Math 675, Homework 5 (Due Friday, November 12, 2015, in class)
(or by 3pm in the box outside EH3086)

(Revised problem numbering) Problems with a star, if there are any, might be harder. Problem 41(d) is not assigned.

38. (*Dirichlet Series Convergence Regions*) Determine as much as you can about the abscissa of absolute convergence $\sigma_a(F)$ and the abscissa of conditional convergence $\sigma_c(F)$ of the following Dirichlet series.

(a) $F_1(s) = \sum_{n=1}^{\infty} \sigma(n) n^{-s}$.

(b) $F_2(s) = \sum_{n=1}^{\infty} (-1)^n \sigma(n) n^{-s}$

(c) $F_3(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}$

(d) $F_4(s) = \sum_{n=1}^{\infty} \lambda(n) n^{-s}$, where $\lambda(n) = (-1)^{\Omega(n)}$. Here it may be helpful to find an expression for $F_4(s)$ in terms of the Riemann zeta function.

39.(*Poisson summation formula-1)*

Establish a finite Poisson summation formula, asserting that for integers $A < B$,

$$\sum_{n=A}^{B}{}' f(n) = \sum_{n=-\infty}^{\infty} \int_A^B f(x) e^{2\pi i n x} dx,$$

where the $'$ in the leftmost sum indicates that the end terms of the sum are to be replaced by $\frac{1}{d} f(A)$ and $\frac{1}{2} f(B)$, respectively, under the following assumption on $f(x)$: $f(x)$ is real-valued, continuous and piecewise monotonic, with only a finite number of monotonic intervals.

40. (*Poisson summation formula-2)*

(a) Let $f(x) \in L^1(\mathbb{R})$. Show that the series

$$\phi(t) := \sum_{n \in \mathbb{Z}} f(n+t)$$

converges for almost all t.

(b) Assume in addition that $f(x)$ is continuous and of bounded variation. Establish the Poisson summation formula in the form

$$\sum_{n \in \mathbb{Z}} f(n+t) = \lim_{N \to \infty} \sum_{-N}^{N} \hat{f}(n) e^{2\pi i t n}.$$

holds for all $t \in [0,1]$. (Writing $\hat{f}(\xi) = \mathcal{F}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$, and $\check{f}(x) = \mathcal{F}^{-1}(x) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x \xi} d\xi$, the classical Poisson sum formula say $\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m)$.)

41. (*Fekete Polynomials*) The *Fekete polynomial* is defined for prime p by

$$F_p(x) = \sum_{j=0}^{p-1} (\frac{j}{p})\, x^j,$$

where $(\frac{n}{p})$ is the Legendre symbol.

(a) Show that

$$|F_p(e^{2\pi ik/p})| = \sqrt{p},$$

for $1 \le k \le p-1$, and that $F(0) = F(1) = 0$.

(b) Determine the functional relation between $F_p(x)$ and its reciprocal polynomial, $F_p^*(x) := x^{p-1}F_p(\frac{1}{x})$.

(c) Show that if $F_p(x)$ has no real zeros in $0 < x < 1$ then the Dirichlet L-function $L(s, \chi_p)$ has no real zero in the critical strip, i.e. $L(\sigma, \chi_p) > 0$ for $0 < \sigma < 1$. Here $\chi_p(n) = (\frac{n}{p})$ and we know $L(1, \chi) > 0$.

(d) (**) Suppose that $F(x) = \sum_{j=1}^{N-1} a_j x^j$, with N odd and all $a_j = \pm 1$. Show that if

$$\max |F(e^{2\pi ik/N})| \le \sqrt{N},$$

holds, then $N = p$ is prime and $F(x) = \pm F_p(x)$.

42. (*Analytic Continuation of Zeta Function-1, by Euler-MacLaurin summation*)

(a) Meromorphically continue the Riemann zeta function $\zeta(s)$ to the half-plane $Re(s) > 0$ with a simple pole at $s = 1$, using Euler-MacLaurin summation.

(b) Repeat, and meromorphically continue it to $Re(s) > -1$.

43. (*Kronecker Symbol*) The *Jacobi symbol* $(\frac{n}{m})$ is defined for all integers n and all positive odd integers $m = \prod_p (p_i)^{e_i}$ by

$$(\frac{n}{m}) = \prod_p (\frac{n}{p_i})^{e_i}.$$

It satisfies an extension of the quadratic reciprocity law (which you may need to solve this problem.)

The *Kronecker symbol* $(\frac{n}{m})$ is defined for all $m \neq 0$ by adding to the product above the two symbols

$$(\frac{n}{-1}) = \begin{cases} 1 & \text{if } n \ge 0 \\ -1 & \text{if } n < 0. \end{cases}$$

$$(\frac{n}{2}) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2 \\ 1 & \text{if } n \equiv \pm 1 \pmod 8, \\ -1 & \text{if } n \equiv \pm 3 \pmod 8. \end{cases}$$

It is also defined for $m = 0$ by the recipe $(\frac{\pm 1}{0}) = 1$ and $(\frac{n}{0}) = 0$ otherwise.

(a) Prove that for $m > 1$, the Kronecker symbol $\chi_m(a) = (\frac{a}{m})$ is a Dirichlet character (mod m).

(b) Determine for which integer a, the Kronecker symbol $\tilde{\chi}_a(m) = (\frac{a}{m})$ is periodic with some period b, and when it is, determine b as a function of a. Then determine for which a is $\tilde{\chi}_a(m)$ a Dirichlet character (mod b).