Corrigendum/Addendum to: Sets of Matrices all Infinite Products of Which Converge

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ABSTRACT

This corrigendum/addendum supplies corrected statements and proofs of some results in our paper appearing in Linear Algebra and its Applications **161** (1992) 227-263. These results concern special kinds of bounded semigroups of matrices. It also reports on progress on the topics of this paper made in the last eight years.

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1. Introduction

Our paper [7] contains a number of errata, which we correct here, of which the following are the most important.

1. The proof of the rightmost inequality in Lemma 3.1 given in [7] is incomplete for some cases where the set of matrices Σ is an infinite set. In §3 we give a proof, due to Olga Holtz, for these remaining cases.

2. The statements of Theorem 4.2 and Theorem 5.1 should read that the projection P_V is an (oblique) projection onto the subspace V away from the subspace $E_1(\Sigma)$, rather than being an orthogonal projection. Theorem 5.1 also requires the additional condition that all generalized 1-eigenspaces of the matrices in Σ be simple, in order to be a necessary and sufficient condition. In §4 we state a corrected version of Theorem 5.1 and supply some additional details of its proof which were omitted in [7, p. 253, line 9].

3. Condition (1) in Lemma 5.2 should read "Sup" rather than "Max."

4. In Theorem 6.1 the condition (C1) needs an additional restriction to be equivalent to the other four conditions, which is that the (generalized) left 1-eigenspaces of each A_j be one-dimensional. A corrected theorem and proof, together with an additional equivalent condition (C1$'$), are given in §5. Corollary 6.1a also requires a similar correction, given in §5.

These errata, and further minor errata listed in §6, were bought to our attention by Olga Holtz, who gave the paper a very careful reading,

We take this opportunity to report in §2 on developments made since our paper appeared in 1992, and to summarize the current state of knowledge on effective computability of various computational questions in this area. There have been over 20 papers published on related subjects. In particular, the two conjectures made in the paper, the Boundedness Conjecture made in [7, p. 246] and the Generalized Spectral Radius Conjecture made in [7, p. 240], were both proved in Berger and Wang [2].

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2. Recent Developments

Our paper [7] studied *RCP sets*, which are sets Σ of $n \times n$ matrices with the *RCP property* that all right infinite products $\lim_{n\to\infty} \mathsf{M}_1\mathsf{M}_2...\mathsf{M}_n$ with M_i drawn from the set Σ converge. The Boundedness Conjecture made in [7, p. 246] asserts that all RCP sets are bounded, i.e. generate a bounded semigroup. This was proved in Berger and Wang [2, Theorem 1]. Thus RCP sets Σ generate a special kind of bounded semigroup $\mathcal{S}_\Sigma$ of matrices. The subject of bounded semigroups of matrices has a long history, tracing back at least to Wielandt [36].

Since 1992 various new matrix norm conditions for a finite set of matrices Σ to have the RCP property or the RCP property with a continuous limit function have been obtained. Some sufficient conditions for a finite set of matrices to have the RCP property, in terms of the existence of a suitable matrix norm were already given in 1990 by Elsner, Kohltracht and Neumann [14]. They actually worked with the LCP property (all left-infinite products converge), but the results are interchangeable with the RCP property by taking transposes of all matrices. In 1997 Elsner and Friedland [13] gave several necessary and sufficient conditions for a set of two matrices to have the LCP property involving matrix norms. Beyn and Elsner [3] gave necessary and sufficient conditions for a finite set of $m \times m$ matrices to be an LCP set having a continuous limit function, in terms of the existence of a suitable matrix norm, with respect to which the matrices in Σ are paracontracting.

Our paper related the RCP property to various notions of spectral radius of a set Σ of $n \times n$ real matrices. Recall that a *matrix norm* is a norm on the set of matrices which is *submultiplicative*, i.e. it satisfies

$$||\mathsf{M}_1\mathsf{M}_2|| \leq ||\mathsf{M}_1||||\mathsf{M}_2||,$$

see [22, p. 358]. (Submultiplicatively is called the *ring property* in [1, p. 8].) Recall that the *joint spectral radius* $\hat{\rho}(\Sigma)$ of a set of $n \times n$ matrices Σ is defined by

$$\hat{\rho}(\Sigma) := \limsup_{k \to \infty} \hat{\rho}_k(\Sigma, ||.||)^{1/k}, \tag{2.1}$$

where

$$\hat{\rho}_k(\Sigma, ||.||) := \sup\{||\mathsf{M}_1 \mathsf{M}_2 \ldots \mathsf{M}_k|| \ : \ \text{all} \ \ \mathsf{M}_i \in \Sigma\}, \tag{2.2}$$

in which $||.||$ is a matrix norm; the $\limsup$ in (2.1) is independent of the choice of matrix norm, as is easily shown, cf. [7, p. 237]. Lemma 3.1 given in §3 below implies that $\limsup$ in (2.1) is in fact $\lim$. If $\rho(\mathsf{M})$ denotes the spectral radius of a complex $n \times n$ matrix M, for any set Σ of $n \times n$ matrices set

$$\bar{\rho}_k(\Sigma) := \sup\{\rho(\mathsf{M}_1 \mathsf{M}_2 \cdots \mathsf{M}_k) \ : \ \text{each} \ \mathsf{M}_j \in \Sigma\}.$$

The *generalized spectral radius* $\bar{\rho}(\Sigma)$ is given by

$$\bar{\rho}(\Sigma) := \limsup_{k \to \infty} \bar{\rho}_k(\Sigma)^{1/k}. \tag{2.3}$$

(In [7] this quantity was denoted $\rho(\Sigma)$.) Recall that Lemma 3.1 of [7] showed that for all sets Σ of $n \times n$ complex matrices one has

$$\bar{\rho}(\Sigma) \leq \hat{\rho}(\Sigma).$$

The Generalized Spectral Radius Conjecture of [7, p. 240] asserts that for finite sets of $n \times n$ matrices the generalized spectral radius and the joint spectral radius are equal. This was proved in Berger and Wang [2, theorem 4], and a different proof was given later by Elsner [12]. Rosenthal and Soltysiak [30] related these two notions of spectral radius to the *geometric joint spectral radius* of Banach algebra sets, showing that for a finite set of elements of a unital complex Banach algebra $\mathcal{A}$ the geometric joint spectral radius is no larger than the corresponding generalized spectral radius. They also prove [30, Theorem 2] that equality of the geometric joint spectral radius and generalized spectral radius holds for all n-tuples of elements in $\mathcal{A}$, for all $n \geq 2$, if and only if $\mathcal{A}/\mathrm{rad}(\mathcal{A})$ is a commutative Banach algebra. This result includes the Berger-Wang result as a special case, where the Banach algebra $\mathcal{A}$ is the set of $n \times n$ complex-valued matrices with its standard Banach norm; in that case $\mathcal{A} = \mathrm{rad}(\mathcal{A})$.

Our paper raised and discussed issues of effective computability for various quantities involving the joint spectral radius, the generalized spectral radius, and the RCP property that all infinite products taken to the right converge, see [7, p.246]. One may formulate the following two basic computational problems, using the standard decision problem format of Garey and Johnson [15, p. 18].

(1) RCP SET

Instance: A finite set $\Sigma = \{\mathsf{A}_0, ..., \mathsf{A}_{m-1}\}$ of $n \times n$ matrices with rational entries.

Question: Is Σ an RCP set?

(2) CONTINUOUS RCP SET

Instance: A finite set $\Sigma = \{\mathsf{A}_0, ..., \mathsf{A}_{m-1}\}$ of $n \times n$ matrices with rational entries.

Question: Is Σ an RCP set that has a continuous limit function?

The decidability of both these problems remains open. Our paper gave necessary and sufficient conditions for both of these problems to have a "yes" answer, given as Theorems 5.1 and 4.2, respectively. These criteria do not yield effective algorithms, see the remark at the end of §4 below. Other necessary and sufficient matrix norm conditions of Beyn and Elsner [3] also do not seem to yield effective decision procedures for either question.

The decidability of CONTINUOUS RCP SET can be related to conjectures concerning the joint spectral radius and generalized spectral radius of a set of matrices. For the joint spectral radius one has the following two computational problems.

(3) UNIT JOINT SPECTRAL RADIUS

Instance: A finite set $\Sigma = \{\mathsf{A}_0, ..., \mathsf{A}_{m-1}\}$ of $n \times n$ matrices with entries algebraic numbers.

Question: Is the joint spectral radius $\hat{\rho}(\Sigma) \leq 1$?

(4) SUBUNIT JOINT SPECTRAL RADIUS

Instance: A finite set $\Sigma = \{\mathsf{A}_0, ..., \mathsf{A}_{m-1}\}$ of $n \times n$ matrices with entries algebraic numbers.

Question: Is the joint spectral radius $\hat{\rho}(\Sigma) < 1$? Equivalently, is Σ an RCP set in which all infinite products are the zero matrix?

In the above problems an *algebraic number* is a complex number satisfying a polynomial equation with integer coefficients, and the input consists of the integer coefficients of such a polynomial and a complex approximation of one root sufficient to specify it. The UNIT JOINT SPECTRAL RADIUS problem is undecidable by a result of Blondel and Tsitsiklis [4, Theorem 1]. It remains an open problem whether SUBUNIT JOINT SPECTRAL RADIUS is decidable.

An effective decision procedure for SUBUNIT JOINT SPECTRAL RADIUS would yield an effective decision procedure for CONTINUOUS RCP SET, because the criterion (3) of Theorem 4.2 of [7] could then be effectively tested. Indeed one can effectively determine the left 1-eigenspace of a rational matrix and determine whether it is simple using algebraic numbers, and one can also test equality of all such eigenspaces. If they are all equal to E_1, Then one projects onto an algebraic subspace V of codimension equal to $\dim(E_1)$, using the oblique projection P_V away from the common 1-eigenspace E_1, to obtain $\Sigma' := \{\mathsf{P}_V \mathsf{M} \mathsf{P}_V \ : \ \mathsf{M} \in \Sigma\}$, and then uses the effective algorithm for SUBUNIT JOINT SPECTRAL RADIUS to determine whether $\hat{\rho}(\Sigma') < 1$, to complete testing criterion (3).

In 1995 Lagarias and Wang [21] formulated a Finiteness Conjecture concerning the generalized spectral radius, as follows.

FINITENESS CONJECTURE. For any finite set Σ of $n \times n$ matrices there exists a finite k such that the generalized spectral radius $\bar{\rho}(\Sigma)$ satisfies $\bar{\rho}(\Sigma) = \bar{\rho}_k(\Sigma)^{1/k}$.

As explained in [21, p. 19], the Finiteness Conjecture would imply that given a finite set Σ of matrices with rational entries, one can effectively decide whether or not the joint spectral radius $\hat{\rho}(\Sigma) < 1$ or $\hat{\rho}(\Sigma) \geq 1$ holds for a finite set Σ of matrices with rational entries, i.e. it would give an effective algorithm for SUBUNIT JOINT SPECTRAL RADIUS. Consider next, the following stronger version of the Finiteness Conjecture, for rational matrices.

EFFECTIVE FINITENESS CONJECTURE. For any finite set Σ of $n \times n$ matrices with rational entries there exists an effectively computable constant $t = t(\Sigma)$ such that the generalized spectral radius $\bar{\rho}(\Sigma)$ satisfies $\bar{\rho}(\Sigma) = \bar{\rho}_t(\Sigma)^{1/t}$.

The results of Blondel and Tsitsiklis [4, Corollary 1] show that the Effective Finiteness Conjecture is false. A related problem previously known to be undecidable is the *mortality problem,* which asks: for a given finite set Σ of $m \times m$ matrices is some finite product of matrices drawn from this set the zero matrix? Miller [23] gives an update and references on this problem.

These results strongly suggests that the Finiteness Conjecture itself is false. A recent preprint of Bousch and Mairesse [6] announces a disproof of the Finiteness Conjecture.

As a final computational problem, we mention the problem of recognizing bounded matrix semigroups.

(5) BOUNDED MATRIX SEMIGROUP (or BOUNDED MATRIX PRODUCTS)

> *Instance:* A finite set $\Sigma = \{\mathsf{A}_0, ..., \mathsf{A}_{m-1}\}$ of $n \times n$ matrices with entries algebraic numbers.
>
> *Question:* Is the matrix semigroup $\mathcal{S}_\Sigma$ generated by Σ a bounded semigroup?

This computational problem was raised in 1987 in a control theory context [34]. It is now known to be undecidable via the reduction of Blondel and Tsitsiklis [4, Theorem 1] to the emptiness problem for probabilistic finite automata.

(6) PFA EMPTINESS

> *Instance:* A finite set Σ of $n \times n$ nonnegative row-stochastic matrices P_i with rational entries, a zero-one n-column vector v, a non-negative $n-$column vector of rational numbers π whose entries sum to one, and a rational number r with $0 < r < 1$.
>
> *Question:* Does no finite product $\mathsf{M} = \mathsf{P}_{i_1} \cdots \mathsf{P}_{i_m}$ of matrices in Σ have $\pi^T \mathsf{M} v > r$? (Equivalently, do all finite products M have $\pi^T \mathsf{M} v \le r$?)

The PFA EMPTINESS problem has been shown to be undecidable, see [4] for a discussion and references.

Problems of convergence of infinite products of matrices can also be formulated as control theory questions. These concern various types of stability of discrete time linear systems which evolve by a matrix multiplication at each step. This viewpoint was taken by Gurvits [16], who obtained many fundamental results for different notions of stability. Further results concerning effective computability and computational difficulty of such stability questions, including NP-completeness results, were obtained in Gurvits [17], [18], Toker and özbay [33] and Tsitsiklis and Blondel [35].

Various RCP sets have limit functions which can be used to construct compactly supported wavelet bases of $\mathbb{R}^n$. The estimation of joint spectral radius of various related sets is important in analyzing the smoothness of the resulting functions, see Daubechies and Lagarias [8]. Other work in this area includes [9].

There is some related literature concerning bounded semigroups of matrices. Given a set Σ of $n \times n$ complex matrices, let

$$\Sigma^k := \{\mathsf{A}_1\mathsf{A}_2\cdots\mathsf{A}_k \ : \ \text{all } \mathsf{A}_j \in \Sigma\}.$$

Dehghan and Rajabilipour [11] show that if Σ^k generates a bounded semigroup, then Σ^m generates a bounded semigroup for all $m \geq k$. Regarding the structure of bounded matrix semigroups, Omladic and Radjavi [24] characterize sets of $n \times n$ complex matrices Σ for which the spectral radius is multiplicative on the semigroup $\mathcal{S}_\Sigma$ they generate, i.e. $\rho(ST) = \rho(S)\rho(T)$ for all $S, T \in \mathcal{S}_\Sigma$. This can be reduced to problem of characterizing such semigroups having a constant spectral radius, normalizable to be 1. They prove [24, Theorem 4.1] that any irreducible semigroup of this kind is necessarily a bounded semigroup, and note this result was proved earlier in Shneperman [31]. Related questions concerning simultaneous triangularizability of matrix semigroups can be found in Dehghan and Rajabilipour [10], Radjavi [26], [27], Radjavi and Rosenthal [28] and Rosenthal and Radjavi [29].

3. Corrected Proof of Lemma 3.1

Lemma 3.1. *For any set of matrices Σ, any $k \geq 1$ and any matrix norm $||.||$,*

$$\bar{\rho}_k(\Sigma)^{1/k} \leq \bar{\rho}(\Sigma) \leq \hat{\rho}(\Sigma) \leq \hat{\rho}_k(\Sigma, ||.||)^{1/k}. \tag{3.1}$$

Proof. The proof in [7] is correct except for the proof that the inequality

$$\hat{\rho}(\Sigma) \leq \hat{\rho}_k(\Sigma, ||.||)^{1/k} \tag{3.2}$$

is valid in those cases when $\hat{\rho}_1(\Sigma, \|\cdot\|) = \infty$. (The condition $\hat{\rho}_1(\Sigma, \|\cdot\|) = \infty$ can only occur when Σ is an infinite set.) The following proof for these remaining cases is due to Olga Holtz.

If $\hat{\rho}_k(\Sigma, \|\cdot\|) = \infty$ holds for all $k \geq 1$, then (3.2) is immediate. Thus we may suppose $\hat{\rho}_k(\Sigma, \|\cdot\|) < \infty$ for some finite k.

We now show by induction on the dimension n ($\Sigma \subseteq \mathbb{C}^{n\times n}$) that the condition $\exists k \in \mathbb{N}$ such that $\hat{\rho}_k(\Sigma, \|\cdot\|) < \infty$ implies that there exists $N \geq 1$ such that $\hat{\rho}_l(\Sigma, \|\cdot\|) < \infty$ for all $l > N$. This property is obviously true for the base case $n = 1$. For the induction step, suppose $n > 1$, that the induction hypothesis holds up to $n-1$, and that $\hat{\rho}_1(\Sigma, \|\cdot\|) = \cdots = \hat{\rho}_{k-1}(\Sigma, \|\cdot\|) = \infty$, and $\hat{\rho}_k(\Sigma, \|\cdot\|) < \infty$, for a given $k > 1$. Let $V := \mathrm{span}_{\mathsf{M}\in\Sigma}[\mathrm{ran}\mathsf{M}]$, where V consists of column vectors. Since $V = \mathrm{span}_{\mathsf{M}\in\Sigma'}[\mathrm{ran}\ \mathsf{M}]$ for some finite subset $\Sigma' \subseteq \Sigma$, any $x \in V$ has the form $x = \sum_{l=1}^{m} \mathsf{M}_{j_l} x_l$ for some finite m and some $\mathsf{M}_{j_1}, \ldots, \mathsf{M}_{j_m} \in \Sigma'$, hence

$$\sup_{\mathsf{M}_{i_1},\ldots,\mathsf{M}_{i_{k-1}}\in\Sigma} \|\mathsf{M}_{i_1}\cdots\mathsf{M}_{i_{k-1}}x\| \leq \hat{\rho}_k(\Sigma, \|\cdot\|)\sum_{l=1}^{m}\|x_l\| < \infty,$$

i.e., the set of all products of $k-1$ factors from Σ is bounded pointwise on V. If $V = \mathbb{C}^n$, this would imply, by Banach-Steinhaus, that

$$\hat{\rho}_{k-1}(\Sigma, \|\cdot\|) = \sup_{\mathsf{M}_{i_1},\ldots,\mathsf{M}_{i_{k-1}}\in\Sigma} \|\mathsf{M}_{i_1}\cdots\mathsf{M}_{i_{k-1}}\| < \infty,$$

This contradicts the definition of k, so we conclude that V must be a proper subspace of $\mathbb{C}^n$. Completing any basis of V to a basis of $\mathbb{C}^n$, we may suppose without loss of generality (by making a suitable similarity transformation to the matrices in Σ) that the matrices in Σ all have the form

$$\mathsf{M} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ 0 & 0 \end{pmatrix} \qquad \forall \mathsf{M} \in \Sigma.$$

As

$$\mathsf{M}_{i_1}\cdots\mathsf{M}_{i_k} = \begin{pmatrix} \mathsf{A}_{i_1}\cdots\mathsf{A}_{i_k} & \mathsf{A}_{i_1}\cdots\mathsf{A}_{i_{k-1}}\mathsf{B}_{i_k} \\ 0 & 0 \end{pmatrix}$$

and any two norms on $\mathbb{C}^{n\times n}$ are equivalent, we have

$$\hat{\rho}_k(\Sigma|_V, \|\cdot\|_s) = \sup \|\mathsf{A}_{i_1}\cdots\mathsf{A}_{i_k}\|_s < \infty,$$

where $\|\cdot\|_s$ denotes the *spectral norm,* which is defined by

$$\|\mathsf{A}\|_s = \sup_{\mathbf{x}\in\mathbb{C}^n,\mathbf{x}\neq 0} \frac{||\mathsf{A}\mathbf{x}||_2}{||\mathbf{x}||_2} = \sigma(\mathsf{A}^*\mathsf{A})^{1/2},$$

where A^* is the conjugate transpose of A, see [22, p. 365]. Since $\dim V < n$ the inductive hypothesis applies to show there exists $N \in \mathbb{N}$ such that

$$\hat{\rho}_l(\Sigma|_V, \|\cdot\|_s) < \infty \qquad \text{for all} \quad l > N.$$

But $\hat{\rho}_k(\Sigma, \|\cdot\|_s) < \infty$ also implies $\sup \|\mathsf{A}_{i_1}\cdots\mathsf{A}_{i_{k-1}}\mathsf{B}_{i_k}\|_s < \infty$, hence

$$\begin{aligned}
&\sup \|\mathsf{A}_{i_1}\cdots\mathsf{A}_{i_l}\|_s < \infty \\
&\sup \|(\mathsf{A}_{i_1}\cdots\mathsf{A}_{i_{l-k}})(\mathsf{A}_{i_{l-k+1}}\cdots\mathsf{A}_{i_{l-1}}\mathsf{B}_{i_l})\|_s < \infty
\end{aligned}$$

which implies $\hat{\rho}_l(\Sigma, \|\cdot\|) < \infty$, for all $l > N+k$ and any norm $\|\cdot\|$. This completes the induction step.

We now verify that the (3.2) holds for any matrix norm $\|\cdot\|$ and any $k \geq 1$. If $\hat{\rho}_k(\Sigma, \|\cdot\|) = \infty$, the inequality is immediate. Otherwise we just saw that $\hat{\rho}_j(\Sigma, \|\cdot\|) < \infty \ \ \forall j > N$ for some $N \in \mathbb{N}$. In particular, there exists l coprime to k s.t. $\hat{\rho}_l(\Sigma, \|\cdot\|) < \infty$. For any $n > kl$ there exist integers t and s such that $n = tk + sl$ with $t > 0$ and $0 \leq s < k$. This implies

$$\hat{\rho}_n(\Sigma, \|\cdot\|)^{1/n} \leq \left(\hat{\rho}_k(\Sigma, \|\cdot\|)^{\frac{n-sl}{k}} \hat{\rho}_l(\Sigma, \|\cdot\|)^s\right)^{1/n} \xrightarrow[n \to \infty]{} \hat{\rho}_k(\Sigma, \|\cdot\|)^{1/k},$$

hence $\hat{\varrho}(\Sigma) \leq \hat{\rho}_k(\Sigma, \|\cdot\|)^{1/k}$. $\square$

Remark. Lemma 3.1 establishes that $\limsup_{k\to\infty}$ in the definition of $\hat{\rho}(\Sigma)$ in (2.1) is in fact $\lim_{k\to\infty}$.

4. Revised Theorem 5.1

A pair of vector spaces (W, V) are *complementing subspaces* of $\mathbb{R}^n$ if $W + V = \mathbb{R}^n$ and $\dim(W) + \dim(V) = n$. Given any pair (W, V) of complementing subspaces there exists a unique (oblique) *projection* P_V *onto* V *away from* W, i.e. $\mathsf{P}_V^2 = \mathsf{P}_V$ with

$$\begin{aligned}
v\mathsf{P}_V &= v \quad \text{if} \quad v \in V\ , \\
w\mathsf{P}_V &= 0 \quad \text{if} \quad w \in W\ .
\end{aligned}$$

Here we view $\mathbb{R}^n$ as a space of row vectors. In the statement of Theorems 4.2 and 5.1 of [7] the projections P_V are projections onto V away from $E_1 = E_1(\Sigma)$.

The statement of Theorem 5.1 in [7] requires a modification, which consists of a strengthening of its condition (2), given below. Given a set Σ of matrices, a finite product $\mathsf{B} = \mathsf{M}_1\mathsf{M}_2\cdots\mathsf{M}_k$ is called a *block* of Σ if $E_1(\Sigma) = \cap_{j=1}^{k} E_1(\mathsf{M}_j)$ but $E_1(\Sigma) \neq \cap_{j=1}^{k-1} E_1(\mathsf{M}_j)$. The set Σ_B consists of all finite products of matrices in Σ which are blocks. The set Σ_B is generally infinite.

Theorem 5.1. *A finite set Σ of $n \times n$ real matrices is a product-bounded RCP set if and only if the following conditions (1)-(3) hold.*

(1) All strict subsets of Σ are product-bounded RCP sets.

(2) Each $A_i \in \Sigma$ has a (generalized) 1-eigenspace $E_1(A_i)$ that is simple, and all $\mathsf{B} \in \Sigma_B$ have $E_1(\mathsf{B}) = E_1(\Sigma)$.

(3) There is a subspace V of $\mathbb{R}^n$ with $E_1(\Sigma) + V = \mathbb{R}^n$, $\dim(V) = n - \dim(E_1(\Sigma))$ such that the set $\mathsf{P}_V\Sigma_B\mathsf{P}_V = \{\mathsf{P}_V\mathsf{B}\mathsf{P}_V : \mathsf{B} \in \Sigma_B\}$ where P_V is projection onto V away from $E_1(\Sigma)$, has joint spectral radius

$$\hat{\rho}(\mathsf{P}_V\Sigma_B\mathsf{P}_V) < 1 \ . \tag{4.1}$$

Remark. It follows by Berger and Wang [2, Theorem 1] that all RCP sets are product-bounded, so Theorem 5.1 actually gives a necessary and sufficient condition for being an RCP set.

Proof. We add some details to the proof in [7].

$\Rightarrow$. The condition in (2) that each $\mathsf{A}_j \in \Sigma$ has a generalized 1-eigenspace $E_1(M)$ that is simple holds because, if some A_i does not have $E_1(\mathsf{A}_i)$ simple, then $\|\mathsf{A}_i^k\| \to \infty$ as $k \to \infty$, contradicting product boundedness. This permits the reduction of Σ by a similarity to the block form

$$\mathsf{A}_j = \begin{bmatrix} \mathsf{I} & \mathbf{0} \\ \mathsf{C}_j & \tilde{\mathsf{A}}_j \end{bmatrix},$$

as given in (5.7) of [7].

$\Leftarrow$. Suppose (1)–(3) hold for Σ, and we must show Σ is an RCP set. The proof up to the final paragraph [7, p. 253, line 7] established that Σ_B is an RCP set and that Σ is product bounded. We note that the condition of simple eigenspaces in (2) was used at the initial step of reducing Σ by a similarity to matrices of the form $\mathsf{A}_j = \begin{bmatrix} \mathsf{I} & \mathbf{0} \\ \mathsf{C}_j & \tilde{\mathsf{A}}_j \end{bmatrix}$ in (5.7), where I corresponds to the space $E_1(\Sigma)$. Thus all matrices $\mathsf{B} \in \Sigma_B$ also have the block form $\mathsf{B} = \begin{bmatrix} \mathsf{I} & \mathbf{0} \\ \mathsf{C}_B & \tilde{\mathsf{B}} \end{bmatrix}$, with $E_1(\mathsf{B}) = E_1(\Sigma_B)$.

We aim to apply Lemma 5.1 to conclude Σ is an RCP set. Its hypotheses (1), (2) hold, and it remains to verify hypothesis (3), which asserts that any infinite product of matrices in Σ_B has all its rows in $E_1(\Sigma_B)$. Let $\mathsf{M}^{(\infty)} := \lim_{k\to\infty} \prod_{j=1}^{k} \mathsf{B}_j$ with all $\mathsf{B}_j \in \Sigma_B$, where the limit exists since Σ_B is an RCP-set. Let $u_i := (0, 0, 1, \dots, 0)$ be the i-th unit vector, with 1 in the i-th position, and we must show $u_i\, \mathsf{M}^{(\infty)} \in E_1(\Sigma)$, for $1 \le i \le n$. Let P_V denote projection onto the subspace V of hypothesis (3) away from $E_1 := E_1(\Sigma)$, and let P_E denote projection away from V, acting on row vectors, so that $\mathsf{P}_V + \mathsf{P}_E = \mathsf{I}$. Given $w \in \mathbb{R}^n$, recursively define $\{e_j \in E_1 : j = 0, 1, 2, \dots\}$ and $\{v_j \in V : j = 0, 1, 2, \dots\}$ by the requirements that $w = e_0 + v_0$ and

$$v_j \mathsf{B}_{j+1} = e_{j+1} + v_{j+1}\ . \tag{4.2}$$

Now applying P_V (resp.P_E) to this equation yields

$$v_{j+1} = v_{j+1}\mathsf{B}_{j+1}\mathsf{P}_V \quad \text{and} \quad e_{j+1} = v_j \mathsf{B}_{j+1}\mathsf{P}_E\,.$$

Since $E_1(\mathsf{B}_j) = E_1$ is a simple 1-eigenspace, we have

$$e\mathsf{B}_j = e \quad \text{for all} \quad e \in E_1\ .$$

Together with (4.2) this yields

$$\begin{aligned} w\mathsf{B}_1\mathsf{B}_2\cdots\mathsf{B}_k &= (e_0 + e_1 + v_1)\mathsf{B}_2\mathsf{B}_3\cdots\mathsf{B}_k \\ &= (e_0 + e_1 + e_2 + v_2)\mathsf{B}_3\cdots\mathsf{B}_k \\ &= e_0 + e_1 + \cdots + e_k + v_k\,. \end{aligned} \tag{4.3}$$

Now $v_j = v_j \mathsf{P}_V$ hence $v_{j+1} = v_j \mathsf{P}_V \mathsf{B}_{j+1} \mathsf{P}_V$, and iterating this relation yields

$$\begin{aligned} w\mathsf{B}_1\mathsf{B}_2\cdots\mathsf{B}_k\mathsf{P}_V &= v_k \\ &= v_0(\mathsf{P}_V\mathsf{B}_1\mathsf{P}_V)(\mathsf{P}_V\mathsf{B}_2\mathsf{P}_V)\cdots(\mathsf{P}_V\mathsf{B}_k\mathsf{P}_V) \\ &= w(\mathsf{P}_V\mathsf{B}_1\mathsf{P}_V)(\mathsf{P}_V\mathsf{B}_2\mathsf{P}_V)\cdots(\mathsf{P}_V\mathsf{B}_k\mathsf{P}_V). \end{aligned} \tag{4.4}$$

By hypothesis (3), the joint spectral radius $\hat{\rho}(\mathsf{P}_V\Sigma\mathsf{P}_V) < 1$. This means there is a submultiplicative matrix norm $\|\cdot\|$ and a finite value l such that

$$\tilde{\rho} := \hat{\rho}_l(\mathsf{P}_V\Sigma\mathsf{P}_V, \|\cdot\|) < 1\,. \tag{4.5}$$

By product boundedness of Σ_B,

$$\|\mathsf{P}_V\mathsf{M}\mathsf{P}_V\| \le C_1 \quad \text{for all} \quad \mathsf{M} \in \Sigma_B. \tag{4.6}$$

For the matrix norm $\|\cdot\|$ there is a constant C_2 such that for all $\mathsf{M} \in Mat_{n\times n}$,

$$\|w\mathsf{M}\| \le C_2\|w\|_2\|\mathsf{M}\| \quad \text{for all} \quad w \in \mathbb{C}^n\,,$$

where $\|w\|_2$ is the l_2-norm. We break the product $\prod_{j=1}^k \mathsf{B}_j$ into $\lfloor k/l \rfloor$ blocks of length l, with at most $l-1$ leftover matrices in the right, then (4.4) gives

$$\begin{aligned} ||w\mathsf{B}_1\mathsf{B}_2\cdots\mathsf{B}_k\mathsf{P}_V||_2 &\le C_2\|w\|_2 \left\| \prod_{j=1}^k (\mathsf{P}_V\mathsf{B}_j\mathsf{P}_V) \right\| \\ &\le (\max(1, C_1))^{l-1} C_2 \|w\|_2 \tilde{\rho}^{\lfloor k/l \rfloor}\,. \end{aligned}$$

For fixed w, letting $k \to \infty$ gives

$$\|w\mathsf{M}^{(\infty)}\mathsf{P}_V\| \le \limsup_{k\to\infty} \left\| \left(w \prod_{i=1}^k \mathsf{B}_i \right) \mathsf{P}_V \right\|_2 = \limsup_{k\to\infty} \|v_k\|_2 = 0\,. \tag{4.7}$$

Applying this with $w = u_i$ for $1 \le i \le n$ yields $\mathsf{M}^{(\infty)}\mathsf{P}_V = \mathbf{0}$ which implies all rows of $\mathsf{M}^{(\infty)}$ are in $E_1(\sigma)$. Thus hypothesis (3) of Lemma 5.1 of [7] holds, and the lemma applies to show that Σ is an RCP set. ■

Remark. The criterion of Theorem 5.2 is computationally effective when Σ_B is a finite set. In general Σ_B is an infinite set, and then this criterion is not effective. It would be desirable to obtain a strengthened criterion of this type that would show that the collection of finite RCP sets with rational entries is recursively enumerable. Does there exist, for every finite RCP set with rational entries, a finite length proof that it is an RCP set?

5. Revised Theorem 6.1

The statement (C1) in Theorem 6.1 of [7] requires a stronger hypothesis. We correct it in (C1) in the revised theorem below, and we also formulate a new equivalent condition (C1′).

Theorem 6.1. *For a finite set $\Sigma = \{\mathsf{P}_0, \mathsf{P}_1, \dots, \mathsf{P}_{m-1}\}$ of $n \times n$ column stochastic nonnegative matrices, the following conditions are equivalent.*

(C1) Σ is an RCP set in which each generalized left 1-eigenspace $\Sigma_1(\mathsf{P}_i)$ is one-dimensional, $0 \le i \le m-1$.

(C1′) Σ is an RCP set having a continuous limit function, whose generalized left 1-eigenspace $E_1(\Sigma)$ is one dimensional.

(C2) All finite products $\mathsf{P}_{d_1} \cdots \mathsf{P}_{d_k}$ are irreducible and aperiodic.

(C3) There exists a finite s such that for all $k \ge s$ all products $\mathsf{P}_{d_1}\mathsf{P}_{d_2} \cdots \mathsf{P}_{d_k}$ are scrambling.

(C4) There exists a finite μ such that for all products $\mathsf{P}_{d_1} \cdots \mathsf{P}_{d_k}$ of length $k \ge \mu$ have a row with all entries nonzero.

(C5) All left-infinite products from Σ are weakly ergodic.

Proof. The implications (C2)$\Leftrightarrow$(C3)$\Leftrightarrow$(C4) and (C5)$\Rightarrow$(C4) are established in [7]. The implication (C2)& (C4)$\Rightarrow$(C1) is given in [7], as the argument there shows that all $E_1(\mathsf{P}_i)$ are one-dimensional.

(C1′)$\Rightarrow$(C1). Theorem 4.2(2) of [7] implies that all matrices P_i in Σ have the same generalized left 1-eigenspace E_1. Now by definition $E_1(\Sigma) := \cap_{j=0}^{n-1} E_1(\mathsf{P}_j) = E_1$. It follows that all $E_1(\mathsf{P}_i) = E_1$ are one-dimensional.

(C1)$\Rightarrow$(C1′). Since all $E_1(\mathsf{P}_i)$ are one-dimensional they are simple eigenspaces. Since $\dim(E_1(\Sigma)) \ge 1$ for any RCP set and $E_1(\Sigma) \subseteq E_1(\mathsf{P}_i)$, it follows that $E_1(\Sigma)$ is one-dimensional, and equal to each $E_1(\mathsf{P}_i)$. By Theorem 2.1 (3) of [7], we have $E_1(\mathsf{B}) = E_1(\Sigma)$ for all finite matrix products B. Next, row stochasticity and nonnegativity of the P_i imply that Σ is product-bounded. By Theorem 5.1 of [7], as amended in §4 above, there is a subspace V of $\mathbb{R}^n$ with $\dim(V) = n-1$ and $E_1(\Sigma) + V = \mathbb{R}^n$, such that if P_V is the projection on V away from $E_1(\Sigma)$ then the set $\mathsf{P}_V\Sigma\mathsf{P}_V = \{\mathsf{P}_V\mathsf{P}_i\mathsf{P}_V : 0 \le i \le m-1\}$ has joint spectral radius $\hat{\rho}(\mathsf{P}_V\Sigma\mathsf{P}_V) < 1$. We have verified that Σ satisfies condition (2) in Theorem 4.2 of [7], so this theorem applies to conclude that Σ has a continuous limit function.

(C1′)⇒(C5). This follows from Corollary 4.2a of [7]. ∎

We reformulate Corollary 6.1a of [7] using the following decision problem.

(7) CONTINUOUS COLUMN-STOCHASTIC RCP SET

Instance: A set $\Sigma = \{\mathsf{P}_0, \dots, \mathsf{P}_{m+1}\}$ of $n \times n$ matrices with rational entries that are nonnegative and column-stochastic.

Question: Is Σ an RCP-set with a continuous limit function, with $E_1(\Sigma)$ being one-dimensional?

Corollary 6.1a. *There is an effective decision procedure for CONTINUOUS COLUMN-STOCHASTIC RCP SET.*

Proof. We can test condition (C4) of Theorem 6.1 effectively. Let $\mathcal{P}_k$ denote the property that each product $\mathsf{P}_{d_1}\mathsf{P}_{d_2}\cdots\mathsf{P}_{d_k}$ of length k from Σ has a row with all entries nonzero. If a set Σ has property $\mathcal{P}_\mu$ then it is easy to see that it has property $\mathcal{P}_k$ for all $k \ge \mu$. Paz [25] showed that if (C4) holds, then it holds for some $\mu \le \mu_n := \frac{1}{2}(3^n - 2^{n+1} + 1)$. Thus it suffices to check if property $\mathcal{P}_\mu$ holds for some μ with $1 \le \mu \le \mu_n$. ∎

6. Other Errata

1. p.228: The displayed formula following (1.2): sup could be replaced by max.
2. p.230, l.6 and the next displayed formula: 'n' should be replaced by 'i'.
3. p.233, Lemma 2.1: The formula following the phrase 'if in addition Σ is finite ... ' actually holds regardless of whether or not Σ is finite.
4. p.233, Theorem 2.1: Here and hereafter by an "eigenspace" what is meant is "generalized eigenspace."

5. p.233, l.-3: Add '$\neq 0$' after 'vectors v_1, v_2'.

6. p.234, l.-7: The word 'finite' in 'For any finite subset … ' is extraneous.

7. p.236: Formula (3.4): the signs '$<$' must be replaced by '$\leq$'.

8. p.237, l.7: '$\hat{\sigma}(\Sigma)$' must read '$\hat{\varrho}(\Sigma)$'; the same happens in formula (4.3).

9. p.239, l.-14: '$\mathsf{W} = \mathsf{A}_i, \ldots, \mathsf{A}_{i_k}$' must read '$\mathsf{W} = \mathsf{A}_{i_1} \cdots \mathsf{A}_{i_k}$'.

10. p.239, l.-10: '$\varrho_j(\Sigma) \leq \ldots$' should read '$\hat{\rho}_j(\Sigma) = \ldots$'.

11. p.239, l.-8: Put a hat on ϱ in $\limsup_{j\to\infty} \varrho_j(\Sigma)^{1/j} \ldots$.

12. p.241: The paragraph next to formula (4.4) could be replaced by the observation 'Directly from (1), $\mathsf{A}^{(\infty)} = \mathbf{0}$'.

13. p.242, l.1: In §4 the term "orthogonal projection" is used erroneously. What is meant is the projection onto V along E_1.

14. p.242, l.12: '$\mathsf{S}^{-1}V\mathsf{S}$' must read '$\mathsf{S}^{-1}V$'.

15. p.243: The rightmost part of (4.7) should be

$$t \cdot \max\{1, \hat{\rho}_1^t(\tilde{\Sigma})\} \frac{\left(\hat{\rho}_t(\tilde{\Sigma})\right)^{\left\lfloor \frac{k+1}{t} \right\rfloor - 1}}{1 - \hat{\rho}_t(\tilde{\Sigma})}.$$

16. p.245, l.9: ' … and taking $C_i = \mathsf{P}_V \mathsf{A}_i \mathsf{P}_{E_1}$, $\tilde{\mathsf{A}}_i = \mathsf{P}_V \mathsf{A}_i \mathsf{P}_V$'. This should be:

$$\mathsf{P}_V \mathsf{A}_i \mathsf{P}_{E_1} = \mathsf{S} \begin{pmatrix} 0 & 0 \\ C_i & 0 \end{pmatrix} \mathsf{S}^{-1}, \qquad \mathsf{P}_V \mathsf{A}_i \mathsf{P}_V = \mathsf{S} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\mathsf{A}}_i \end{pmatrix} \mathsf{S}^{-1}.$$

17. p.245, l.-2: Add a tilde over Σ in '$\hat{\rho}_2(\Sigma, \|\cdot\|)$'.

18. p.246: Formula (5.1) should also contain $\forall k \in N$.

19. p.248, l.11: 'i' must be replaced by 'j' in 'If $A^{(l)} = \prod_{j=1}^{l} A_{d_i}$'. The same typo occurs on p.249, l.-11.

20. p.249, l.9: the factor $(1 - \hat{\rho}_t(\Sigma))^{-1}$ must not be in the formula.

21. p.250:, Formula (5.5): the second '$\leq$' must be '$=$'.

22. p.251, l.11: 'If conditions (1)–(3)' should be 'If condition (3)'.

23. p.252, l.-9: instead of '$\|\mathsf{P}_j\|_s \leq \varrho_1(\Sigma_B)$' there should be '$\|\mathsf{D}_j\| \leq \hat{\varrho}_1(\Sigma_B)$'.

24. p.252: Formula (5.11) should contain m instead of r.

25. p.252: Formula (5.12): drop the last factor and replace the first by $t\Delta^{2t}$.

26. p.253, ll.8-9: 'Lemma 5.2' should read 'Lemma 5.1'. See §4 above.

27. p.254, l.4: 'Section 4' should read 'Section 5'.

28. p.254, l.9: 'equal' should read 'constant'.

29. p.256, Fig.2: the ordinate of the point separating segment 2 from segment 3 should be $\frac{1}{3}y_n^- + \frac{2}{3}y_n^+$.

30. p.259: The caption for Fig.3 should contain $f(2x+1)$, $f(2x+2)$, $f(2x+3)$ rather than $f(2x-1)$ etc. or, alternatively, the preceding equation should contain $-$'s rather than $+$'s.

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