}$ and $\mathbf{y}$ (where $\mathbf{x}$ and $\mathbf{y}$ are lattice points if m_0 is odd and half lattice points if m_0 is even) have identical symbol patterns so do the cubes of side $m_0 + 1$ with centers at $\mathbf{x} + (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and $\mathbf{y} + (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, since otherwise the number of symbol patterns occurring in cubes of side $m_0 + 1$ would be at least one greater than the number occurring in cubes of side m_0. The same holds when $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is replaced by $(\pm\frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$ for any choice of signs. Consequently the symbol pattern on any cube of side m_0 determines uniquely the symbol pattern on the cube of side $m_0 + 2$ with the same center. (This cube consists of the original cube surrounded by an extra shell of lattice points.) Hence $N_S(m_0 + 2) = N_S(m)$ and it follows by induction that the symbol pattern on any cube of side m_0 determines the pattern on the whole of $\mathbb{Z}^n$. Take any vector $\boldsymbol{\lambda} \in \mathbb{Z}^n$. Since there are only finitely many symbols there must be two cubes of side m_0 with the same symbol pattern such that one is a translate of the other by $p\boldsymbol{\lambda}$, for some $p \in \mathbb{Z}$. Since each cube determines the same global pattern, $p\boldsymbol{\lambda}$ is a period of S. Since it has periods in every lattice direction, S is fully periodic.

It follows that if S is not fully periodic then $N_S(m)$ is a strictly increasing function of m, and hence

$$N_S(m) \geq N_S(1) + m - 1 = A + m - 1.$$

□

Remark. The one-dimensional case of Theorem 4.1 is well known and is due to Morse and Hedlund [21, Theorem 7.4] in 1938. The proof above is an extension of the one-dimensional proof. The result is best possible in all cases: one-dimensional Sturmian sequences S have $A = 2$ and $N_S(m) = m + 1$ for all $m > 1$, and examples in higher dimensions are given by patterns that are Sturmian in one coordinate direction and constant in all other coordinate directions. There exist optimal examples for all other values of A as well.

5. Linear Repetitivity and Ideal Crystals

In [17] we studied *linearly repetitive* sets, which are Delone sets X for which there is some constant c with $M_X(T) < cT$ for all T. There are many examples of aperiodic linearly repetitive sets associated to self-similar constructions. Theorem 2.2 asserts that linear repetitivity, with a sufficiently small constant, is sufficient to force a set X to be an ideal crystal, and Theorem 2.3 asserts that linear repetitivity with a slightly larger constant, depending on the dimension, forces at least one period.

Proof of Theorem 2.2. Without loss of generality we may suppose that $\mathbf{0} \in X$. Consider the patch $\mathcal{P} := \mathcal{P}_X(\mathbf{0};T) = B(\mathbf{0};T) \cap X$, where T satisfies (2.2), and let $\Sigma_{\mathcal{P}} := \{\mathbf{y} \in X : \mathcal{P}_X(\mathbf{y};T) \sim \mathcal{P}\}$ be the set of centers of T-patches of X translation-equivalent to $\mathcal{P}$.

We first note that any vector $\mathbf{p} \in \Sigma_{\mathcal{P}} \cap B(\mathbf{0};\frac{2}{3}T)$ is a period of X. This is because $\mathbf{x} \in \mathcal{P}$ and $\mathbf{p} \in \Sigma_{\mathcal{P}}$ implies $\mathbf{x}+\mathbf{p} \in \mathcal{P}+\mathbf{p} = \mathcal{P}_X(\mathbf{p};T) \subset X$. If $\mathbf{x} \in \mathcal{P}_X(\mathbf{0};\frac{1}{3}T)$, then $\mathbf{x}+\mathbf{p} \in B(\mathbf{0};T) \cap X = \mathcal{P}$ and so, for all $\mathbf{y} \in \Sigma_{\mathcal{P}}$,

$$\mathbf{x} \in \mathcal{P}_X(\mathbf{y};\tfrac{1}{3}T) \quad \Rightarrow \quad \mathbf{x}+\mathbf{p} \in \mathcal{P}_X(\mathbf{y};T) \subset X.$$

Since $M_X(T) < \frac{1}{3}T$ the balls $B(\mathbf{y};\frac{1}{3}T)$ with $\mathbf{y} \in \Sigma_{\mathcal{P}}$ cover $\mathbb{R}^n$ and therefore $\mathbf{p}$ is a period of X.

It remains to show that $B(\mathbf{0};\frac{2}{3}T)$ contains n linearly independent vectors $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ of $\Sigma_{\mathcal{P}}$. Let B_1 be any ball of radius $\frac{1}{3}T$ with $\mathbf{0}$ on its boundary. Since $\frac{1}{3}T > M_X(T)$, B_1 contains a non-zero point $\mathbf{p}_1$ of $\Sigma_{\mathcal{P}}$. When $\mathbf{p}_1, \dots, \mathbf{p}_i$ have been chosen let B_{i+1} be any ball of radius $\frac{1}{3}T$ that touches the linear subspace $\langle \mathbf{p}_1, \dots, \mathbf{p}_i \rangle$ at $\mathbf{0}$. Then there is a vector $\mathbf{p}_{i+1}$ of $\Sigma_{\mathcal{P}}$ in B_{i+1}, which is necessarily linearly independent of $\mathbf{p}_1, \dots, \mathbf{p}_i$. Continuing in this way we find n linearly independent vectors $\mathbf{p}_1, \dots, \mathbf{p}_n$ which all lie in $B(\mathbf{0};\frac{2}{3}T)$, since each $B_i \subset B(\mathbf{0};\frac{2}{3}T)$. □

Proof of Theorem 2.3. Let $\mathcal{P}$ and $\Sigma_{\mathcal{P}}$ be as in the proof of Theorem 2.2. Since $M_X(T)$ is at least as large as the covering radius of $\Sigma_{\mathcal{P}}$ the argument used in the proof of that theorem shows that if there are two distinct points $\mathbf{x}, \mathbf{y} \in \Sigma_{\mathcal{P}}$ with

$$\|\mathbf{x}-\mathbf{y}\| < T - M_X(T) \tag{5.1}$$

then $\mathbf{x}-\mathbf{y}$ is a period of X. But the packing radius of $\Sigma_{\mathcal{P}}$ is $\leq M_X(T)/\kappa(n)$, so we can certainly find $\mathbf{x}, \mathbf{y} \in \Sigma_{\mathcal{P}}$ satisfying (5.1) provided

$$2M_X(T)/\kappa(n) < T - M_X(T),$$

that is, provided (2.3) holds. □

6. Non-crystals of Low Patch Complexity

Here we give examples that limit the extent to which the constants in Theorems 2.1, 2.2 and 2.3 can be increased. We first point out that, for trivial reasons, the constant $\frac{1}{2R}$ on the right of (2.1) cannot be improved and the constant $\frac{1}{3}$ on the right of (2.2) cannot be increased beyond $\frac{1}{2}$ for general Delone sets. This is because for any Delone set Y with Delone constants r, R we have $N_Y(T) = 1 = \frac{1}{2r}T$ and $M_Y(T) = R = \frac{R}{2r}T$ when $T = 2r$. When $n = 1$ there are non-crystals with $\frac{R}{r}$ arbitrarily close to 1, so $\frac{1}{2r}$ can be made as close as we like to $\frac{1}{2R}$ and $\frac{R}{2r}$ as close as we like to $\frac{1}{2}$. Higher dimensional examples can be found by taking $X = Y \times \eta\mathbb{Z}^{n-1}$ with small η. T hen X is a non-crystal since Y is, $N_X(T) = N_Y(T)$, and $M_X(T)$ is very close to $M_Y(T)$, for $T2r_Y$, when η is small.

The growth rates of $N_X(T)$ and $M_X(T)$ as $T \to \infty$ are more fundamental characteristics of X than the sizes of these functions at a particular value of T. Consequently we give below examples that combine near-minimal sizes of $N_X(T)$ and $M_X(T)$ for a particular T with small linear growth rates as $T \to \infty$. In particular, these examples are linearly repetitive, which implies that they have a great deal of regularity in the form of the existence of uniform patch frequencies and of an approximately linear address map, as shown in [17, Theorems 6.1, 7.1]. Most of our examples have the additional property of being Meyer sets, where a *Meyer set* is a Delone set X whose set of differences $X - X$ is also a Delone set, see [14], [19].

We begin with one-dimensional examples and then extend them to higher dimensions by the technique of taking the direct product with a lattice of small mesh.

Theorem 6.1 *For each $\epsilon > 0$ there exists a one-dimensional aperiodic Meyer set $Y = Y(\epsilon)$ which is linearly repetitive and has the following properties.*

(i) There is a $T^* > 0$ with

$$N_Y(T^*) < \Big(\frac{1}{2} + \epsilon\Big)\frac{T^*}{R} \quad \textit{and} \quad M_Y(T^*) < \Big(\frac{1}{2} + \epsilon\Big)T^*. \tag{6.1}$$

(ii) There is a $T_0 > 0$ such that, for all $T > T_0$,

$$N_Y(T) < (1 + \epsilon)\frac{T}{R} \quad \textit{and} \quad M_Y(T) < (\tau + 1 + \epsilon)T, \tag{6.2}$$

where $\tau = \frac{1}{2}(1 + \sqrt{5}) \approx 1.6180$.

Proof. (i) As described above, to satisfy (i) it is sufficient to construct a set Y with $\kappa_Y = \frac{R}{r} < 1 + 2\epsilon$ and to take $T^* = 2r$. We shall take for Y a set of the form

$$Y = Y(\alpha; \delta) = \{m + \delta\langle m\alpha\rangle : m \in \mathbb{Z}\},$$

where α is irrational, $|\delta| < \frac{1}{2}$, and $\langle x\rangle = x - \lfloor x\rfloor$ is the fractional part of x. This set is *almost linear*, in the sense of [15], so is a Meyer set by Theorem 5.1 of [15]. The irrationality of α ensures that Y is aperiodic. The packing radius r_Y of Y satisfies

$$r_Y \geq \frac{1}{2}(1 - |\delta|)$$

while the covering radius R_Y satisfies

$$R_Y \leq \frac{1}{2}(1 + |\delta|), \tag{6.3}$$

so

$$\kappa_Y = \frac{R_Y}{r_Y} \leq \frac{1 + |\delta|}{1 - |\delta|},$$

which is $< 1 + 2\epsilon$ when $|\delta| < \frac{1}{2}\epsilon$.

(ii) To bound $N_Y(T)$ for large T we note that the length of the mth interval of $Y = Y(\alpha; \delta)$ is

$$1 + \delta(\langle(m+1)\alpha\rangle - \langle m\alpha\rangle) = 1 + \delta\alpha - \delta(\lfloor(m+1)\alpha\rfloor - \lfloor m\alpha\rfloor),$$

so the intervals of Y are of two lengths and the arrangement of these lengths corresponds to the arrangement of the terms of the Beatty sequence (or Sturmian word)

$$S := \{\lfloor(m+1)\alpha\rfloor - \lfloor m\alpha\rfloor\}.$$

For investigating the symbolic dynamics of Y, the two lengths can be replaced by two abstract symbols, A and B say. The study of such symbolic sequences goes back to the eighteenth century but the emergence of a coherent body of theory dates from the two papers [21, 22] of Morse and Hedlund in 1938 and 1940. Coven and Hedlund [5] showed that for $m \geq 1$ there are exactly $m+1$ distinct words of length m in S when α is irrational. Since the number of intervals that are contained in a given T-patch of Y, or else overlap it sufficiently for the overlap to determine their length, is at most[2] $2\lfloor T+|\delta|\rfloor$, this result shows that

$$N_Y(T) \leq 2\lfloor T+|\delta|\rfloor + 1 \leq (1+|\delta|)\frac{T}{R} + 1 + 2|\delta| \tag{6.4}$$

for all $T > 0$.

The analogue in symbolic dynamics of the repetitivity function is the recurrence function, $R(m)$, introduced by Morse and Hedlund. As in §1, we work in terms of the function $M_S(m) = R_S(m) - m$, the maximum distance between the leading symbols of two successive identical words of length m in the symbolic sequence S. It is proved in [22, p.2] (see also [1], for example) that if $\{q_k\}$ is the sequence of denominators of the continued fraction convergents of α and $q_{k-1} \leq m < q_k$ then $M_S(m) = q_k + q_{k-1}$. If the continued fraction has bounded partial quotients (as happens when α is a quadratic irrational, for example) then $q_k + q_{k-1} < (b+2)q_{k-1}$, where b is an upper bound for the partial quotients, so S is linearly repetitive and hence Y is too. To get the bound in (6.2) we take $\alpha = \tau$, so that $b = 1$ and $q_k = F_k$, the kth Fibonacci number. Recalling that the number of identifiable intervals in any T-patch is at most $2\lfloor T+|\delta|\rfloor$, we then have

$$2M_Y(T) \leq F_k + F_{k-1} + |\delta| \tag{6.5}$$

where k is chosen so that

$$F_{k-1} \leq 2\lfloor T+|\delta|\rfloor < F_k, \tag{6.6}$$

where the last term on the right of (6.5) is to take account of the fact that the distance between two points of Y differs from the number of intervals separating them by at most $|\delta|$. Using the identity

$$F_k = \tau F_{k-1} + \left(\frac{-1}{\tau}\right)^{k-1}$$

and the lower bound in (6.6) we obtain

$$M_Y(T) \leq (\tau+1)T + \frac{1}{2} + \left(\tau + \frac{3}{2}\right)|\delta| \tag{6.7}$$

for all $T > 0$. If we choose $|\delta| < \frac{1}{2}\epsilon$ and $T_0 > 3/\epsilon$ then (6.3), (6.4) and (6.7) give (6.2). □

Remark. We note in passing that the set $Y(\tau; -1/\sqrt{5})$ is a is a scaled version of the well known Fibonacci quasicrystal. It can be shown to satisfy $M_Y(T) \leq (\tau+1)T + (\tau+1)/2\sqrt{5}$, where the constant term is half the length of the long interval. In this case it can be shown that this bound for $M_Y(T)$ is optimal in the sense that there exist arbitrarily large T for which it is achieved.

It is now straightforward to extend Theorem 6.1 to the n-dimensional case by taking direct products with lattices of small mesh.

[2] Perhaps more transparently, we could use instead the upper bound $2\lceil T+|\delta|\rceil$ for the total number of intervals that meet the T-patch, at the cost of slightly larger constant terms in (6.4) and (6.7).

Theorem 6.2 *In $\mathbb{R}^n$, for each $\epsilon > 0$ there exists a non-crystalline Meyer set $X = X(\epsilon)$ which is linearly repetitive and has the following properties.*

(i) There is a $T^* > 0$ with

$$N_X(T^*) < \Big(\frac{1}{2} + \epsilon\Big)\frac{T^*}{R} \quad \textit{and} \quad M_X(T^*) < \Big(\frac{1}{2} + \epsilon\Big)T^*. \tag{6.8}$$

(ii) There is a $T_0 > 0$ such that, for all $T > T_0$,

$$N_X(T) < (1 + \epsilon)\frac{T}{R} \quad \textit{and} \quad M_X(T) < (\tau + 1 + \epsilon)T. \tag{6.9}$$

Proof. We take $Y = Y(\epsilon)$ to be the one-dimensional set constructed in Theorem 6.1 and

$$X := Y \times \eta\mathbb{Z}^{n-1},$$

where η is a sufficiently small positive constant. For any η, X is an n-dimensional linearly repetitive Meyer set which is a non-crystal, but has an $(n-1)$-dimensional lattice of periods. Because of these periods

$$N_X(T) = N_Y(T) \tag{6.10}$$

for all T. We also have

$$M_X(T)^2 \leq M_Y(T)^2 + M_{\eta\mathbb{Z}^{n-1}}(T)^2 = M_Y(T)^2 + \frac{1}{4}\eta^2(n-1) \tag{6.11}$$

for all T, and

$$R_X^2 \leq R_Y^2 + \frac{1}{4}\eta^2(n-1), \tag{6.12}$$

where R_X and R_Y are the covering radii of X and Y, respectively. Now (6.10), (6.11) and (6.12), together with Theorem 6.1 (with $\frac{1}{2}\epsilon$ in place of ϵ), give (6.8) and (6.9) when η is small enough, with T^* and T_0 as in Theorem 6.1. □

Remark. For the examples $X(\epsilon)$ the constants on the right of the inequalities (6.8) and (6.9) simultaneously approach $\frac{1}{2R}$, $\frac{1}{2}$, $\frac{1}{R}$ and $\tau + 1$. The first of these is optimal in all dimensions, by Theorem 2.1, and it is conceivable that the others are also individually optimal. We note that if the second value, $\frac{1}{2}$, is optimal then, since $c(n) \leq \frac{1}{2}$ for all n, every Delone set X satisfying the hypothesis of Theorem 2.3 is not only periodic but also an ideal crystal.

We end this section by deriving attainable small values for the complexity of aperiodic (rather than merely non-crystalline) Delone sets. Note that, since $1 < \kappa(n) \leq 2$ for $n \geq 2$, the constants in (6.13) below are larger than those in (i) of Theorem 6.2, as is to be expected, but are also smaller than those in (ii) of Theorem 6.2.

Theorem 6.3 *(i) In each dimension $n \geq 1$ and for each $\epsilon > 0$, there exists an aperiodic Delone set $X = X(\epsilon)$ for which there is a $T > 0$ such that*

$$N_X(T) < \Big(\frac{1}{2}\kappa(n) + \epsilon\Big)\frac{T}{R} \quad \textit{and} \quad M_X(T) < \Big(\frac{1}{2}\kappa(n) + \epsilon\Big)T. \tag{6.13}$$

(ii) In each dimension $n \geq 1$ and for each $\epsilon > 0$, there exists an aperiodic repetitive Meyer set $Z = Z(\epsilon)$ having uniform patch frequencies and for which there is a $T > 0$ such that

$$N_Z(T) < \Big(\frac{1}{2}\kappa_L(n) + \epsilon\Big)\frac{T}{R} \quad \textit{and} \quad M_Z(T) < \Big(\frac{1}{2}\kappa_L(n) + \epsilon\Big)T. \tag{6.14}$$

Proof. (i) Take a Delone set X with $\kappa_X < \kappa(n) + \epsilon$. We may suppose that X is aperiodic by moving one point of it by a small enough amount not to disturb this inequality but in such a way as to destroy any global periods. Choose $T = 2r_X$, so that $N_X(T) = 1$ and $M_X(T) = R_X$. Then (6.13) holds since

$$\Big(\frac{1}{2}\kappa(n) + \epsilon\Big)T > \Big(\frac{R_X}{2r_X} + \frac{1}{2}\epsilon\Big)2r_X > R_X.$$

(ii) Uniform patch frequencies are defined as in [17]. Take a lattice $\Lambda \in \mathbb{R}^n$ with $\kappa_\Lambda < \kappa_L(n) + \epsilon$ and let

$$Z = \{\boldsymbol{\lambda} + \langle\phi(\boldsymbol{\lambda})\rangle\mathbf{v} : \boldsymbol{\lambda} \in \Lambda\},$$

where $\mathbf{v} \in \mathbb{R}^n$ and ϕ is a linear functional on $\mathbb{R}^n$. By choosing $||\mathbf{v}||$ small enough (for a fixed ϕ) we can ensure that the packing and covering radii of Z remain close enough to the corresponding values for Λ for $\kappa_Z < \kappa_L(n) + \epsilon$ to hold. Then, as in the proof of (i), (6.14) is satisfied with $T = 2r_Z$. For the other properties of Z we note that, up to an invertible linear transformation, Z is a cut-and-project set with lattice $\Lambda \times \mathbb{Z} \subset \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$, the hyperplane $x_{n+1} = \phi(x_1, \ldots, x_n)$ as "physical space" and the line joining $(\mathbf{0}, 0)$ to $(-\mathbf{v}, -1 - \phi(\mathbf{v}))$ as "internal space", the window being the half-open interval, closed at the $(\mathbf{0}, 0)$ end, with these points as end points. (The invertible linear transformation in question is projection on the $\mathbb{R}^n$ factor, its inverse being given by $\mathbf{x} \mapsto (\mathbf{x}, \phi(\mathbf{x}).)$ The facts that Z is a repetitive Meyer set with uniform patch frequencies that is aperiodic can now be derived from Propositions 2.21 and 2.23 of [23] provided certain conditions are satisfied; namely, (a) the physical space contains no non-zero vector of the lattice $\Lambda \times \mathbb{Z}$, (b) the physical space is not contained in any subspace generated by n lattice vectors, and (c) the internal space also contains no non-zero lattice vector. Of these, (a) amounts to choosing ϕ so that $1, \phi(\boldsymbol{\lambda}_1), \ldots, \phi(\boldsymbol{\lambda}_n)$ are linearly independent over $\mathbb{Q}$, where $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n$ is a basis of Λ. Then (b) holds too, since the physical space, having dimension n, cannot lie in a subspace generated by n lattice vectors unless it contains those lattice vectors. Finally, any lattice vector in internal space has the form $(t\mathbf{v}, t(1 + \phi(\mathbf{v}))$ with $t\boldsymbol{\delta} \in \Lambda$ and $t(1 + \phi(\mathbf{v})) \in \mathbb{Z}$, so the internal space will contain no non-zero lattice vectors if $\mathbf{v}$ is chosen to avoid the countably many vectors $(m - \phi(\boldsymbol{\lambda}))^{-1}\boldsymbol{\lambda}$ with $\boldsymbol{\lambda} \in \Lambda$, $m \in \mathbb{Z}$ and λ and m not both zero. $\square$

Note that while the set Z in (ii) of Theorem 6.3 is repetitive, and hence also of finite local complexity, the set X in (i) may have $N_X(T)$ or $M_X(T)$ infinite for large T.

Consider dimension $n = 24$ as an example, where $\kappa_L(24) \leq \sqrt{2}$. By Theorem 2.3 if $X \subset \mathbb{R}^{24}$ satisfies

$$M_X(T) < \frac{10\sqrt{3} - 12}{13}T \approx 0.4093T$$

for some $T > 0$ then X has a non-zero period. Theorem 6.3(ii) shows that there exist aperiodic sets X in $\mathbb{R}^{24}$ with uniform patch frequencies that satisfy $M_X(T) < cT$ for some $T > 0$, for any $c > 1/\sqrt{2} \approx 0.7071$.

7. Delone Packing-Covering Constant

Recall that the *Delone packing-covering constant* $\kappa(n)$ in n dimensions is the infimum of $\frac{R}{r}$ taken over all n-dimensional Delone sets and the *lattice packing-covering constant* $\kappa_L(n)$ is the infimum of $\frac{R}{r}$ over all n-dimensional lattices. In this section we prove various facts about $\kappa(n)$ and $\kappa_L(n)$. We first show that these infimums are attained.

Theorem 7.1 *In each dimension n there exists a Delone set X whose Delone constants (r, R) satisfy $\frac{R}{r} = \kappa(n)$, and a lattice Λ whose Delone constants satisfy $\frac{R}{r} = \kappa_L(n)$.*

Proof. For each $m \in \mathbb{N}$ let

$$\mathcal{S}_m = \{S \subset \mathbb{R}^n : |\mathbf{x}| \leq m \text{ and } |\mathbf{x} - \mathbf{y}| \geq 1 \text{ for all } \mathbf{x}, \mathbf{y} \in \mathcal{S}\}.$$

Then each $S \in \mathcal{S}_m$ contains only finitely many points and the Hausdorff metric

$$d(S_1, S_2) = \max_{\mathbf{x} \in S_1} \min_{\mathbf{y} \in S_2} |\mathbf{x} - \mathbf{y}| + \max_{\mathbf{x} \in S_2} \min_{\mathbf{y} \in S_1} |\mathbf{x} - \mathbf{y}|$$

makes $\mathcal{S}_m$ a compact metric space. After suitable scalings we can find a sequence $\{X_i\}$ of Delone sets in $\mathbb{R}^n$ having $r \geq 1$ for all i and R tending to $\kappa(n)$ as $i \to \infty$. The sets $X_i \cap B(\mathbf{0}; 1)$ belong to the compact space $\mathcal{S}_1$, so we can find a convergent subsequence

$$\{S_{1i}\} = \{X_{1i} \cap B(\mathbf{0}; 1)\}$$

which we can choose so that $d(S_{1i}, S_{11}) < 1$ for all i. Similarly the sets $X_{1i} \cap B(\mathbf{0}; 2)$ belong to $\mathcal{S}_2$, and we can find a subsequence

$$\{S_{2i}\} = \{X_{2i} \cap B(\mathbf{0}; 2)\}$$

with $d(S_{2i}, S_{21}) < 1/2$ for all i. Continuing in this way, for each $j \in \mathbb{N}^+$ we can find a subsequence $\{X_{ji}\}_{i=1}^{\infty}$ of $\{X_{j-1,i}\}_{i=1}^{\infty}$ such that

$$d(X_{ji} \cap B(\mathbf{0}; j), X_{j1} \cap B(\mathbf{0}; j)) < \frac{1}{j} \quad \text{for all } i \in \mathbb{N}^+.$$

The sequence of Delone sets $\{X_{j1}\}_{j=1}^{\infty}$ now converges to a unique set $X \subset \mathbb{R}^n$ that has $r \geq 1$ and $R \leq \kappa(n)$, and by the definition of $\kappa(n)$ there is equality in both places.

The proof for the lattice case is similar, noting that a limit of lattices that is a Delone set is a lattice. $\square$

We next give upper and lower bounds for $\kappa(n)$, which are due to Ryshkov [26].

Proposition 7.1 *In each dimension n the Delone packing-covering constant $\kappa(n)$ satisfies*

$$\sqrt{\frac{2n}{n+1}} \leq \kappa(n) \leq 2. \tag{7.1}$$

Proof. The lower bound for $\kappa(n)$ follows by combining Rogers's upper bound for the packing density [33, Theorem 7.1] and Coxeter, Few and Rogers's lower bound for the covering thickness [33, Theorem 3.4] of equal balls in $\mathbb{R}^n$. The former asserts that for every $\epsilon > 0$ the number of balls of radius r that can be packed into a ball of radius l is $< (1+\epsilon)\sigma_n(l/r)^n$ when l is large enough, where σ_n is the proportion of an n-dimensional regular simplex that is covered by balls centered at its vertices that reach to the mid-points of its sides. The latter asserts that the number of balls of radius R it takes to cover a ball of radius l is $> (1-\epsilon)\tau_n(l/R)^n$ when l is large enough, where τ_n is the proportion of the simplex (counted with multiplicity) that is covered by balls centered at its vertices that reach to its centroid. (These latter balls just cover the simplex). It follows from these bounds that the packing radius r and covering radius R of any set $X \subset \mathbb{R}^n$ satisfy

$$\frac{R}{r} \geq \left(\frac{\tau_n}{\sigma_n}\right)^{1/n} = \sqrt{\frac{2n}{n+1}},$$

the expression on the right being the ratio of the circumradius to half the edge length of a regular n-simplex.

To establish the upper bound for $\kappa(n)$, take a packing of $\mathbb{R}^n$ with unit spheres, to which no sphere can be added without overlap. The sphere centers then form a Delone set X with $r \geq 1$. We claim that $R \leq 2$. If not, and $R > 2$, then there would exist some point at a distance at least 2 from all sphere centers, and a unit sphere placed at this point would overlap no sphere in the packing, a contradiction. □

The lower bound in Proposition 7.1 is sufficient to establish that $\kappa(1) = \kappa_L(1) = 1$ and $\kappa(2) = \kappa_L(2) = 2/\sqrt{3}$, with the optimal configurations in dimensions 1 and 2 being $\mathbb{Z}$ and the hexagonal lattice A_2. This gives $c_1 = \frac{1}{3}$ and $c_2 = \frac{1}{2}(\sqrt{3}-1) \approx 0.3660$. In three dimensions we have the bounds

$$\sqrt{\frac{3}{2}} \approx 1.2247 \leq \kappa(3) \leq \kappa_L(3) = \sqrt{\frac{5}{3}} \approx 1.2910,$$

the upper bound coming from the body-centered cubic lattice A_3^*. In twenty-four dimensions we have the remarkably close bounds

$$\frac{4\sqrt{3}}{5} \approx 1.3856 \leq \kappa(24) \leq \kappa_L(24) \leq \sqrt{2} \approx 1.4142, \tag{7.2}$$

with the upper bound coming from the Leech lattice. (The values of the packing density and covering thickness of the Leech lattice are given in [4].) Note that $\kappa(n)$ and $\kappa_L(n)$ are probably not monotonically increasing in n.

The lower bound in Proposition 7.1 also yields the asymptotic bound

$$\liminf_{n\to\infty} \kappa(n) \geq \sqrt{2}.$$

This asymptotic bound can be further improved by using the current best upper bound for sphere packing density (the Kabatiansky-Levenshtein bound) as $n \to \infty$, instead of the Rogers bound, which yields

$$\liminf_{n\to\infty} \kappa(n) \geq 2^{0.5990} \approx 1.5146,$$

cf. [4, Chapters 1 and 9], [33, Theorem 8.2].

The upper bound of Proposition 7.1 shows that the constant $c(n) = \frac{\kappa(n)}{\kappa(n)+2} \leq \frac{1}{2}$, hence Theorem 2.3 gives at most a very slight dimension-dependent improvement on Theorem 2.2.

The lattice packing-covering constant $\kappa_L(n)$ satisfies the lower bound in (7.1) *a fortiori*. In [26] Ryshkov gives the upper bound

$$\kappa_L(n) \leq \sqrt{\frac{n+2}{3}}.$$

(Recall that $\kappa_L(n)$ is, by definition, twice Ryshkov's constant.) This is known to hold with equality in dimensions 1, 2 and 3. There seems to be very little known about upper bounds for $\kappa_L(n)$, and we raise the following question.[3]

Question. Is there a constant C such that $\kappa_L(n) \leq C$ for all n, or is $\kappa_L(n)$ unbounded as $n \to \infty$?

One reason for interest in the question is that in any dimension n for which $\kappa_L(n) > 2$ (should such dimensions exist) the densest sphere packing with equal spheres cannot be a lattice packing.

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