Smooth Solutions to the ABC Equation

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Credits

- This talk reports on joint work with K. Soundararajan (Stanford)

- graphics in this talk were taken off the web using Google images.
  (Google search methods use number(s) theory.)

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Table of Contents

1. (Contents of Talk)-1

- The talk considers the ABC equation

$$A + B + C = 0.$$

  This is a homogeneous linear Diophantine equation.

- We study multiplicative properties of the solutions (A, B, C), i.e. solutions with restrictions on their prime factorization.

Contents of Talk-2

- The height of a triple (A, B, C) is

$$H := H(A, B, C) = \max\{|A|, |B|, |C|\}$$

- The radical of a triple (A, B, C) is

$$R := R(A, B, C) = \prod_{p|ABC} p$$

- The smoothness of a triple (A, B, C) is

$$S := S(A, B, C) = \max\{p : p \text{ divides } ABC\}.$$

Contents of Talk-3

- The ABC Conjecture concerns the relation of the height and the radical of relatively prime triples (A, B, C).

- We consider the relation of the height and the smoothness of relatively prime triples.

- We formulate the XYZ Conjecture concerning this relation.

An Example

- $(A, B, C) = (2401, -2400, -1)$

- $2401 = 7^4$
  $2400 = 2^5 \cdot 3 \cdot 5^2$
  $1 = 1$

- The height is $H = 2401$.

- The radical is $R = 2 \cdot 3 \cdot 5 \cdot 7 = 210$.

- The smoothness is $S = 7$.

Another Example

- $(A, B, C) = (2^{n+1}, -2^n, -2^n)$
- The height is $H = 2^{n+1}$.
- The radical is $R = 2$.
- The smoothness is $S = 2$.
- Relative primality condition needed: Without it get infinitely many solutions with small radical R (resp. smoothness S), arbitrarily large height H.

The Basic Problem

Problem. How small can be the smoothness S be, as a function of the height H,

so that:

There are infinitely many relatively prime triples (A, B, C) with these values satisfying $A + B + C = 0$?

Nomenclature

- A smooth number is a number all of whose prime factors are “small.” This means all prime factors at most y, where y is the smoothness bound.

- Some authors call such numbers friable. In English, this means: brittle, easily crumbled or crushed into powder.

ABC Conjecture

- ABC Conjecture. There is a positive constant α_1 such that:

- For any $\epsilon > 0$ there are

  (a) infinitely many relatively prime solutions (A, B, C) with radical

  $$R \leq H^{\alpha_1 + \epsilon}$$

  (b) finitely many relatively prime solutions (A, B, C) with radical

  $$R \leq H^{\alpha_1 - \epsilon}.$$

- **Remark.** Most versions of ABC Conjecture assert $\alpha_1 = 1$.

XYZ Equation

- To avoid confusion with ABC Conjecture, we define the XYZ equation to be:

- X+Y+Z=0.

XYZ Conjecture

- XYZ Conjecture. There is a positive constant α_0 such that the XYZ equation $X + Y + Z = 0$ has:

- For any $\epsilon > 0$ there are

  (a) infinitely many relatively prime solutions (X, Y, Z) with smoothness

  $$S \leq (\log H)^{\alpha_0 + \epsilon}$$

  (b) finitely many such solutions (X, Y, Z) with smoothness

  $$S \leq (\log H)^{\alpha_0 - \epsilon}.$$

- **Question.** What should be the threshold value α_0?

Counting Smooth Numbers

Definition. $\Psi(x, y)$ counts the number of integers $\leq x$ all of whose prime factors $p \leq y$.

Notation. The quantity

$$u := \frac{\log x}{\log y}$$

is very important in characterizing the size of $\Psi(x, y)$.

Dickman Rho function

- $\rho(u)$ is a continuous function with $\rho(u) = 1$ for $0 \leq u \leq 1$, determined by the difference-differential equation

$$u\rho'(u) = -\rho(u-1).$$

  It is positive and rapidly decreasing on $1 \leq u < \infty$.

- The Dickman ρ-function is named after Karl Dickman, in his only published paper:

- "On the frequency of numbers containing prime factors of a certain relative magnitude," Arkiv för Math., Astron. och Fysik **22A** (1930), 1–14.

- Dickman showed (heuristically) that

$$\Psi\left(x, x^{\frac{1}{\beta}}\right) \sim \rho(\beta)x.$$

Logarithmic Scale

- For $y = x^\beta$ a positive proportion of integers below x have all prime factors smaller than y.

- We consider smoothness bounds y where there are only some positive power x^γ of integers below x having factors smaller than y. This scale is $y = (\log x)^\alpha$.

- For $\alpha > 1$ there holds

$$\Psi(x, (\log x)^\alpha) \sim x^{1-\frac{1}{\alpha}+o(1)}$$

- There is a threshold at $\alpha = 1$, below which $\Psi(x, y) = O(x^\epsilon)$; it qualitatively changes behavior.

Heuristic Argument-1

- "Claim". The threshold value in the XYZ Conjecture ought to be $\alpha_0 = \frac{3}{2}$.

- Heuristic Argument. (a) Pick (A, B) to be y-smooth numbers. There are $\Psi(x, y)^2$ choices. Assume these give mostly distinct values of $C = -(A + B)$.

- (b) The probability that a random C is y-smooth is $\frac{\Psi(x,y)}{x}$.
  Reasonable chance of at least one "hit" would require (assuming independence)

$$\Psi(x, y)^3 > x.$$

- (c) Thus take y so that $\Psi(x, y) = x^{\frac{1}{3}+\epsilon}$.
  Then $1 - \frac{1}{\alpha} = \frac{1}{3}$ so $\alpha = \frac{3}{2}$ and:

$$y = (\log x)^{\frac{3}{2}+\epsilon}.$$

Heuristic Argument-2

- Claim 2. The threshold value in the XYZ Conjecture for relatively prime solutions to $A + B = 1$ should be $\alpha_0 = 2$.

- Heuristic Argument. (a) Pick A to be a y-smooth numbers. There are $\Psi(x, y)$ choices. Assume these give mostly distinct values of $B = -(A - 1)$.

- (b) The probability that a random B is y-smooth is $\frac{\Psi(x,y)}{x}$.
Reasonable chance of at least one "hit" would require (assuming independence)

$$\Psi(x, y)^2 > x.$$

- (c) Thus take y so that $\Psi(x, y) = x^{\frac{1}{2}+\epsilon}$.
Then $1 - \frac{1}{\alpha} = \frac{1}{2}$ so $\alpha = 2$ and:

$$y = (\log x)^{2+\epsilon}.$$

Example Revisited

- $(A, B, C) = (2401, -2400, -1)$.

- $\log 2401 \approx 7.783$

- $(\log 2401)^2 \approx 60.584$

- $S = 7$.

- Is this a “lucky” example? Numerically the matching value of $\alpha = 1$, not $\alpha_0 = 2$ as in the heuristic.

Main Result

- Theorem (Alphabet Soup Theorem)
  $ABC + GRH$ implies XYZ.

- This is a conditional result. It has two (unequal) parts.

- Lower Bound Theorem
  ABC Conjecture $\Longrightarrow$ the XYZ constant $\alpha_0 \geq 1$.

- Upper Bound Theorem
  Generalized Riemann Hypothesis (GRH) $\Longrightarrow$ the XYZ constant $\alpha_0 \leq 8$.

Main Result: Comments

- The exact constant α_0 is not determined by the Alphabet Soup Theorem, only its existence is asserted.

- Lower Bound Theorem assuming ABC Conjecture: This is Easy Part.

- Upper Bound Theorem assuming GRH: This is Hard Part.

  Stronger result: Get asymptotic formula for number of primitive solutions, for $\alpha_0 > 8$. Use Hardy-Littlewood method (circle method).

2. Lower Bound assuming ABC Conjecture

- *ABC* Conjecture.
  For each $\epsilon > 0$ there are only finitely many relatively prime triples (A, B, C) having
  $$R \leq H^{1-\epsilon}.$$

- Recall:

- The height of a triple (A, B, C) is
  $$H := H(A, B, C) = \max\{|A|, |B|, |C|\}$$

- The radical of a triple (A, B, C)
  $$R := R(A, B, C) = \prod_{p|ABC} p$$

Remarks: Lower Bound

- The ABC Conjecture implies many things: (asymptotic) Fermat's Last Theorem, etc.

- It is a powerful hammer, here we use it to crack something small.

- XYZ Lower bound is a very easy consequence of ABC Conjecture.

(Conditional) Lower Bound Theorem

- Theorem. (XYZ Lower Bound) Assuming the ABC Conjecture, the constant α_0 in the XYZ Conjecture satisfies.

$$\alpha_0 \geq 1.$$

- Note: This lower bound $\alpha_0 \geq 1$ is exactly at the value $(\log x)^\alpha$ where the behavior of $\Psi(x, y)$ changes.

Lower Bound Theorem-Proof

- (a) The radical R and smoothness S of any (A, B, C) are related by

$$R = \prod_{p|ABC} p \leq \prod_{p \leq S} p$$

- (b) This easily gives

$$R \leq \exp\left(S(1 + o(1))\right),$$

since $\prod_{p \leq y} p = e^{y(1+o(1))}$.

- (c) Argue by contradiction. Suppose, for fixed $\epsilon > 0$, have infinitely many solutions

$$S \leq (1 - \epsilon) \log H.$$

Combine with (b) to get, for such solutions,

$$R \leq e^{(1-\epsilon) \log H (1+o(1))} \ll H^{1-\frac{1}{2}\epsilon}.$$

This contradicts the ABC Conjecture.

Unconditional Lower Bound

- Theorem. For each $\epsilon > 0$ there are only finitely many relatively prime solutions to $A + B + C = 0$ having height H and smoothness S satisfying

$$S \leq (3 - \epsilon) \log \log H$$

- **Proof.** Similar to above, but using unconditional result:

- Theorem. (Cam Stewart and Kunrui Yu) There is a constant c_1 such that any primitive solution to $A + B + C = 0$ has height H and radical R satisfying

$$H \leq \exp\left(c_1 R^{\frac{1}{3}} (\log R)^{\frac{1}{3}}\right).$$

Cameron L. Stewart

Kunrui Yu

3. Upper Bounds assuming GRH

- We assume:

- Generalized Riemann Hypothesis. (GRH) All the zeros of the Riemann zeta function and all Dirichlet L-functions $L(s, \chi)$ inside the critical strip $0 < Re(s) < 1$ have real part $\frac{1}{2}$.

- **Note.** We allow imprimitive Dirichlet characters so the L-functions may have complex zeros on the line $Re(s) = 0$.

Upper Bound Theorem

- Theorem. (Height-Smoothness Upper Bound) If the GRH holds, then for each $\epsilon > 0$ there are infinitely many primitive solutions (A, B, C) for which the height H and smoothness S satisfy

$$S \leq (\log H)^{8+\epsilon}.$$

- Corollary. (XYZ Conjecture Upper Bound) If the GRH holds, then the constant $\alpha_{\mathbf{0}}$ in the XYZ Conjecture satisfies

$$\alpha_0 \leq 8.$$

Remarks on Upper Bound Theorem-1

- Proof establishes a stronger result: An asymptotic formula counting the number of (weighted) solutions (X, Y, Z).

- Approach to this uses the Circle Method, combined with the Saddle-Point method of Hildebrand-Tenenbaum for counting y-smooth numbers below a bound x.

- The method counts all solutions, without imposing the relative primality condition. An inclusion-exclusion argument is needed at the end to get relative primality.

Remarks on Upper Bound Theorem-2

- Crucial new ingredient for minor arcs: An expansion of additive characters in terms of a set of multiplicative functions.

- The additive characters are the exponentials in the circle method.

- A "spanning set" of multiplicative functions (depending on a parameter y) are Dirichlet characters of small conductor and multiplicative functions $g_t(n) = n^{it}$ for some range of t. (The latter are "continuous spectrum").

Technical Main Theorem: Weighted Sums-1

- Let $\Phi(x)$ be a "weight function" compactly supported on $(0, \infty)$.

- Main case: A "bump function" that is the constant 1 on $[\epsilon, 1-\epsilon]$ and is 0 outside $[\frac{1}{2}\epsilon, 1-\frac{1}{2}\epsilon]$.

- Think:

$$\Phi(x) := \chi_{[0,1]}(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1, \\ 0 & \text{if } \quad x \ge 1 \end{cases}$$

(not compactly supported)

Technical Main Theorem: Weighted Sums-2

- Weighted Sum to Estimate

$$N_\Phi(x,y) := \sum_{\substack{X,Y,Z \in \mathcal{S}(y) \\ X+Y=Z}} \Phi(\frac{X}{x})\Phi(\frac{Y}{x})\Phi(\frac{Z}{x}).$$

- Here $\mathcal{S}(y)$ = the set of all integers with no prime factor larger than y.

- For "bump function" this sum only detects solutions with $max(|X|,|Y|,|Z|) \le x$.

Technical Main Theorem: Weighted Sums-4

- The Hildebrand-Tenenbaum saddle-point method evaluates a certain contour integral by integrating on a saddle point line $Re(s) = c$, which lies in critical strip.

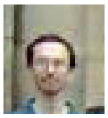
- Given range x and smoothness bound y, the associated saddle point value c is the (unique) positive solution to the equation

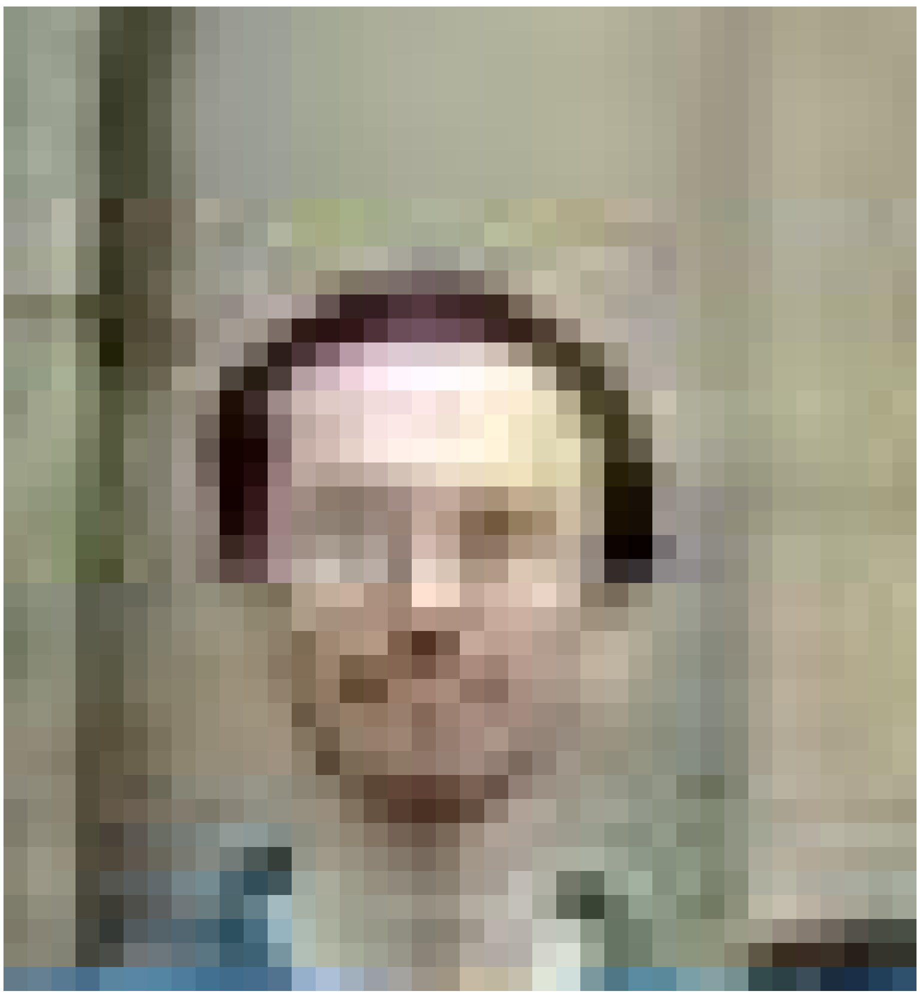
$$\sum_{p \le y} \frac{\log p}{p^c - 1} = \log x.$$

- If $y = (\log x)^\alpha$, with $\alpha > 3$, then the saddle point value is

$$c = 1 - \frac{1}{\alpha} + O\left(\frac{\log\log\log x}{\log\log x}\right).$$

Adolf J. Hildebrand





(An Elusive Sighting...)

Gérald Tenenbaum

Technical Main Theorem: Statement

- Theorem. (Counting Weighted Solutions) For all (x, y) with $y = (\log x)^{\alpha}$ with

$$8 + \epsilon \le \alpha \le 8 + \frac{1}{\epsilon},$$

($\epsilon > 0$ fixed) the weighted sum $N_{\Phi}(x, y)$ counting all solutions (including imprimitive ones) is given by main term:

$$\mathfrak{S}_f(1 - \frac{1}{\alpha})\mathfrak{S}_{\infty}(\Phi, 1 - \frac{1}{\alpha})\frac{\Psi(x, y)^3}{x}$$

which contains two singular series, and by remainder term $R(x, y)$:

$$R(x, y) = O\left(\frac{\Psi(x, y)^3}{x} \cdot \frac{1}{(\log \log x)^{1-\epsilon}}\right)$$

Technical Main Theorem: Singular Series

- Finite Place Singular Series $\mathfrak{S}(c) :=$

$$\prod_p \left(1 + \frac{p-1}{p(p^{3c-1}-1)} \left(\frac{p-p^c}{p-1}\right)^3\right)$$

(converges for $Re(c) > \frac{2}{3}$, diverges at $c = \frac{2}{3}$.)

- Archimedean Singular Series $\mathfrak{S}(\Phi, c) :=$

$$c^3 \int\int \Phi(t_1)\Phi(t_2)\Phi(t_1+t_2)(t_1t_2(t_1+t_2))^{c-1}dt_1dt_2.$$

(Converges for $Re(c) > \frac{2}{3}$, diverges at $c = \frac{2}{3}$, for step function $\Phi = \chi_{[0,1]}(t)$)

- Singular series are positive for real $c > \frac{2}{3}$, i.e. for $\alpha > 3$!

Inclusion-Exclusion Theorem-1

- Theorem. (Counting Primitive Solutions) Let the weighted sum $N^*_\Phi(x,y)$ count primitive solutions only. For all (x,y) with $y = (\log x)^\alpha$ satisfying

$$8 + \epsilon < \alpha < 8 + \frac{1}{\epsilon},$$

(ϵ fixed) the weighted sum $N^*_\Phi(x,y)$ is given by the main term

$$\frac{1}{\zeta(2-\frac{3}{\alpha})}\mathfrak{S}_f(1-\frac{1}{\alpha})\mathfrak{S}_\infty(\Phi, 1-\frac{1}{\alpha})\frac{\Psi(x,y)^3}{x}$$

in which the remainder term $R(x,y)$ satisfies

$$R(x,y) = O\left(\frac{\Psi(x,y)^3}{x} \cdot \frac{1}{(\log\log x)^{1-\epsilon}}\right)$$

Technical Main Theorem 2: Inclusion-Exclusion-2

- The inclusion-exclusion factor

$$\frac{1}{\zeta(2-\frac{3}{\alpha})}$$

  is positive for $\alpha > 3$. However it is 0 at $\alpha = 3$.

- Interpretation.
  For $\alpha > 3$ a positive fraction of all integer solutions are primitive integer solutions.
  But for $0 < \alpha < 3$ a zero fraction of all integer solutions are primitive solutions.

- Heuristic: Expect infinitely many primitive solutions only for $\alpha > \frac{3}{2}$.

Hardy-Littlewood Method-1

- Weighted Counting Function Identity

$$N(x, y; \Phi) = \int_0^1 E(x, y, \beta)^2 E(x, y; -\beta) d\beta,$$

  where integrand is ...

- Weighted Exponential sum.

$$E(x, y; \beta) := \sum_{n \in \mathcal{S}(y)} e(n\beta)\Phi(\frac{n}{x})$$

- Here

$$e(\beta) = exp(2\pi i \beta).$$

Hardy-Littlewood Method-2

- Estimate the integral

$$N(x, y; \Phi) = \int_0^1 E(x, y, \beta)^2 E(x, y; -\beta) d\beta,$$

- Idea (a) Cut the integration interval $[0, 1]$ up into subintervals $I(\frac{a}{q})$, indexed by rational numbers $\frac{a}{q}$ having small denominators: $q < \sqrt{x}$.

- (b) The integrand is large in small neighborhoods of $\frac{a}{q}$ with small denominators, $q \leq x^{1/4}$, these are the major arcs $J(\frac{a}{q})$. Estimate their size exactly, get the main term of asymptotics.

- (c) Show the integrand is "small" everywhere else, the minor arcs, get the remainder term.

Hardy-Littlewood Method-3

- Cutoff Exponent δ. This is an adjustable parameter. $0 < \delta < 1/4$, fixed in advance.

- Major Arcs. $1 \le q \le x^{1/4}$, take

$$J(\frac{a}{q}) := \{\beta : |\beta - \frac{a}{q}| \le \frac{1}{x^{1-\delta}}\}$$

(As δ gets smaller, these intervals get smaller. But they stlll give main term!)

- Minor Arcs. Everything else. For denominators $x^{1/4} \le q \le x^{1/2}$ it is the whole Farey interval $I(\frac{a}{q})$.

For denominators $1 \le q \le x^{\frac{1}{4}}$ it is part of Farey interval not covered by $J(\frac{a}{q})$, which generally consists of two pieces.

Multiplicative Character Decomposition-1

- Weighted Exponential sum Consider:

$$E(x, y; \beta) := \sum_{n \in \mathcal{S}(y)} e(n\beta)\Phi(\frac{n}{x})$$

  (This involves the additive character $f(n) := e(n\beta)$)

- Idea: Expand in "basis" of multiplicative characters. These include:
  (a) Dirichlet characters $\chi(n)$ (mod q) for general integer modulus q, may be imprimitive or primitive character.
  (b) continuous characters $\psi_t(n) := n^{it}$.

- This is overdetermined set, extract suitable subset, depending on parameters (x, y).

Multiplicative Character Decomposition-2

- Dirichlet Character Decomposition. For $\beta = \frac{a}{b}$, given n set $d := (n, q)$. Then

$$e(n\beta) = \frac{1}{\phi(\frac{q}{d})} \sum_{\chi (\bmod \frac{q}{d})} \tau(\bar{\chi}) \chi(\frac{na}{d})$$

where $\tau(\chi)$ is a Gauss sum.

- This decomposition preserves L^2-norm:

$$\sum_{\chi (\bmod \frac{q}{d})} \frac{|\tau(\chi)|^2}{\phi(\frac{q}{d})^2} = 1.$$

Multiplicative Character Decomposition-3

- Transformed Weight Function Given weight function $\Phi(x)$, form the Laplace-Mellin transform

$$\hat{\Phi}(s,\lambda) := \int_0^\infty \Phi(w)e(\lambda w)w^{s-1}dw$$

- Size Lemma 1. This function $\hat{\Phi}(s,\lambda)$ is "small" except when $|s| \approx |\lambda|$

- Size Lemma 2. This function satisfies the L^1-smallness bound

$$\int_{-\infty}^{\infty} |\hat{\Phi}(c+it,\lambda)|dt << (1+|\lambda|)^{1/2+\epsilon}.$$

Multiplicative Character Decomposition-4

- Continuous Character Decomposition.
  The function $f(n) = e(n\beta)$

$$e(n\beta)\Phi(\frac{n}{x}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widehat{\Phi}(s, \beta x)(\frac{x}{n})^s ds.$$

- This decomposition preserves L^2-norm:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{\Phi}(c+it, \beta x)|^2 = \int_{-\infty}^{\infty} |\Phi(e^u)e(\beta x e^u)e^{cu}|^2 du.$$

Exponential Sums with Dirichlet Characters

- Reduction of Problem. For $\beta = \frac{a}{q} + \gamma$, can express exponential sum $E(x, y; \beta)$ as a combination of generalized exponential sums $E(x, y; \chi, \gamma)$ over Dirichlet characters (mod q). Here...

- Generalized Weighted Exponential sum
For $\chi(n)$ a Dirichlet character (mod q),

$$E(x, y; \chi, \gamma) := \sum_{n \in \mathcal{S}(y)} e(n\gamma)\chi(n)\Phi(\frac{n}{x})$$

Partial Euler Product

- Now use Dirichlet series for smooth numbers...

- Partial Euler Product

$$\zeta(s; y) := \prod_{p \le y} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n \in \mathcal{S}(y)} n^{-s}.$$

- Partial Euler Product with Dirichlet Character

$$L(s; \chi, y) := \prod_{p \le y} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \sum_{n \in \mathcal{S}(y)} \chi(n) n^{-s}.$$

Relation to (Partial) L-Functions

- To estimate $E(x, y; \chi, \gamma)$ use...

- Inverse Mellin Integral Formula

$$E(x, y : \chi, \gamma) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s; \chi, y) \hat{\Phi}(s, \gamma x) x^s ds.$$

- Idea. $L(s; \chi, y)$ behaves somewhat like $L(s; \chi)$, part way into the critical strip. Actually compare $\log L(s; \chi, y)$ and $\log L(s, \chi)$. Use the GRH to control the error, shift contour to the line $Re(s) = 1/2 + \epsilon$.

Minor Arcs Estimate

- Theorem. (Minor Arcs Estimate) Assume GRH. Then:

  (1) For non-principal Dirichlet character $\chi(\text{mod } q)$, and for $|\gamma|x < x^{1/2}$,

  $|E(x, y : \chi, \gamma)| << (1+|\gamma|x)^{1/2+1/2\epsilon} x^{1/2+1/2\epsilon}$

  (2) For principal character $\chi(\text{mod } q)$, and $y < x^{\delta}$, and $x^{\delta} \leq |\gamma|x < x^{1/2}$, the same estimate holds.

- (1) applies to major arcs as well. So get:

  Corollary. Only the principal characters (mod q) for "small" q make large contribution to the major arcs!

Minor Arcs Estimate-Comments

- Estimate achieves a power savings in x.
- Main loss from converting L^2-estimate to L_1-estimate.
- The minor arcs method is crucial to the method.

Minor Arcs Estimate-Proof

- Inverse Mellin Transform Formula

  $$E(x, y : \chi, \gamma) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} L(s; \chi, y)\hat{\Phi}(s, \gamma x)x^s ds.$$

- Idea. Goal is to upper bound absolute value of integral. The quantity $|\hat{\Phi}(s, \gamma x)|$ is large only in narrow region, must control size of $|L(s; \chi, y)|$ there. Relate it to L-function $L(s; \chi)$.

- Control of estimates in q-aspect (Dirichlet characters) and the T-aspect (continuous characters) used.

Major Arcs Formula

- Only Principal Characters Matter. Main term formulas involve only integrals against partial Euler product $L(s; \chi_{0,q}, y)$ + a small error term. This simplifies the expression for the major arcs.

- The formulas for main term are obtained by Hildebrand-Tenenbaum saddle-point method contour integral shift.

- The GRH is invoked (again) to show the simplified "main term" inside the major arcs sums is dominant contribution.

- Singular Series formulas. These arise naturally from the sum of main terms over q, the Farey fractions. The archimedean integral decouples from the finite primes.

3. Final Remarks

- Linear Diophantine Equations.

  This $GL(1)$ method works because have first degree equations.

- This approach doesn't work for higher degree equations, where the circle method was originally applied (Waring's problem).

Extensions of the Method

- One can expect variants of this method to apply to:

- (a) Nontrivial coefficients in the ABC-equation

$$aA + bB + cC = 0.$$

- (b) Fixed side congruence conditions to be imposed on A, B, C.

- (c) Systems of several linear homogeneous Diophantine equations.

Unconditional Upper Bound?

- It might be possible to remove GRH assumption.

- Main Obstacle. Need Minor Arc estimates with some power of x savings.

- Cannot shift contour near critical line. Hope to use zero density results to show "most" L-functions in the sums contribute a small amount. (Contours shifted by zero locations.)

- If method works, expect to get a (much) worse upper bound on α_0.

Unconditional Lower Bound?

- No ideas!

The Last Slide...

Thank you for your attention!