

The Lerch Zeta Function and Related Functions

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J. C. Lagarias and W.-C. Winnie Li , The Lerch Zeta Function
I. Zeta Integrals, Forum Math, 24 (2012), 1–48.

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IV. Two-variable Hecke operators, in preparation.

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Topics Covered

- Part I. History: Lerch Zeta and Lerch Transcendent
- Part II. Basic Properties
- Part III. Multi-valued Analytic Continuation
- Part IV. Consequences
- Part V. Lerch Transcendent
- Part VI. Two variable Hecke operators

Part I. Lerch Zeta Function: History

- The Lerch zeta function is:

$$\zeta(s, a, c) := \sum_{n=0}^{\infty} \frac{e^{2\pi i n a}}{(n + c)^s}$$

- The Lerch transcendent is:

$$\Phi(s, z, c) = \sum_{n=0}^{\infty} \frac{z^n}{(n + c)^s}$$

Thus

$$\zeta(s, a, c) = \Phi(s, e^{2\pi i a}, c).$$

Special Cases-1

- Hurwitz zeta function (1882)

$$\zeta(s, 0, c) = \zeta(s, c) := \sum_{n=0}^{\infty} \frac{1}{(n+c)^s}.$$

- Periodic zeta function (Apostol (1951))

$$e^{2\pi i a} \zeta(s, a, 1) = F(a, s) := \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^s}.$$

Special Cases-2

- Fractional Polylogarithm

$$z \Phi(s, z, 1) = Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

- Riemann zeta function

$$\zeta(s, 0, 1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

History-1

- [Lipschitz \(1857\)](#) studies general Euler integrals including the Lerch zeta function
- [Hurwitz \(1882\)](#) studied Hurwitz zeta function.
- [Lerch \(1883\)](#) derived a three-term functional equation.
(Lerch's Transformation Formula)

$$\begin{aligned}\zeta(1-s, a, c) = (2\pi)^{-s}\Gamma(s) & \left(e^{\frac{\pi i s}{2}} e^{-2\pi i a c} \zeta(s, 1-c, a) \right. \\ & \left. + e^{-\frac{\pi i s}{2}} e^{2\pi i c(1-a)} \zeta(s, c, 1-a) \right).\end{aligned}$$

History-2

- de Jonquiere (1889) studied the function

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{x^n}{n^s},$$

sometimes called the **fractional polylogarithm**, getting integral representations and a functional equation.

- Barnes (1906) gave contour integral representations and method for analytic continuation of functions like the Lerch zeta function.

History-3

- Further work on functional equation: [Apostol \(1951\)](#), [Berndt \(1972\)](#). Also: [Weil 1976](#), *Elliptic curves according to Eisenstein and Kronecker*, covers functional equation, Eisenstein-Kronecker-Lerch series.
- Work on value distribution: [Garunkštis \(1996\)](#), (1997), (1999), [Laurinčikas \(1997\)](#), (1998), (2000), [Laurinčikas and Matsumoto \(2000\)](#). Work up to 2002 summarized in [L. & G.](#) book on the Lerch zeta function.
- Other books: [Kanemitsu, Tsukada](#), *Vistas of Special Functions*, [Chakraborty, Kanemitsu, Tsukada](#), *Vistas of Special Functions, Volume II* (2007, 2010).

Objective 1: Analytic Continuation

- **Objective 1.** Analytic continuation of Lerch zeta function and Lerch transcendent in three complex variables.
- **Relevant Work:** Kanemitsu, Katsurada, Yoshimoto (2000) gave a single-valued analytic continuation of Lerch transcendent in three complex variables: they continued it to various large simply-connected domain(s) in $\mathbb{C}^3$.
- We obtain a continuation to a **multivalued function** on a maximal domain of holomorphy in 3 complex variables.

Objective 2: Extra Structures

- **Objective 2.** Determine effect of analytic continuation on other structures: **difference equations** (non-local), **linear PDE** (local), and **functional equation**.
- Behavior at **special values**: $s \in \mathbb{Z}$.
- Behavior near **singular values** $a, c \in \mathbb{Z}$; these are “singularities” of the three-variable analytic continuation.

Objectives: Singular Strata

- The values $a, c \in \mathbb{Z}$ give (non-isolated) singularities of this function of three complex variables. What is the behavior of the function approaching the singular strata?
- The Hurwitz zeta function and Periodic zeta function lie on “singular strata” of real codimension 2. The Riemann zeta function lies on a “singular stratum” of real codimension 4.
- There also is analytic continuation in the s -variable on the singular strata (in many cases, perhaps all cases).

Part II. Basic Structures

Important structures of the Lerch zeta function include:

1. Functional Equation(s).
2. Differential-Difference Equations
3. Linear Partial Differential Equation
4. Integral Representations

Four Term Functional Equation-1

- **Defn.** Let a and c be real with $0 < a < 1$ and $0 < c < 1$. Set

$$L^{\pm}(s, a, c) := \zeta(s, a, c) \pm e^{-2\pi ia} \zeta(s, 1-a, 1-c).$$

Formally:

$$L^{+}(s, a, c) = \sum_{n=-\infty}^{\infty} \frac{e^{2\pi ina}}{|n+c|^s}.$$

- **Defn.** The completed function

$$\hat{L}^{+}(s, a, c) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L^{+}(s, a, c)$$

and the completed function

$$\hat{L}^{-}(s, a, c) := \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L^{-}(s, a, c).$$

Four Term Functional Equation-2

- **Theorem** (Weil (1976)) Let $0 < a, c < 1$ be real. Then:

(1) The completed functions $\hat{L}^+(s, a, c)$ and $\hat{L}^-(s, a, c)$ extend to **entire functions** of s and satisfy the functional equations

$$\hat{L}^+(s, a, c) = e^{-2\pi iac} \hat{L}^+(1-s, 1-c, a)$$

and

$$\hat{L}^-(s, a, c) = i e^{-2\pi iac} \hat{L}^-(1-s, 1-c, a).$$

(2) These results extend to $a = 0, 1$ and/or $c = 0, 1$. If $a = 0, 1$ then $\hat{L}^+(s, a, c)$ is a **meromorphic function** of s , with simple poles at $s = 0, 1$. In all other cases these functions are **entire functions** of s .

Functional Equation Zeta Integrals

- Part I of our papers obtains a generalized functional equation for Lerch-like zeta integrals containing a test function. (This is in the spirit of [Tate's thesis](#).)
- These equations relate a integral with test function $\phi(x)$ at point s to integral with Fourier transform $\hat{f}(\xi)$ of test function at point $1 - s$.
- The self-dual test function $f_0(x) = e^{-\pi x^2}$ yields the function $\hat{L}^+(s, a, c)$. The eigenfuctions $f_n(x)$ of the oscillator representation yield similar functional equations: Here $f_1(x)$ yields $\frac{1}{\sqrt{2\pi}}\hat{L}^-(s, a, c)$.

Differential-Difference Equations

- The Lerch zeta function satisfies two differential-difference equations.

- (Raising operator)

$$\frac{\partial}{\partial c} \zeta(s, a, c) = -s \zeta(s + 1, a, c).$$

- Lowering operator)

$$\left(\frac{1}{2\pi i} \frac{\partial}{\partial a} + c \right) \zeta(s, a, c) = \zeta(s - 1, a, c)$$

- These operators are non-local in the s -variable.

Linear Partial Differential Equation

- The Lerch zeta function satisfies a **linear PDE**:

$$\left(\frac{1}{2\pi i} \frac{\partial}{\partial a} + c\right) \frac{\partial}{\partial c} \zeta(s, a, c) = -s \zeta(s, a, c).$$

- The (formally) skew-adjoint operator

$$\Delta_L := \frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + c \frac{\partial}{\partial c} + \frac{1}{2} I$$

has

$$\Delta_L \zeta(s, a, c) = -\left(s - \frac{1}{2}\right) \zeta(s, a, c).$$

Integral Representations

- The Lerch zeta function has two different integral representations, generalizing integral representations in Riemann's original paper.
- Riemann's formulas are:

$$\int_0^\infty \frac{e^{-t}}{1 - e^{-t}} t^{s-1} dt = \Gamma(s) \zeta(s)$$

and, formally,

$$\int_0^\infty \vartheta(0; it^2) t^{s-1} dt \text{ " = " } \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

where

$$\vartheta(0; \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}.$$

Integral Representations

- The generalizations to Lerch zeta function are

$$\int_0^\infty \frac{e^{-ct}}{1 - e^{2\pi ia} e^{-t}} t^{s-1} dt = \Gamma(s) \zeta(s, a, c)$$

and

$$\int_0^\infty e^{\pi c^2 t^2} \vartheta(a + ict^2, it^2) t^{s-1} dt = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s, a, c).$$

where

$$\vartheta(z; \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau} e^{2\pi i n z}.$$

Part III. Analytic Continuation for Lerch Zeta Function

- **Theorem.** $\zeta(s, a, c)$ analytically continues to a **multivalued function** over the domain

$$\mathcal{M} = (s \in \mathbb{C}) \times (a \in \mathbb{C} \setminus \mathbb{Z}) \times (c \in \mathbb{C} \setminus \mathbb{Z}).$$

It becomes single-valued on the maximal abelian cover of $\mathcal{M}$.

- The monodromy functions giving the multivaluedness are computable. For fixed s , they are built out of the functions

$$\phi_n(s, a, c) := e^{2\pi i n a} (c - n)^{-s}, \quad n \in \mathbb{Z}.$$

$$\psi_{n'}(s, a, c) := e^{2\pi c(a - n')} (a - n')^{s-1} \quad n' \in \mathbb{Z}.$$

Analytic Continuation-Features

- The manifold $\mathcal{M}$ is invariant under the symmetries of the functional equation: $(s, a, c) \mapsto (1 - s, 1 - c, a)$.
- **Fact.** The four term functional equation extends to the maximal abelian cover by analytic continuation. It expresses a non-local symmetry of the function.

Lerch Analytic Continuation: Proof

- **Step 1.** The first integral representation defines $\zeta(s, a, c)$ on the simply connected region

$$\{0 < \operatorname{Re}(a) < 1\} \times \{0 < \operatorname{Re}(c) < 1\} \times \{0 < \operatorname{Re}(s) < 1\}.$$

Call it the **fundamental polycylinder**.

- **Step 2a.** Weil's four term functional equation extends to fundamental polycylinder by analytic continuation. It leaves this polycylinder invariant.
- **Step 2b.** Extend to **entire function** of s on fundamental polycylinder in (a, c) -variables, together with the four-term functional equation.

Lerch Analytic Continuation: Proof -2

- **Step 3.** Integrate single loops around $a = n, c = n'$ integers, using contour integral version of first integral representation to get initial monodromy functions

Here **monodromy functions** are difference (functions) between a function and the same function traversed around a closed path. They are labelled by elements of $\pi_1(\mathcal{M})$.

- **Step 4.** The monodromy functions themselves are multivalued, but in a simple way: Each is multivalued around a single value $c = n$ (resp. $a = n'$). They can therefore be labelled with the place they are multivalued. (This gives functions $\phi_n, \psi_{n'}$)

Lerch Analytic Continuation: Proof -3

- **Step. 5.** Iterate to the full homotopy group in (a, c) -variables by induction on generators; use fact that a -loop homotopy commutes with c -loop homotopy.
- **Step. 6.** Explicitly calculate that the monodromy functions all vanish on the commutator subgroup $[\pi_1(\mathcal{M}), \pi_1(\mathcal{M})]$ of $\pi_1(\mathcal{M})$. This gives single-valuedness on the maximal abelian covering of $\mathcal{M}$.

Exact Form of Monodromy Functions-1

- At points $c = m \in \mathbb{Z}$,

$$M_{[Y_m]}(Z) = c_1(s) e^{2\pi i m a} (c - m)^{-s}$$

in which

$$c_1(s) = 0 \quad \text{for } m \geq 1,$$

$$c_1(s) = e^{2\pi i s} - 1 \quad \text{for } m \leq 0.$$

Also

$$M_{[Y_m]-1}(Z) = -e^{2\pi i s} M_{[Y_m]}(Z).$$

$$M_{[Y_m]\pm k}(Z) = \frac{e^{\pm 2\pi i k s} - 1}{e^{\pm 2\pi i s} - 1} M_{[Y_m]\pm 1}(Z).$$

Exact Form of Monodromy Functions-2

- At points $a = m \in \mathbb{Z}$,

$$M_{[X_m]}(Z) = c_2(s) e^{2\pi i c(a-m)} (a-m)^{s-1}$$

where

$$c_2(s) = -\frac{(2\pi)^s e^{\frac{\pi i s}{2}}}{\Gamma(s)}.$$

Also

$$M_{[X_m]-1}(Z) = -e^{2\pi i s} M_{[X_m]}(Z)$$

$$M_{[X_m]^{\pm k}}(Z) = \frac{e^{\mp 2\pi i k s} - 1}{e^{\mp 2\pi i s} - 1} M_{[X_m]^{\pm 1}}(Z).$$

Extended Lerch Analytic Continuation

- **Theorem.** $\zeta(s, a, c)$ analytically continues to a multivalued function over the (larger) domain

$$\mathcal{M}^\# = (s \in \mathbb{C}) \times (a \in \mathbb{C} \setminus \mathbb{Z}) \times (c \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}).$$

Here the extra points $c = 1, 2, 3, \dots$ are glued into $\mathcal{M}$. The extended function is single-valued on the maximal abelian cover of $\mathcal{M}^\#$.

- The manifold $\mathcal{M}^\#$ is **not invariant** under the four term Lerch functional equation. This is a **broken symmetry**.

Part IV. Consequences

We determine the effect of analytic continuation on the other properties of the Lerch zeta function above.

1. **Functional Equation.** This is inherited by analytic continuation on $\mathcal{M}$ but not on $\mathcal{M}^\sharp$.
2. **Differential-Difference Equations.** These equations lift to the maximal abelian cover of $\mathcal{M}$. However they are not inherited individually by the monodromy functions.

Consequences: Other Properties

(3) **Linear PDE**. This lifts to the maximal abelian cover. That is, this PDE is **equivariant** with respect to the covering map. The monodromy functions are all solutions to the PDE.

For fixed s the monodromy functions give an infinite dimensional vector space of solutions to this PDE.
(View this vector space as a **direct sum**.)

Consequences: Special Values

- **Theorem.** The monodromy functions **vanish identically** when $s = 0, -1, -2, -3, \dots$. That is, for these values of s the value of the Lerch zeta function is well-defined on the manifold $\mathcal{M}$, without lifting to the maximal abelian cover.
- The functional equation takes $s \rightarrow 1 - s$. At nonnegative integer values of $s = 1, 2, \dots$ there is nontrivial monodromy. However at these points the monodromy functions satisfy linear dependencies that cancel out of the functional equation.

Consequences: Special Values-2

- It is well known that at the special values $s = 0, -1, -2, \dots$ the Lerch zeta function simplifies to a rational function of c and $e^{2\pi ia}$.
- **p-adic interpolation** is possible at the special values $s = 0, -1, -2, -3, \dots$. This can be obtained from the analytic continuation as a **continuous limit**, taking $c \rightarrow 1+$, using the periodic zeta function. (It does not seem to be possible using the Hurwitz zeta function.)

Approaching Singular Strata

- There are (sometimes!) **discontinuities** in the Lerch zeta function's behavior approaching a singular stratum: these depend on the value of the s -variable.

Observation. Only the **real part** of the s -variable matters.
Three regimes:

$$\operatorname{Re}(s) < 0; \quad 0 \leq \operatorname{Re}(s) \leq 1; \quad \operatorname{Re}(s) > 1.$$

Part V. Lerch Transcendent

We determine the effect of analytic continuation on the
Lerch transcendent

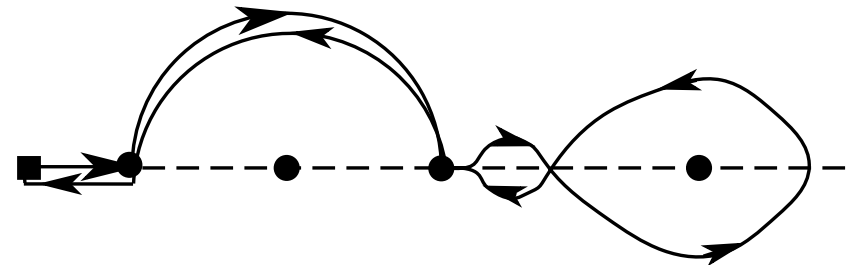
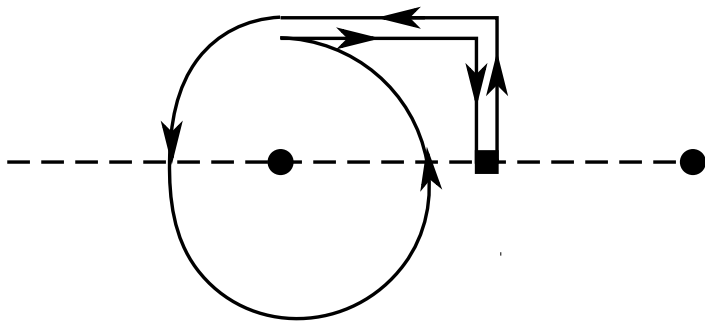
$$\Phi(s, z, c) = \sum_{n=0}^{\infty} \frac{z^n}{(n+c)^s}.$$

We make the change of variable $z = e^{2\pi i a}$ so that

$$a = \frac{1}{2\pi i} \log z.$$

This introduces extra multivaluedness: a is a multivalued function of z .

Image under $z = e^{2\pi i a}$ of loop around $a = 0$,
using basepoints $a = \frac{1}{2}$ and $z = -1$.



Polylogarithm

- The Lerch transcendent (essentially) specializes to the m -th order **polylogarithm** at $c = 1$, $s = k \in \mathbb{Z}_{>0}$.

$$L_m(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^k} = z\Phi(k, z, 1).$$

- The m -th order polylogarithm satisfies an $(m + 1)$ -st order **linear ODE in the complex domain**. This equation is **Fuchsian** on the Riemann sphere, i.e. it has regular singular points. These are located at $\{0, 1, \infty\}$.
- The point $c = 1$ is on a **regular stratum**. This uses the **extended analytic continuation**, which is not invariant under the functional equation.

Analytic Continuation for Lerch Transcendent

- **Theorem.** $\Phi(s, z, c)$ analytically continues to a multivalued function over the domain

$$\mathcal{N} = (s \in \mathbb{C}) \times (z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}) \times (c \in \mathbb{C} \setminus \mathbb{Z}).$$

It becomes single-valued on a two-fold solvable cover of $\mathcal{N}$.

- The monodromy functions giving the multivaluedness are explicitly computable, but complicated.

Monodromy Functions for Lerch Transcendent

For fixed s , the monodromy functions are built out of the functions

$$\phi_n(s, z, c) := z^n (c - n)^{-s}, \quad n \in \mathbb{Z}.$$

and

$$f_n(s, z, c) := e^{\pi i(s-1)} e^{2\pi i n c} z^{-c} \left(n - \frac{1}{2\pi i} \operatorname{Log} z\right)^{s-1} \quad \text{if } n \geq 1.$$

$$f_n(s, z, c) := e^{2\pi i n c} z^{-c} \left(\frac{1}{2\pi i} \operatorname{Log} z - n\right)^{s-1} \quad \text{if } n \leq 0.$$

taking $z^{-c} = e^{-c \operatorname{Log} z}$. where $\operatorname{Log} z$ denotes a branch of the logarithm cut along the positive real axis.

Functional Equations: Lerch Transcendent

The Lerch transcendent satisfies functional equations inherited from the Lerch zeta function. However they are multivalued!

Set

$$L_{\Phi}^{+}(s, z, c) := \Phi(s, z, c) + (1/z)\Phi(s, 1/z, 1 - c).$$

Then functional equation becomes

$$\hat{L}_{\Phi}^{+}(s, z, c) = z^c \hat{L}_{\Phi}^{+}(1 - s, e^{-2\pi i c}, (1/2\pi i) \log a).$$

($\hat{L}$ is completed L -function.)

Differential-Difference Equations: Lerch Transcendent

- The Lerch transcendent satisfies two differential-difference equations. These operators are non-local in the s -variable.

- (Raising operator) Let $D_{\Phi}^{+} := \frac{\partial}{\partial c}$. Then

$$D_{\Phi}^{+} \Phi(s, z, c) = -s \Phi(s + 1, z, c).$$

- Lowering operator) Let $D_{\Phi}^{-} := \left(z \frac{\partial}{\partial z} + c \right)$. Then

$$D_{\Phi}^{-} \Phi(s, z, c) = \Phi(s - 1, z, c)$$

Linear Partial Differential Equation: Lerch Transcendent

As with Lerch zeta function, the Lerch transcendent satisfies a linear PDE:

$$D_{\Phi}^{-} D_{\Phi}^{+} \Phi(s, z, c) = -s \Phi(s, a, c).$$

These operators satisfy Canonical Commutation relations:

$$D_{\Phi}^{+} D_{\Phi}^{-} - D_{\Phi}^{-} D_{\Phi}^{+} = I.$$

The (formally) skew-adjoint operator

$$\tilde{\Delta}_L := \frac{1}{2} \left(D_{\Phi}^{-} D_{\Phi}^{+} + D_{\Phi}^{+} D_{\Phi}^{-} \right)$$

has

$$\tilde{\Delta}_L \Phi(s, z, c) = -\left(s - \frac{1}{2}\right) \Phi(s, z, c).$$

Specialization to Polylogarithm-1

- For integer $s = k \geq 1$, and c as a parameter, the function $z\Phi(k, z, c)$ gives a deformation of the polylogarithm:

$$Li_k(z, c) := \sum_{m=0}^{\infty} z^m (m + c)^k.$$

- Viewing c as fixed, it satisfies the **Fuchsian ODE** $D_c Li_k(z, c) = 0$ where the differential operator is:

$$D_c := z^2 \frac{d}{dz} \left(\frac{1-z}{z} \right) \left(z \frac{d}{dz} + c - 1 \right)^m.$$

- The **singular stratum** points are $c = 0, -1, -2, -3, \dots$

Specialization to Polylogarithm-2

- A basis of solutions for each regular stratum point is

$$\{Li_m(z, c), z^{1-c}(\log z)^{m-1}, z^{1-c}(\log z)^{m-2}, \dots, z^{1-c}\}.$$

- The monodromy of the loop $[Z_0]$ on this basis is:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & e^{-2\pi ic} & e^{-2\pi ic} \frac{2\pi i}{1!} & \dots & e^{-2\pi ic} \frac{(2\pi i)^{m-2}}{(m-2)!} & e^{-2\pi ic} \frac{(2\pi i)^{m-1}}{(m-1)!} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & e^{-2\pi ic} & e^{-2\pi ic} \frac{2\pi i}{1!} \\ 0 & 0 & 0 & \dots & 0 & e^{-2\pi ic} \end{pmatrix}.$$

The monodromy of the loop $[Z_1]$ is unipotent and independent of c .

Specialization to Polylogarithm-3

- A basis of solutions for each singular stratum point is

$$\{Li_m^*(z, c), z^{1-c}(\log z)^{m-1}, z^{1-c}(\log z)^{m-2}, \dots, z^{1-c}\}$$

- The monodromy of the loop $[Z_0]$ in this basis is:

$$\begin{pmatrix} 1 & \frac{2\pi i}{1!} & \frac{(2\pi i)^2}{2!} & \dots & \frac{(2\pi i)^{m-1}}{(m-1)!} & \frac{(2\pi i)^m}{m!} \\ 0 & 1 & \frac{2\pi i}{1!} & \dots & \frac{(2\pi i)^{m-2}}{(m-2)!} & \frac{(2\pi i)^{m-1}}{(m-1)!} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \frac{2\pi i}{1!} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

It is unipotent. The monodromy of the loop $[Z_1]$ is unipotent and independent of c .

Specialization to Polylogarithm-4

Some observations:

- The monodromy representation (of π_1 of the Riemann sphere minus $0, 1, \infty$) is **upper triangular**, and is unipotent exactly when c is a positive integer (regular strata) or c is a negative integer (singular strata!).
- The equation makes sense on the singular strata. The equation remains Fuchsian on the singular strata. The monodromy representation continues to be unipotent, paralleling the positive integer case (polylogarithms at $c = 1$). However it takes a **discontinuous jump** at these points. (It appears unfixable by change of basis).

Part VI. Two variable Hecke Operators

- In Part IV we introduce and study two-variable “Hecke operators”

$$T_m(F)(a, c) := \frac{1}{m} \sum_{j=0}^{m-1} F\left(\frac{a + k}{m}, mc\right)$$

- We show that **formally** these operators commute, and they also commute with the differential operator Δ_L . These operators are non-local, and **dilate** in the c -direction while **contract and shift** in the a -direction.
- For fixed s the LZ function is a **simultaneous eigenfunction** of these operators, with eigenvalue m^{-s} for T_m .

Two-variable Hecke operators-2

- There is a generalization to the Lerch zeta function of Milnor's 1983 result characterizing the Hurwitz zeta function $\zeta(s, z)$ as a simultaneous eigenfunction of "Kubert operators":

$$T_m(F)(z) = \frac{1}{m} \sum_{j=0}^{m-1} F\left(\frac{z + k}{m}\right)$$

- The two-variable Hecke operators exchange the a and c -variables. This exchanges s and $1 - s$, compared to Milnor's results.

Two-variable Hecke operators-3

- The two-variable Hecke operators induce a Hecke action on the vector space of all monodromy functions of the analytically continued Lerch zeta functions.
- **Observation.** This action resembles part of the Bost-Connes Hecke algebra action in the C^* -dynamical system they found whose partition function is the Riemann zeta function.
- The action can be suitably completed to give the full Bost-Connes Hecke algebra.

Summary-1

- A number of interesting extra structures associated with the Lerch zeta function, and related these to analytic continuation.
- **Observation.** The analytic continuation of Lerch zeta fails at values a, c integers, which are the most interesting values where the Hurwitz and Riemann zeta functions appear. These are **singular points**. Understanding the behavior as the singular points are approached may shed new light on these functions.

Summary-2

- Does the Lindelöf hypothesis hold for the Lerch zeta function? (Question raised by [Garunkštis-Steuding](#)).

Observation. If so, Lindelöf hypothesis would hold also for all the multivalued branches as well, because the monodromy functions are all of slow growth in the t -direction.

Thank You!